Proof

The following proof is based on that outlined by Segal in [4], page 80.

We begin by observing that if $D = \dfrac{\mathrm{d}}{\mathrm{d}\,s}$ and $f(s)$ is a smooth function of the real variable s, then

$$F(D)_{|s=0} f(s) = \int_0^1 f(s)\ \mathrm{d}\,s. \tag{2.2}$$

This holds since the Taylor expansion of a smooth function g satisfies

$$\sum_{k \geqslant 1} \frac{1}{k!} D^k g(s) = g(s+1) - g(s),$$

hence taking $g(s) = \int f(s)\ \mathrm{d}\,s$ to be an indefinite integral of f we obtain

$$\sum_{k \geqslant 0} \frac{1}{(k+1)!} D^k f(s) = g(s+1) - g(s).$$

Evaluating at $s = 0$ gives Equation (2.2).

The matrix-valued function

$$\varphi(s) = \exp(sA) B \exp((1-s)A)$$

can be shown (for example, using Theorem 2.12) to satisfy

$$\varphi(s) = \exp(sA) B \exp(-sA) \exp(A) = \exp(s \operatorname{ad} A)(B)\ \exp(A).$$

So

$$F(D)(\varphi(s)) = \left(\sum_{k \geqslant 0} \frac{((s+1)^{k+1} - s^{k+1})}{(k+1)!} (\operatorname{ad} A)^k \right) (B) \exp(A),$$

giving

$$\begin{aligned} F(D)(\varphi(s))_{|s=0} &= \left(\sum_{k \geqslant 0} \frac{1}{(k+1)!} (\operatorname{ad} A)^k \right) (B) \exp(A) \\ &= F(\operatorname{ad} A)(B) \exp(A). \end{aligned}$$

We also have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}_{|t=0} \exp(A + tB) = \int_0^1 \varphi(s)\ \mathrm{d}\,s,$$

which is obtained by expanding the left hand side as a power series in $A + tB$ and differentiating, then using the identity

$$\int_0^1 \frac{s^m (1-s)^n}{m!\,n!}\ \mathrm{d}\,s = \frac{1}{(m+n+1)!}$$

for $m, n \geqslant 0$ to identify this with the right hand side. The desired formula now follows by combining the last two results. □