Contemporary Mathematics
Volume 19, 1983

ON WEAKLY ALMOST COMPLEX MANIFOLDS WITH VANISHING DECOMPOSABLE CHERN NUMBERS.

ANDREW BAKER

Abstract:

We describe the subgroup of the complex bordism ring consisting of elements with only the indecomposable Chern monomial giving a non-zero Chern number. In dimensions $4k + 2$ we recover results of Ray, and in dimensions $4k$ we prove a conjecture of Dyer.

In this note we will investigate the subgroup of the complex bordism ring MU_* consisting of classes for which the only non-zero Chern number is that coming from the top dimensional Chern class. In dimensions of form $4k + 2$, we recover results of [Ra]; in dimensions of form $4k$, we prove an old conjecture of E. Dyer [Dy].

Theorem Let $X_n \in MU_{2n}$ be in the subgroup of elements for which all decomposable Chern numbers are zero. Then X_n is a generator if and only if $c_n(X_n)$ takes the value (up to sign)

2, if $n = 1$,

$(2k)!$, if $n = 2k + 1, k > 1$,

$d_k(2k - 1)!$, if $n = 2k$, where d_k is the denominator of $B_{2k}/2k$, for B_{2k} the $2k$-th Bernoulli number.

1980 Mathematics Subject Classification. 55R50, 57R77

Our method is to construct certain families of coaction primitives in K_*MU which by the Hattori-Stong Theorem must come from π_*MU. We give a construction for generators in dimensions $4k+2$ (some of which are given in [Ra]), but are unable to do this in the remaining cases.

This work grew out of the author's Ph.D. thesis together with joint work with Nigel Ray and Francis Clarke which will be described in [Ba-Cl-Ra-Sc] - see also [Ba-Ra].

For all notation, basic definitions, etc. see [Ad]. Throughout, E will denote a commutative ring spectrum equipped with a given complex orientation $x^E \in E^2(CP^\infty_+)$ and we assume that E_* is torsion-free. We also have a basis $\beta^E_0 = 1, \beta^E_1, \beta^E_2, \cdots$ of $E_*(CP^\infty_+)$ consisting of the duals of powers of x^E. The left coaction (left unit)

$$\psi_E\colon E_*(CP^\infty_+) \longrightarrow (E \wedge E)_*(CP^\infty_+)$$

is described by

(1) $$\psi_E \beta^E(T) = \beta^R(\exp^R(\log^L T))$$

where $\beta^E(T) = \sum_{0<j} \beta^E_j T^j$ for a formal indeterminate T, $\exp^E$ and $\log^E$ denote the exponential and logarithmic series for the E-theory formal group law, and R,L denote the images of those under the left and right units.

We can replace T by $\exp^E T$ and get

(2) $$\psi_E \beta^E(\exp^E T) = \beta^R(\exp^R T)$$

- so $\beta^E(\exp^E T)$ is primitive. In fact the coefficients are of form $\frac{1}{n!}(\beta^E_1)^n$. See [Se], [Mi].

Now use the canonical map $CP^\infty \longrightarrow BU$ to embed the above in $E_*(BU_+)$. It is well known that we can usefully identify β^E_n with the elementary symmetric function $\Sigma t_1 \cdots t_n$ - hence

(3) $$\beta^E(T) = \prod_i (1 + t_i T)$$

We also have the Newton functions Σt^n_1 which correspond to elements

$s_n^E \in E_{2n}(BU_+)$ (these are of course diagonal primitives). Next we can apply the natural logarithm power series ℓn to give

$$\ell n\ \beta^E(T) = \sum_{1\leq k} \frac{(-1)^{k-1}}{k} s_k^E T^k$$

which on replacing T by $\exp^E T$ becomes

$$\ell n\ \beta^E(\exp^E T) = \sum_{1\leq k}\sum_{1\leq n} \frac{(-1)^{k-1}}{k} \frac{A^E(n,k)}{n!} s_k^E T^n \tag{4}$$

Here we define $A^E(n,k) \in E_{2(n-k)}$ by

$$(\exp^E T)^k = \sum_{k\leq n} \frac{A^E(n,k)}{n!} T^n \tag{5}$$

(actually care needs to be taken if E_* has torsion in which case we use the universality of MU for complex orientations). It turns out that $\frac{A^E(n,k)}{k!} \in E_{2(n-k)}$ (exercise for the reader). For $E = K$, $A^K(n,k) = k!\,S(n,k)u^{n-k}$, where $S(n,k)$ is a Stirling number of the second kind and $K_* = \mathbb{Z}[u,u^{-1}]$ - in this case $\exp^K T = \frac{e^{uT} - 1}{u}$.

Now define $\Sigma_n^E \in E_{2n}(BU)$ by

$$\Sigma_n^E = \sum_k \frac{(-1)^{k-1}}{k} A^E(n,k)s_k^E \tag{6}$$

From (2) and (4) we have

$$\psi_E \sum_{1\leq n} \frac{\Sigma_n^E}{n!} T^n = \sum_{1\leq n} \frac{\Sigma_n^R}{n!} T^n \tag{7}$$

- so Σ_n^E is primitive (in fact in the sense of the Hopf algebra structure as well).

We leave the reader to verify that Σ_n^E could also be defined inductively using the Bott homomorphism

$$B_*: E_*(BU) \longrightarrow E_{*+2}(BU).$$

We have

$$\Sigma_{n+1}^E = B_*\Sigma_n^E \tag{8}$$

(9) $\Sigma_n^E = \underline{e}(u^n)$ where $\underline{e}$ denotes the E-theory Hurewicz homomorphism and $u^n \in \pi_{2n}BU = K_{2n}$ is the usual generator. This approach is taken in [Ba] - see [Ad] for details.

We now introduce the E-theory Thom isomorphism

$$\Phi^E: E_*(BU_+) \longrightarrow E_*MU$$

Let $\Phi^E\beta_n^E = b_n^E$, $\Phi^E s_n^E = p_n^E$, $\Phi^E\Sigma_n^E = \bar{\Sigma}_n^E$. Then we can again identify b_n^E with $\Sigma t_1\cdots t_n$, and p_n^E with Σt_1^n.

The coaction becomes

$$\psi_E b^E(T) = \frac{1}{T}\exp^R(\log^L T)\cdot b^R(\exp^R(\log^L T)) \tag{10}$$

since under the homomorphism induced by the inclusion $MU(1) \longrightarrow \Sigma^2 MU$ we have $\beta_n^E \longmapsto b_{n-1}^E$. Beware - our $b^E(T)$ denotes $\sum_{0<j} b_j^E T^j$, not as in [Ad]! Again replace T by $\exp^E T$:

$$\psi_E b^E(\exp^E T) = \left(\frac{\exp^R T}{\exp^L T}\right)\cdot b^R(\exp^R T) \tag{11}$$

Now applying ℓn we obtain

$$\psi_E \sum_{1<n} \frac{\bar{\Sigma}_n^E}{n!} T^n = \sum_{1<n} \frac{\bar{\Sigma}_n^E}{n!} + \ell n\left(\frac{\exp^R T}{T}\right) - \ell n\left(\frac{\exp^L T}{T}\right) \tag{12}$$

We obtain a primitive series by adding $\ell n \dfrac{\exp^E T}{T}$ to the $\bar{\Sigma}$ series:

$$\psi_E \sum_{1<n} \frac{\bar{\Sigma}_n^E}{n!} T^n + \ell n\left(\frac{\exp^E T}{T}\right) = \sum_{1<n} \frac{\bar{\Sigma}_n^R}{n!} T^n + \ell n\left(\frac{\exp^R T}{T}\right). \tag{13}$$

Specialising to $E = MU$, we recall that

$$MU_* \otimes \mathbb{Q} = H_*MU \otimes \mathbb{Q}$$

$$= \mathbb{Q}[b_1^H, b_2^H \cdots]$$

and $\exp^{MU} T = Tb^H(T)$. Hence

$$\ell n\left(\frac{\exp^{MU}T}{T}\right) = \ell n \prod_i (1 + t_i T).$$

So $\bar{\Sigma}_n^{MU} + (-1)^{n-1}(n-1)!\, p_n^H \in MU_*MU \otimes \mathbb{Q}$ is primitive and we need only determine exactly which multiples are in MU_*MU, hence come from π_*MU via $\underline{mu}$. To do this let $E = K$, and note that

$$\frac{\exp^K T}{T} = \frac{e^{uT} - 1}{uT}$$

Differentiating (13) with respect to T tells us that

(14) $$\sum_{1<n} \frac{\bar{\Sigma}_n^K}{(n-1)!} T^{n-1} + \frac{1}{T}\left[\frac{1}{2} uT + \sum_{2<j} \frac{B_j}{j!} u^j T^j\right]$$

is primitive. So we obtain as indivisible primitives in K_*MU the following (this makes use of the fact that $B_{2k+1} = 0$ if $k > 0$):

(15) $$2\bar{\Sigma}_1^K + u \in K_2MU$$

$$\bar{\Sigma}_{2k+1}^K \in K_{4k+2}MU$$

$$d_k\left(\bar{\Sigma}_{2k}^K + \frac{B_{2k}}{2k} u^{2k}\right) \in K_{4k}MU$$

where d_k is as in the statement of the Theorem.

Returning to the case $E = MU$ we now have

(16) $$2\bar{\Sigma}_1^{mu} + 2p_1^H \in MU_2MU$$

$$\bar{\Sigma}_{2k+1}^{mu} + (2k)!\, p_{2k+1}^H \in MU_{4k+1}MU$$

$$d_k\, \bar{\Sigma}_{2k}^{mu} - d_k(2k-1)!\, p_{2k}^H \in MU_{4k}MU$$

as our indivisible primitives. In fact, these must come from elements of π_*MU whose integral homology Hurewicz images are

$$2p_1^H,\ (2k)!\, p_{2k+1}^H,\ - d_k(2k-1)!\, p_{2k}^H.$$

Since $s_n^H = (\Phi^H)^{-1} p_n^H$ is dual to c_n^H in the monomial basis for $H^*(BU_+)$, the result about Chern numbers follows.

Hence we have proved the Theorem.

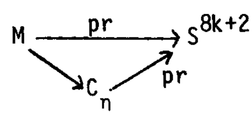
By [Ra], we can construct $(S^{4k+2}) \in MU_{4k+2}$ by suitably choosing a normal U-structure on S^{4k+2} so that

$$\underline{k}(S^2) = 2\overline{\Sigma}_1^K + u$$

$$\underline{k}(S^{2k+2}) = \overline{\Sigma}_{4k+3}^K$$

$$\underline{k}(S^{8k+2}) = 2\overline{\Sigma}_{4k+1}^K, \quad \text{if} \quad k > 1.$$

We can realise $\frac{1}{2}(S^{8k+2})$ by taking <u>any</u> $8k+2$ dimensional bounding U-manifold M which is not a Spin manifold and using the existence of a stable factorisation



where pr denotes projection onto the top cell, and C_η denotes the mapping cone of the Hopf map $\eta \in \pi_1^S$. The details are routine applications of ideas in [Ra-Sw-Ta] and appear in [Ba]. The simplest such manifold is $CP^2 \times S^{8k-2}$.

Unfortunately, we have no general construction in dimension $4k$.

<u>Remarks</u> 1) The above can be used to calculate the e-invariant of an element in the image of the J-homomorphism. The details involve the observation that the mapping cone is a Thom space over a sphere, the fact that

$$\tau_* \overline{\Sigma}_n^K = \frac{B_n}{n}(v^n - u^n) \in K_{2n}K$$

where $\tau\colon MU \to K$ is the canonical Todd orientation, and $v \in K_2K$ is $\eta_R u$. We leave the reader to verify this.

2) R. Stong has pointed out to the author that the above Theorem has appeared in "On the homotopy groups of BPL and PL/O", by G. Brumfiel, Ann. of Math. 88 (1968).

3) The referee has given a variation of the above proof which makes more use of [Mi], and emphasises the role of formal group theory.

Bibliography

[Ad] J. F. Adams, Stable homotopy and generalised homology, University of Chicago Press, Chicago, 1974.

[Ba] A. Baker, Some geometric filtrations on bordism groups, (Ph.D. Thesis) University of Manchester, 1980.

[Ba-Cl-Ra-Sc] A. Baker, F. Clarke, N. Ray, L. Schwartz, "The stable homotopy of BU" in preparation.

[Ba-Ra] A. Baker, N. Ray, "Some infinite families of U-hypersurfaces", to appear in Math. Scand, 50 (1982), 149-66.

[Dy] E. Dyer, "Chern characters of certain complexes", Math. Zeit. 80 (1963) 363-73.

[Mi] H. R. Miller, "Universal Bernoulli numbers and the S^1-transfer", to appear in "Current Trends in Alg. Top." AMS.

[Ra] N. Ray, "Bordism J-homomorphisms", Ill. J. Math. 18, (1974), 290-309.

[Ra-Sw-Ta] N. Ray, R. M. Switzer, L. Taylor, "G-structures, G-bordism, and univeral manifolds", Mem. Amer. Math. Soc., 193 (1977) 1-27.

[Se] D. M. Segal "The cooperation on $MU_*(CP^\infty)$ and $MU_*(HP^\infty)$ and the primitive generators", J. of Pure and App. Alg. 14 (1979) 315-22.

University of Chicago, Chicago, Illinois 60637
June 1982.