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Minimal atomic S-modules and S-algebras

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1 Nuclear spectra and cores

All spectra are localized at a prime $p > 0$.

A cellular spectrum X of finite type is a *Hurewicz complex* if it has no cells of negative dimension, one 0-cell and $\pi_0 X = H_0 X \neq 0$; thus X is (-1)-connected and $H_0(X; \mathbb{F}_p) = \mathbb{F}_p$.

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A map $f\colon Y \longrightarrow X$ of Hurewicz complexes is a *monomorphism* if it induces an isomorphism on $\pi_0(\ )$ and a monomorphism on $\pi_*(\ )$.

A Hurewicz complex is *nuclear* if for all $n \geqslant 0$ its $(n+1)$-skeleton X_{n+1} is obtained from the n-skeleton X_n as the mapping cone of $j_n\colon J_n \longrightarrow X_n$ from a wedge of n-spheres J_n satisfying

$$\ker[j_{n*}\colon \pi_n J_n \longrightarrow \pi_n X_n] \subseteq p \cdot \pi_n J_n. \tag{1.1}$$

A monomorphism $f\colon Y \longrightarrow X$ is a *core* if Y is nuclear. Every X has such a core, but Y may not be unique up to homotopy.

2 Some general results

A Hurewicz complex X is
• *irreducible* if every monomorphism $Y \longrightarrow X$ from a Hurewicz complex is an equivalence;
• *atomic* if any map $X \longrightarrow X$ inducing an isomorphism on $\pi_0(\ )$ is an equivalence.

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• *minimal atomic* if it is atomic and every monomorphism $Y \longrightarrow X$ from an atomic Hurewicz complex Y is an equivalence.

Theorem 2.1. *Let X be a Hurewicz complex.*

(i) *If X is nuclear complex then every core of X is an equivalence.*

(ii) *If X is nuclear then it is irreducible and atomic.*

(iii) *X is minimal atomic iff it is equivalent to a nuclear complex.*

(iv) *X is irreducible iff it is minimal atomic.*

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Priddy [6] noted that the condition (1.1) is equivalent to triviality of the Hurewicz homomorphisms $h\colon \pi_n X_n \longrightarrow H_n(X_n;\mathbb{F}_p)$ for all $n \geqslant 1$.

Definition 2.2. *A Hurewicz complex X has* no mod p detectable homotopy *if the Hurewicz homomorphism $h\colon \pi_n X \longrightarrow H_n(X;\mathbb{F}_p)$ is trivial for all $n \geqslant 1$.*

Definition 2.3. *A complex X is* minimal *if for each n,*

$$H_n(X_n;\mathbb{F}_p) \xrightarrow{\cong} H_n(X_{n+1};\mathbb{F}_p) = H_n(X;\mathbb{F}_p).$$

The next result is a variation on an old result of Cooke [3].

Theorem 2.4. *Every finite type connective p-local complex is equivalent to a minimal complex.*

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Our next result characterizes nuclear complexes in a useful way.

Theorem 2.5 (Nuclear Test). *A Hurewicz complex is nuclear if and only if it is minimal and has no mod p detectable homotopy.*

Using the nuclear test it is easy to identify lots of minimal atomic spectra. For example, if $H^*(X;\mathbb{F}_p)$ is monogenic over the Steenrod algebra then X has no mod p detectable homotopy. Hence $H^*(BP\langle n\rangle;\mathbb{F}_p)$ is always minimal atomic for $0 \leqslant n \leqslant \infty$ and the natural map $BP\langle\infty\rangle = BP \longrightarrow MU_{(p)}$ is a core.

At $p = 2$, the spectra

$$ko,\ ku,\ tmf = eo_2,\ BoP,\ \mathbb{C}\mathrm{P}^\infty,\ \mathbb{H}\mathrm{P}^\infty,\ \mathbb{R}\mathrm{P}^\infty_{-1}$$

are all minimal atomic. $\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$ is atomic but not minimal atomic.

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Sketch proof of the Nuclear Test

Suppose that X is defined inductively by attaching cells so that for each n there is a cofibre exact sequence

$$J_n \xrightarrow{j_n} X_n \longrightarrow X_{n+1}$$

with J_n a finite wedge of n-spheres. Then there is a diagram of long exact sequences containing the following portion whose vertical maps are Hurewicz homomorphisms and $\overline{H}_*$ stands for mod p homology. Notice that the vertical arrow for ΣJ_n is surjective.

(2.1)

$$\begin{array}{ccccccc} \pi_{n+1}X_{n+1} & \longrightarrow & \pi_{n+1}\Sigma J_n & \longrightarrow & \pi_n X_n & \longrightarrow & \pi_n X_{n+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \to \overline{H}_{n+1}X_{n+1} & \longrightarrow & \overline{H}_{n+1}\Sigma J_n & \longrightarrow & \overline{H}_n X_n & \longrightarrow & \overline{H}_n X_{n+1} \end{array}$$

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If X is nuclear then (1.1) holds for every n. A diagram chase shows that the left-most arrow is 0 since the composition

$$\pi_{n+1}X_{n+1} \longrightarrow \pi_{n+1}\Sigma J_n \longrightarrow \overline{H}_{n+1}\Sigma J_n$$

is 0. This shows that X has no mod p detectable homotopy. As the vertical map for X_n is 0, every element of $\overline{H}_{n+1}\Sigma J_n$ lifts to $\pi_{n+1}\Sigma J_n$, hence we see that the boundary map $\overline{H}_{n+1}\Sigma J_n \longrightarrow \overline{H}_n X_n$ is 0 and so X is minimal.

Conversely, if X is minimal and has no mod p detectable homotopy it is nuclear.

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3 S-algebras

From [4] there is a good category of spectra with symmetric monoidal smash product whose unit is the sphere spectrum S. For example, the category of S-modules $\mathscr{M}_S$ in which every object satisfies $S \wedge M \cong M$ (rather than just having $S \wedge M \simeq M$). The derived homotopy category $\mathscr{D}_S$ is equivalent to Boardman's, so the usual homotopy theory of spectra is equivalent to the homotopy theory of S-modules.

An *S-algebra* is an S-module R with product $R \wedge_S R \longrightarrow R$ which is strictly associative and unital in $\mathscr{M}_S$; R is commutative if the obvious commutativity diagram commutes. This is stronger than the notion of a *ring spectrum* in which the associativity and unital conditions only hold in $\mathscr{D}_S$.

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Let R be a commutative S-algebra. An R-module is an S-module with product $R \wedge_S M \longrightarrow M$ which is strictly associative and unital. The category of R-modules $\mathscr{M}_R$ has a strictly associative and unital tensor product that sends M and N to $M \wedge_R N$ and passing to the derived homotopy category $\mathscr{D}_R$. An *R-algebra* is an R-module A with a product $A \wedge_R A \longrightarrow A$ that is strictly associative and unital in $\mathscr{M}_R$. A is an *R-ring spectrum* if it has a product $A \wedge_R A \longrightarrow A$ which is associative and unital in $\mathscr{D}_R$.

For a spectrum X there is a free R-module $\mathbb{F}_R X \simeq R \wedge X$ which is characterized by a universal property. When R is commutative, there is a *free commutative R-algebra*

$$\mathbb{P}_R X = \bigvee_{k \geqslant 0} R \wedge X^{(k)}/\Sigma_k$$

which is the analogue of a polynomial ring over a commutative ring. The map $X \longrightarrow *$ induces an augmentation $\mathbb{P}_R X \longrightarrow \mathbb{P}_R * = R$.

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If A is a commutative R-algebra and $f\colon X \longrightarrow A$ is a map, we can form the diagram of spectra

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr] & \\ A & & \mathrm{C}X \end{tikzcd}$$

and by the universal property we get a diagram of R-algebras

$$\begin{tikzcd} & \mathbb{P}_R X \arrow[dl, "\widetilde{f}"'] \arrow[dr] & \\ A & & \mathbb{P}_R \mathrm{C}X \end{tikzcd}$$

The pushout of this is $A \wedge_{\mathbb{P}_R X} \mathbb{P}_R \mathrm{C}X$.

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If X is a wedge of S^n's we view the resulting R-algebra as the result of attaching $\mathbb{P}_R \mathrm{C}X$, viewed as a bunch of '$(n+1)$-cell objects', to A. This allows us to define cellular and CW objects by starting with S and inductively attaching cells.

For example, given a prime p, using $p\colon S^0 \longrightarrow S^0 \simeq S$ we have $A = S \wedge_{\mathbb{P}_R S^0} \mathbb{P}_R \mathrm{C}S^0$. This is a commutative S-algebra and $\pi_0 A = \mathbb{F}_p$. When $p = 2$, this is a substitute for the Moore spectrum $\mathrm{M}(2)$ which is *not* a commutative ring spectrum. There is a Künneth spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{S_*[x]}({}_pS_*, S_*) \Longrightarrow \pi_{p+q} A,$$

where ${}_pS_* = S_* = \pi_* S$, x has degree 0 and acts on ${}_pS_*$ through multiplication by p.

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A cellular p-local commutative S-algebra R is *nuclear* if its $(n+1)$-skeleton $R_{[n+1]}$ is defined inductively starting with $R_{[0]} = S$ and using a map $k_n \colon K_n \longrightarrow R_{[n]}$ from a finite wedge of n-spheres which satisfies the condition

$$\ker[k_{n*} \colon \pi_n K_n \longrightarrow \pi_n R_{[n]}] \subseteq p \cdot \pi_n K_n, \tag{3.1}$$

analogous to that of (1.1), and then forming

$$R_{[n+1]} = R_{[n]} \wedge_{\mathbb{P}_S K_n} \mathbb{P}_S \mathrm{C} K_n.$$

We will sketch a theory of *nuclear* and *minimal atomic* commutative S-algebras analogous to the one for spectra. A central ingredient is a suitable homology theory defined on pairs of commutative S-algebras B/A (*i.e.*, S-algebras A, B together a morphism $A \longrightarrow B$).

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4 Topological André-Quillen theory

Let A be a commutative S-algebra and let B be a commutative A-algebra. Then (assuming these are q-cofibrant) there is a functor

$$\bar{\mathrm{h}}\mathscr{C}_A \rightsquigarrow \mathscr{D}_B; \quad B \longmapsto \Omega_{B/A}.$$

This has various good properties. For example, if C is a commutative B-algebra, there is a cofibration sequence of C-modules

$$\Omega_{B\wedge_A C/C} \wedge_B C \longrightarrow \Omega_{B/A} \longrightarrow \Omega_{C/A}.$$

Also, as $B \wedge_A C$-modules there are equivalences

$$\Omega_{B\wedge_A C/C} \simeq \Omega_{B/A} \wedge_A C, \quad \Omega_{B\wedge_A C/A} \simeq \Omega_{B/A} \wedge_A C \vee B \wedge_A \Omega_{C/A}.$$

For a spectrum X,

$$\Omega_{\mathbb{P}_R X/R} \simeq (\mathbb{P}_R X) \wedge X, \quad \Omega_{R/\mathbb{P}_R X} \simeq R \wedge X.$$

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Topological André-Quillen homology and *cohomology* of B/A with coefficients in a B-module M are defined by

$$\mathrm{TAQ}_n(B/A; M) = \pi_n(\Omega_{B/A} \wedge_B M),$$
$$\mathrm{TAQ}^n(B/A; M) = \pi_n(F_B(\Omega_{B/A}, M)).$$

If A and B are connective and $\Bbbk = A_0 = B_0$, there is an Eilenberg-Mac Lane object $H\Bbbk$ which is a commutative B-algebra. For an $\Bbbk$-module N we define

$$\mathrm{HAQ}_n(B/A; N) = \pi_n(\Omega_{B/A} \wedge_B HN),$$
$$\mathrm{HAQ}^n(B/A; N) = \pi_n(F_B(\Omega_{B/A}, HN)).$$

When $N = \Bbbk$ we define

$$\mathrm{HAQ}_n(B/A) = \mathrm{HAQ}_n(B/A; \Bbbk),$$
$$\mathrm{HAQ}^n(B/A) = \mathrm{HAQ}^n(B/A; \Bbbk).$$

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For a connective B-algebra with $C_0 = \Bbbk$ there is a long exact sequence

$$\cdots \longrightarrow \mathrm{HAQ}_k(B/A; N) \longrightarrow \mathrm{HAQ}_k(C/A; N) \longrightarrow \mathrm{HAQ}_k(C/B; N)$$
$$\longrightarrow \mathrm{HAQ}_{k-1}(B/A; N) \longrightarrow \cdots$$

and a similar long exact sequence for cohomology.

Some interesting examples are provided by Basterra & Mandell. Let X be connective spectrum and let $\Omega^\infty X \longrightarrow BO$ be an infinite loop map with associated Thom spectrum T. Then

$$\Omega_{T/S} \simeq T \wedge X.$$

For example, $\Omega_{MU/S} \simeq MU \wedge \Sigma^2 ku$ and $\mathrm{HAQ}_k(MU/S) = H_{k-2}(ku)$.

Computations of topological André-Quillen homology for cellular S-algebras can be done in a similar way to cellular homology of CW complexes. Our next results contain the crucial observations required for this.

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Theorem 4.1. *For any n,*

$$\mathrm{HAQ}_k(\mathbb{P}_A S^n/A) = \mathrm{HAQ}_{k+1}(A/\mathbb{P}_A S^n) = \begin{cases} \Bbbk = A_0 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 4.2. *If R is a connective CW commutative S-algebra then $R_{[n]} \longrightarrow R_{[n+1]}$ induces a map $\mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S)$ which is an isomorphism if $k < n$ and an epimorphism if $k = n$. Furthermore there is an exact sequence*

$$\begin{aligned} 0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S) &\longrightarrow \mathrm{HAQ}_{n+1}(S/\mathbb{P}_S K_n) \\ &\longrightarrow \mathrm{HAQ}_n(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_n(R_{[n+1]}/S) \to 0 \end{aligned}$$

in which

$$\mathrm{HAQ}_{n+1}(S/\mathbb{P}_S K_n) \cong \pi_n K_n.$$

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We also have a commutative diagram with exact columns which is analogous to (2.1).

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ \pi_{n+1} R_{[n+1]} & \xrightarrow{\overline{\theta}_{n+1}} & \mathrm{HAQ}_{n+1}(R_{[n+1]}/S; \mathbb{F}_p) \\ \downarrow & & \downarrow \\ \pi_{n+1}\Sigma K_n & \xrightarrow{\text{epi}} & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]}; \mathbb{F}_p) \\ \downarrow & & \downarrow \\ \pi_n R_{[n]} & \xrightarrow[\overline{\theta}_n]{} & \mathrm{HAQ}_n(R_{[n]}/S; \mathbb{F}_p) \end{array} \tag{4.1}$$

Using these ideas we are led to analogues of our results for minimal atomic and nuclear S-modules.