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## Cores of spectra and a construction of $BoP$

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## References

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# 1 Nuclear spectra and cores

All spectra are localized at a prime $p > 0$.

A CW spectrum $X$ of finite type is a *Hurewicz complex of dimension* $n_0$ if it has no cells of dimension less than $n_0$, one $n_0$-cell and $\pi_{n_0} X \neq 0$; thus $X$ is $(n_0 - 1)$-connected and $H_{n_0}(X; \mathbb{F}_p) = \mathbb{F}_p$.

For Hurewicz complexes $X', X$, $f \colon X' \longrightarrow X$ is a *monomorphism* if $f_*$ is an isomorphism on $\pi_{n_0}(\ )$ and a monomorphism on $\pi_*(\ )$.

$X$ is *nuclear* if for each $n \geqslant n_0$, the $(n+1)$-skeleton $X_{n+1}$ is the mapping cone of a map $j_n \colon J_n \longrightarrow X_n$ from a wedge of $n$-spheres $J_n$ for which

(1.1) $$\ker(j_{n*} \colon \pi_n J_n \longrightarrow \pi_n X_n) \subseteq p \cdot \pi_n J_n.$$

A monomorphism $f \colon X' \longrightarrow X$ is a *core* if $X'$ is nuclear. Every such Hurewicz complex $X$ has a core.

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# 2 Some general results

A Hurewicz complex $X$ of dimension $n_0$ is *irreducible* if every monomorphism $X' \longrightarrow X$ is an equivalence, while it is *atomic* if any map $X \longrightarrow X$ inducing an isomorphism on $\pi_{n_0}(\ )$ is an equivalence. If $X$ is atomic then it is *minimal* if every map $X' \longrightarrow X$ from an atomic Hurewicz complex $X'$ of dimension $n_0$ which induces an isomorphism on $\pi_{n_0}(\ )$ and a monomorphism on $\pi_*(\ )$ is an equivalence.

**Proposition 2.1.** *Let $X$ be a Hurewicz complex.*

(i) *If $X$ is nuclear then it is irreducible and atomic.*

(ii) *$X$ is nuclear if and only if it is minimal atomic.*

(iii) *If $X$ is nuclear and $f \colon X' \longrightarrow X$ is a core, then $f$ is an equivalence.*

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Priddy [Pr] noted that the condition of (1.1) is equivalent to triviality of the Hurewicz homomorphism $h\colon \pi_{n+1}X_{n+1} \longrightarrow H_{n+1}(X_{n+1};\mathbb{F}_p)$.

**Proposition 2.2 (The nuclear test).** *Let $X$ be Hurewicz of dimension $n_0$ satisfying the following two conditions.*

(A) *The Hurewicz homomorphism $h\colon \pi_n X \longrightarrow H_n(X;\mathbb{F}_p)$ is trivial for $n > n_0$;*

(B) *For each $n$, inclusion of the $n$-skeleton into the $(n+1)$-skeleton induces an isomorphism*

$$H_n(X_n;\mathbb{F}_p) \xrightarrow{\cong} H_n(X_{n+1};\mathbb{F}_p) = H_n(X;\mathbb{F}_p).$$

*In particular, this holds if the cells of $X$ occur in dimensions differing by at least 2.*

*Then $X$ is nuclear.*

*Conversely, if $X$ is nuclear then condition* (A) *is satisfied.*

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## 3 Some examples

**Example 3.1.** $BP$ is nuclear and the natural map $BP \longrightarrow M\mathrm{U}_{(p)}$ is a core. For any core $X \longrightarrow M\mathrm{U}_{(p)}$, $X \simeq BP$. In particular, Priddy's spectrum $BP'$ is equivalent to $BP$.

Let $\zeta_3 \downarrow \mathbb{H}\mathrm{P}^\infty$ be the bundle associated to the adjoint representation of $S^3$.

**Example 3.2.** For the prime $p = 2$, $\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$, $\Sigma^\infty\mathbb{H}\mathrm{P}^\infty$ and $\Sigma^\infty M\zeta_3$ are nuclear. At an odd prime $p$, there is a non-trivial splitting

$$\Sigma^\infty\mathbb{C}\mathrm{P}^\infty_{(p)} \simeq W_{p,1} \vee W_{p,2} \vee \cdots \vee W_{p,p-1},$$

and each of the $W_{p,r}$ is nuclear.

**Example 3.3.** At $p = 2$, $\Sigma^\infty\mathbb{R}\mathrm{P}^\infty$ is atomic but not nuclear.

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**Theorem 3.4.** *For $m \geqslant 1$, $BP\langle m\rangle$ is nuclear.*

**Corollary 3.5.** *The natural map $BP\langle 1\rangle \longrightarrow ku_{(p)}$ is a core.*

The proof of this Theorem is more involved since $H_*BP\langle m\rangle$ is not concentrated in even degrees alone. However condition (B) of the nuclear test still holds as does condition (A). In the proof we use a modification of an folk result from [Co]. It seems likely that every core of $ku_{(p)}$ is equivalent to $BP\langle 1\rangle$.

**Lemma 3.6.** *Let $X$ be an $(n_0-1)$-connected spectrum of finite type and $(D_*,\partial)$ be a chain complex of free abelian groups with $D_n = 0$ if $n < n_0$ and $\Phi\colon H_*(D_*,\partial) \xrightarrow{\cong} H_*X$. Then there is a $\varphi\colon X' \longrightarrow X$ from a cellular spectrum with cellular chain complex $(\mathrm{C}_*(X',\mathbb{Z}),\mathrm{d})$ and a chain isomorphism $\theta\colon (D_*,\partial) \xrightarrow{\cong} (\mathrm{C}_*(X',\mathbb{Z}),\mathrm{d})$ for which the composite $\varphi_*\circ\theta$ induces $\Phi$. An analogous result holds for a $p$-local cellular spectrum and a chain complex of free $\mathbb{Z}_{(p)}$-modules.*

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# 4 Pengelley's spectrum *BoP*

Pengelley [Pe] constructed an atomic spectrum $BoP$ which is a retract of $M\mathrm{SU}_{(2)}$ and satisfies the conditions of the nuclear test.

**Proposition 4.1.** *$BoP$ is nuclear and any retraction $BoP \longrightarrow M\mathrm{SU}_{(2)}$ is a core.*

It is not clear if every core $X \longrightarrow M\mathrm{SU}_{(2)}$ satisfies $X \simeq BoP$. The following observation may be important in understanding this question. The proof uses a result attributed to Barratt on Toda brackets in $ko_*$.

**Lemma 4.2.** *Let $X$ be a 2-local Hurewicz complex of dimension 0 with inclusion of the bottom cell $w_0\colon S^0 \longrightarrow X$ and let $q\colon X \longrightarrow ko_{(2)}$ be a map giving a homotopy factorization $S^0 \xrightarrow{w_0} X \xrightarrow{q} ko_{(2)}$. If $\nu \in \pi_3 S^0$, $\sigma \in \pi_7 S^0$ satisfy $\nu x = 0 = \sigma x \in \pi_* X$ for every $x \in \pi_* X$, then $q_*\colon \pi_* X \longrightarrow \pi_* ko$ is an epimorphism.*

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Here is another construction of $BoP$. Starting with $BoP'_0 = S^0$, inductively define $BoP'_{2n} = BoP'_{2n+1}$ and a map $g'_{n+1}\colon BoP'_{2n+1} \longrightarrow ko$ by attaching a wedge of $2n$-cells to $BoP'_{2n-1}$ non-trivially as in (1.1) so that the cofibre sequence

$$J'_{2n-1} \xrightarrow{j'_{2n-1}} BoP'_{2n-1} \longrightarrow BoP'_{2n}$$

satisfies

$$\operatorname{im} j'_{2n-1*} = \ker(g_{n*}\colon \pi_{2n-1}BoP'_{2n-1} \longrightarrow \pi_{2n-1}ko).$$

It is straightforward to see that $g'_n$ extends to a map $g'_{n+1}$. This defines a nuclear spectrum $BoP'$ with a map $g'\colon BoP' \longrightarrow ko$ extending the unit $S^0 \longrightarrow ko$. An application of Lemma 4.2 now gives

**Lemma 4.3.** *$g'\colon BoP' \longrightarrow ko$ induces an epimorphism on $\pi_*(\ )$.*

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Let $g\colon BoP \longrightarrow ko$ be the natural map.

**Theorem 4.4.** *There exist $f\colon BoP \longrightarrow BoP'$, $f'\colon BoP' \longrightarrow BoP$ which induce isomorphisms on $\pi_0(\ )$. Hence $ff'$ and $f'f$ are equivalences and $BoP \simeq BoP'$.*

The following diagram may not be homotopy commutative since our proof leaves open the possibility of phantom maps obstructions; however, on applying $\pi_*(\ )$ it yields a commutative diagram of abelian groups.

$$\begin{tikzcd} BoP \arrow[r, "f"] \arrow[dr, "g"'] & BoP' \arrow[r, "f'"] \arrow[d, "g'"] & BoP \arrow[dl, "g"] \\ & ko & \end{tikzcd}$$

By obstruction theory there is a map $M\mathrm{SU}_{(2)} \longrightarrow BoP'$ inducing an isomorphism on $\pi_0(\ )$; it does not seem easy to obtain a splitting $BoP' \longrightarrow M\mathrm{SU}_{(2)}$ for this map although from [Pe], such maps exist.