hence a solution of Equation (2.8) is

$$\mathbf{v}(t) = \exp(tA)\mathbf{v}_0. \tag{2.10}$$

In fact, this is the unique solution.

Example 2.19

For a skew symmetric matrix $S \in \mathrm{M}_n(\mathbb{R})$ and non-zero $\mathbf{v}_0 \in \mathbb{R}^n$, the differential equation

$$\mathbf{v}'(t) = S\mathbf{v}(t), \quad \mathbf{v}(0) = \mathbf{v}_0.$$

has a solution for which

$$|\mathbf{v}(t)| = |\mathbf{v}_0| \quad (t \in \mathbb{R}).$$

Proof

By Equation (3.13), for all $t \in \mathbb{R}$ we have $\exp(tS) \in \mathrm{SO}(n)$, hence

$$|\mathbf{v}(t)| = |\exp(tS)\mathbf{v}(0)| = |\mathbf{v}(0)|.$$

Notice that since $|\mathbf{v}(t)|^2 = |\mathbf{v}(0)|^2$ is a constant,

$$\begin{aligned} 2\mathbf{v}(t) \cdot \mathbf{v}'(t) &= \mathbf{v}(t) \cdot \mathbf{v}'(t) + \mathbf{v}'(t) \cdot \mathbf{v}(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}\,t}(\mathbf{v}(t) \cdot \mathbf{v}(t)) \\ &= \frac{\mathrm{d}}{\mathrm{d}\,t}|\mathbf{v}(t)|^2 \\ &= 0. \end{aligned}$$

This shows that $\mathbf{v}'(t)$ is tangent to the sphere centred at the origin and of radius $|\mathbf{v}(0)|$ at the point $\mathbf{v}(t)$. $\square$

Here is an explicit example of this type.

Example 2.20

When $n = 3$ and $S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$, the differential equation

$$\mathbf{v}'(t) = S\mathbf{v}(t), \quad \mathbf{v}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

has the solution

$$\mathbf{v}(t) = \frac{1}{5}\begin{bmatrix} 2 - 2\cos\sqrt{5}t \\ 2\sqrt{5}\sin\sqrt{5}t \\ 1 + 4\cos\sqrt{5}t \end{bmatrix}.$$

Proof

One approach to computing $\exp(tS)$ involves determining the eigenvalues of S and then diagonalising it over $\mathbb{C}$; the details are left as an exercise for the reader. We obtain the special orthogonal matrix

$$\exp(tS) = \frac{1}{5}\begin{bmatrix} 4 + \cos\sqrt{5}t & \sqrt{5}\sin\sqrt{5}t & 2 - 2\cos\sqrt{5}t \\ -\sqrt{5}\sin\sqrt{5}t & 5\cos\sqrt{5}t & 2\sqrt{5}\sin\sqrt{5}t \\ 2 - 2\cos\sqrt{5}t & -2\sqrt{5}\sin\sqrt{5}t & 1 + 4\cos\sqrt{5}t \end{bmatrix},$$

giving for the required solution

$$\mathbf{v}(t) = \exp(tS)\mathbf{v}(0) = \frac{1}{5}\begin{bmatrix} 2 - 2\cos\sqrt{5}t \\ 2\sqrt{5}\sin\sqrt{5}t \\ 1 + 4\cos\sqrt{5}t \end{bmatrix}.$$

□

When the matrix A in Equation (2.8) is diagonalisable, this approach works well, provided an explicit diagonalisation can actually be found. In particular, symmetric, skew symmetric, hermitian and skew hermitian matrices are always diagonalisable over $\mathbb{C}$, as are matrices without multiple eigenvalues. For more general matrices with multiple eigenvalues it may be necessary to use Jordan forms as discussed in Section 2.2. This is illustrated in the next example.

Example 2.21

If we take

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{v}_0 \neq \mathbf{0},$$

then the differential equation

$$\mathbf{v}'(t) = A\mathbf{v}(t), \quad \mathbf{v}(0) = \mathbf{v}_0$$

has solution

$$\begin{bmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix}\mathbf{v}_0.$$