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An undergraduate approach to Lie theory

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1 Smooth manifolds and Lie groups

A continuous $g\colon V_1 \longrightarrow V_2$ with $V_k \subseteq \mathbb{R}^{m_k}$ open is called *smooth* if it is infinitely differentiable. If g is a homeomorphism $g\colon V_1 \longrightarrow V_2$, it is called a *diffeomorphism* if g and g^{-1} are both smooth.

Let M be a separable Hausdorff topological space.

Definition 1 *A homeomorphism $f\colon U \longrightarrow V$ where $U \subseteq M$ and $V \subseteq \mathbb{R}^n$ are open subsets, is called an n-chart for U.*

If $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ is an open covering of M and $\mathcal{F} = \{f_\alpha\colon U_\alpha \longrightarrow V_\alpha\}$ is a collection of charts, then $\mathcal{F}$ is called an atlas for M *if, whenever $U_\alpha \cap U_\beta \neq \emptyset$, the function*

$$f_\beta \circ f_\alpha^{-1}\colon f_\alpha(U_\alpha \cap U_\beta) \longrightarrow f_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.

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$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dr, "f_\beta"] & \\ f_\alpha(U_\alpha \cap U_\beta) \arrow[ur, "f_\alpha^{-1}"] \arrow[rr, "f_\beta \circ f_\alpha^{-1}"'] & & f_\beta(U_\alpha \cap U_\beta) \end{tikzcd}$$

We will sometimes denote such an atlas by $(M,\mathcal{U},\mathcal{F})$ and refer to it as a *smooth manifold of dimension* or *smooth n-manifold.*

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Definition 2 *A* smooth map $h\colon (M,\mathcal{U},\mathcal{F}) \longrightarrow (M',\mathcal{U}',\mathcal{F}')$ *is a continuous map $h\colon M \longrightarrow M'$ such that for each pair α, α' with $h(U_\alpha) \cap U'_{\alpha'} \neq \emptyset$, the composite map*

$$f'_{\alpha'} \circ h \circ f_\alpha^{-1}\colon f_\alpha(h^{-1}U'_{\alpha'}) \longrightarrow V'_{\alpha'}$$

is smooth. Such a map is a diffeomorphism *if it is a homeomorphism which has a smooth inverse.*

$$\begin{tikzcd} f_\alpha(h^{-1}U'_{\alpha'}) \arrow[r, "f'_{\alpha'} \circ h \circ f_\alpha^{-1}"] \arrow[d, "f_\alpha^{-1}"'] & V'_{\alpha'} \arrow[d, "{f'_{\alpha'}}^{-1}"] \\ h^{-1}U'_{\alpha'} \arrow[r, "h"'] & h(U_\alpha) \cap U'_{\alpha'} \end{tikzcd}$$

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Definition 3 *A* Lie group *is a smooth manifold equipped with smooth maps* $\mu\colon G \times G \longrightarrow G$, $\mathrm{inv}\colon G \longrightarrow G$ *for which* (G, μ, inv) *is a group.*

A Lie homomorphism *is a smooth map* $h\colon G \longrightarrow G'$ *which is a group homomorphism. A* Lie isomorphism *is a* Lie homomorphism *that is also a diffeomorphism. A subgroup* H *of* G *which is a submanifold is called a* Lie subgroup *denoted* $H \leqslant G$.

As important examples, consider

$$\begin{aligned}
\mathrm{GL}_n(\mathbb{R}) &= \{A \in \mathrm{M}_n(\mathbb{R}) : \det A \neq 0\}, \ \dim \mathrm{GL}_n(\mathbb{R}) = n^2, \\
\mathrm{SL}_n(\mathbb{R}) &= \{A \in \mathrm{M}_n(\mathbb{R}) : \det A = 1\} \subseteq \mathrm{GL}_n(\mathbb{R}), \ \dim \mathrm{SL}_n(\mathbb{R}) = n^2 - 1, \\
\mathrm{GL}_n(\mathbb{C}) &= \{A \in \mathrm{M}_n(\mathbb{C}) : \det A \neq 0\}, \ \dim \mathrm{GL}_n(\mathbb{C}) = 2n^2, \\
\mathrm{SL}_n(\mathbb{C}) &= \{A \in \mathrm{M}_n(\mathbb{C}) : \det A = 1\} \subseteq \mathrm{GL}_n(\mathbb{C}), \ \dim \mathrm{SL}_n(\mathbb{C}) = 2n^2 - 2.
\end{aligned}$$

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Definition 4 *A* matrix group *is a closed subgroup of some* $\mathrm{GL}_n(\mathbb{R})$.

All of the above are examples of real matrix groups, since there are smooth embeddings $\mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_{2n}(\mathbb{R})$. Other examples are

$$\begin{aligned}
\mathrm{O}(n) &= \{A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I\}, \\
\mathrm{SO}(n) &= \{A \in \mathrm{O}(n) : \det A = 1\}, \\
\mathrm{U}(n) &= \{A \in \mathrm{GL}_n(\mathbb{C}) : A^* A = I\}, \\
\mathrm{SU}(n) &= \{A \in \mathrm{U}(n) : \det A = 1\}.
\end{aligned}$$

Theorem 5 *Every matrix group* $G \leqslant \mathrm{GL}_n(\mathbb{R})$ *is a Lie subgroup.*

Corollary 6 *The exponential map restricted to the Lie algebra of a matrix group* G *maps into* G, $\exp\colon \mathfrak{g} \longrightarrow G$, *and is a local diffeomorphism at the identity, hence* $\dim G = \dim \mathfrak{g}$.

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The above examples have the following dimensions.

$$\dim \mathrm{O}(n) = \dim \mathrm{SO}(n) = \binom{n}{2}, \ \dim \mathrm{U}(n) = n^2, \ \dim \mathrm{SU}(n) = n^2 - 1.$$

Moreover,

$$\begin{aligned} \mathfrak{o}(n) = \mathfrak{so}(n) &= \text{Sk-Sym}_n(\mathbb{R}) \\ &\quad (n \times n \text{ real skew-symmetric matrices}), \\ \mathfrak{u}(n) &= \text{Sk-Herm}_n(\mathbb{C}) \\ &\quad (n \times n \text{ complex skew-hermitian matrices}), \\ \mathfrak{su}(n) &= \text{Sk-Herm}_n^0(\mathbb{C}) \\ &\quad (n \times n \text{ complex skew-hermitian matrices of trace } 0). \end{aligned}$$

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Let $\Bbbk = \mathbb{R}, \mathbb{C}$. For an $n \times n$ matrix A,

$$\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|.$$

Then for any A, the series

$$\sum_{k \geqslant 0} \frac{1}{k!} A^k$$

is absolutely convergent since $\|A^k\|/k! \to 0$ as $k \to \infty$. Hence we can define the exponential function

$$\exp\colon \mathrm{M}_n(\Bbbk) \longrightarrow \mathrm{GL}_n(\Bbbk); \quad \exp(A) = \sum_{k \geqslant 0} \frac{1}{k!} A^k.$$

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Given a matrix group $G \leqslant \mathrm{GL}_n(\mathbb{R})$ we can differentiate a curve $\alpha\colon (-\varepsilon, \varepsilon) \longrightarrow G$ and define

$$\alpha'(t) = \lim_{h \to 0} \frac{1}{h} \left(\alpha(t+h) - \alpha(t)\right)$$

whenever this limit exists. If $\alpha'(t)$ is defined for all $t \in (-\varepsilon, \varepsilon)$ then α is said to be *differentiable.* In particular we can define the *tangent space* to G at $A \in G$ by

$$\mathrm{T}_A\, G = \{\alpha'(0) : \alpha \text{ differentiable } G,\ \alpha(0) = A\}.$$

Then $\mathrm{T}_A\, G$ is a real vector subspace of $\mathrm{M}_n(\mathbb{R})$. Also, $\mathfrak{g} = \mathrm{T}_I\, G$ is closed under the Lie bracket operation $[X, Y]$. Left multiplication by A gives a linear isomorphism $\mathrm{T}_I\, G \longrightarrow \mathrm{T}_A\, G$.

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When A, B commute,

$$\exp(A + B) = \exp(A)\exp(B).$$

Also the function

$$\alpha\colon \mathbb{R} \longrightarrow \mathrm{GL}_n(\Bbbk); \quad \alpha(t) = B\exp(tA)$$

is the unique solution of the differential equation

$$\alpha'(t) = \alpha(t)A, \quad \alpha(0) = B.$$

Moreover, for every A, the *derivative* of exp at A is the linear map

$$\mathrm{d}\exp_A\colon\ \mathrm{M}_n(\mathbb{R}) \longrightarrow \mathrm{M}_n(\mathbb{R}); \quad X \longmapsto \lim_{h \to 0} \frac{1}{h}\left(\exp(A + hX) - \exp(A)\right)$$

which is actually an isomorphism, so exp is everywhere a local diffeomorphism.

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Theorem 7 *If $U, V \in \mathrm{M}_n(\mathbb{R})$, the following identities are satisfied.*

$$\exp(U+V) = \lim_{r\to\infty} \left(\exp((1/r)U)\exp((1/r)V)\right)^r ;$$

[Trotter Product Formula]

$$\exp([U,V]) =$$

$$\lim_{r\to\infty} \left(\exp((1/r)U)\exp((1/r)V)\exp(-(1/r)U)\exp(-(1/r)V)\right)^{r^2} .$$

[Commutator Formula]

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2 Maximal tori in compact connected Lie groups

A *torus* is a compact connected abelian Lie group. It can be shown that every torus is isomorphic to some

$$\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r \cong (S^1)^r = S^1 \times \cdots \times S^1.$$

If G is a Lie group then a *maximal torus in* G is a torus $T \leqslant G$ in G which is not contained in any other torus in G.

Theorem 8 *Let G be a compact connected Lie group. If $g \in G$, there is an $x \in G$ such that $g \in xTx^{-1}$, i.e., g is conjugate to an element of T. Equivalently,*

$$G = \bigcup_{x\in G} xTx^{-1}.$$

Corollary 9 *Every maximal torus in G is a maximal abelian*

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subgroup.

Corollary 10 *If $T, T' \leqslant G$ are maximal tori then they are conjugate in G, i.e., there is a $y \in G$ such that $T' = yTy^{-1}$.*

Corollary 11 *Let G be a compact, connected matrix group. Then the exponential map $\exp\colon \mathfrak{g} \longrightarrow G$ is surjective.*

A key idea in proving these is

Proposition 12 *Every torus T has a topological generator, i.e., there is an element $t \in T$ with $\langle t \rangle$ dense in T.*

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3 A Lie group that is not a matrix group

Let

$$U = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, x \in \mathbb{R} \right\}, \quad N = \left\{ \begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} \lhd U.$$

The *Heisenberg group* $H = U/N$ is a 3-dimensional Lie group with an exact sequence

$$1 \to T \longrightarrow H \longrightarrow \mathbb{R} \times \mathbb{R} \to 1,$$

where $T \cong \mathbb{T}$ is central and contained in the commutator subgroup.

Proposition 13 *Every Lie homomorphism $\varphi\colon H \longrightarrow \mathrm{GL}_n(\mathbb{R})$ has non-trivial kernel.*

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References

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[3] S. Sternberg, Group Theory and Physics, Cambridge University Press (1994).