L-complete Hopf algebroids

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Derived functors of completion

Let $(R, \mathfrak{m})$ be a commutative Noetherian regular local ring and let $\mathscr{M} = \mathscr{M}_R$ be the category of left R-modules and $\otimes = \otimes_R$.
For an R-module M, the $\mathfrak{m}$-*adic completion* is

$$M_{\mathfrak{m}} = \lim_k M/\mathfrak{m}^k M = \lim_k (R/\mathfrak{m}^k \otimes_R M).$$

This defines an additive functor $(-)_{\mathfrak{m}} \colon \mathscr{M} \longrightarrow \mathscr{M}$ which is neither left nor right exact. Instead we can define its left derived functors L_s for $s \geqslant 0$ as follows. Given a module M, choose a projective resolution $P_\bullet \longrightarrow M \to 0$ and form the complex $(P_\bullet)_{\mathfrak{m}}$:

$$\cdots \longrightarrow (P_s)_{\mathfrak{m}} \longrightarrow (P_{s-1})_{\mathfrak{m}} \longrightarrow (P_0)_{\mathfrak{m}} \to 0.$$

Now define

$$L_s M = H_s((P_\bullet)_{\mathfrak{m}}).$$

Then there are natural transformations

$$\mathrm{Id} \xrightarrow{\eta} L_0 \longrightarrow (-)_{\mathfrak{m}} \longrightarrow R/\mathfrak{m} \otimes_R (-).$$

where the two right hand natural transformations are epimorphic for each module, and L_0 is idempotent, *i.e.*, $L_0^2 \cong L_0$.

L-complete modules

Can assume R is *complete*, $R_{\mathfrak{m}} = R$. A module M is *L-complete* if $M \cong L_0 M$. Get a full subcategory $\widehat{\mathscr{M}} \subseteq \mathscr{M}$ of L-complete modules. There is a symmetric monoidal product on $\widehat{\mathscr{M}}$,

$$M \widehat{\otimes} N = L_0(M \otimes N).$$

- If M is L-complete then $L_s M = 0$ for $s > 0$.
- For any N, each $L_s N$ is L-complete.
- If M is finitely generated then

  $$L_0 M \cong M_{\mathfrak{m}}, \quad M \otimes L_0 N \cong L_0(M \otimes N),$$
  $$R/\mathfrak{m}^k \otimes L_0 N \cong R/\mathfrak{m}^k \otimes N = N/\mathfrak{m}^k N.$$

  A module has *bounded* $\mathfrak{m}$*-torsion* if it is annihilated by some power of $\mathfrak{m}$. Every such module is L-complete.
- Nakayama Lemma: If M is L-complete, $\mathfrak{m}M = M \Longrightarrow M = 0$.
- $\widehat{\mathscr{M}}$ has enough projectives which are the *pro-free modules* $F_{\mathfrak{m}} = L_0 F$ for F a free module. $\widehat{\otimes}$-flat modules are pro-free. If P is flat then $L_0 P = P_{\mathfrak{m}}$ and is pro-free. $\widehat{\mathscr{M}}$ has no interesting injectives since an injective would be $\mathfrak{m}$-divisible.

Determining the derived functors L_s

Theorem

For any module M, there is a natural short exact sequence

$$0 \to \lim_r \operatorname{Tor}^R_{s+1}(R/\mathfrak{m}^r, M) \longrightarrow L_s M \longrightarrow \lim_r \operatorname{Tor}^R_s(R/\mathfrak{m}^r, M) \to 0.$$

If $\mathfrak{m} = (u_1, \ldots, u_n)$ where the sequence u_i is regular, then there is a certain module $R/\mathfrak{m}^\infty$ for which

$$L_s M = \operatorname{Ext}^{n-s}(R/\mathfrak{m}^\infty, M).$$

Hence $L_s = 0$ if $s > n$.

Colimits in $\widehat{\mathscr{M}}$

A filtered directed system of L-complete modules M_α has a colimit $\operatorname{colim}_\alpha M_\alpha$ in $\mathscr{M}$ and $L_0(\operatorname{colim}_\alpha M_\alpha)$ is its colimit in $\widehat{\mathscr{M}}$. The functor $\{M_\alpha\} \mapsto L_0(\operatorname{colim}_\alpha M_\alpha)$ has left derived functors $\operatorname{colim}^{(s)}$ which can be identified by

$$\operatorname*{colim}_\alpha{}^{(s)} M_\alpha = L_s(\operatorname*{colim}_\alpha M_\alpha).$$

Notice that $\operatorname*{colim}_\alpha{}^{(s)} = 0$ if $s > n$. For coproducts, we also have $\operatorname*{colim}_\alpha{}^{(n)} = 0$ (Hovey).

Hopf algebroids

Let $\mathscr{C}$ be a category with the properties required for the following to make sense. A *groupoid object in* $\mathscr{C}$, $(\mathcal{O}, \mathcal{G}, c, d, i, \mu, \gamma)$, consists of objects $\mathcal{O}, \mathcal{G}$ and morphisms

$$\mathcal{O} \xrightarrow{i} \mathcal{G} \overset{c}{\underset{d}{\rightrightarrows}} \mathcal{O} \qquad \mathcal{G} \sqcap_{\mathcal{O}} \mathcal{G} \xrightarrow{\mu} \mathcal{G} \qquad \mathcal{G} \xrightarrow{\gamma} \mathcal{G}$$

satisfying the conditions of a groupoid (=small category in which all morphisms are invertible). Here we define $\mathcal{G} \sqcap_{\mathcal{O}} \mathcal{G}$ from the pullback diagram

$$\begin{tikzcd}
\mathcal{G} \sqcap_{\mathcal{O}} \mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[d, "d"] \\
\mathcal{G} \arrow[r, "c"] & \mathcal{O}
\end{tikzcd}$$

Dually, a *cogroupoid in object in* $\mathscr{C}$ is a groupoid object in the opposite category $\mathscr{C}^{\mathrm{o}}$.

Cogroupoids in commutative rings

Let $\Bbbk$ be a fixed commutative ring (e.g., $\mathbb{Z}$) and let $\mathscr{C} = \mathscr{C}_{\Bbbk}$ be the category of commutative unital $\Bbbk$-algebras. The opposite category $\mathscr{C}^{\mathrm{o}}$ is the category of $\Bbbk$-schemes, i.e., covariant corepresentable functors

$$\mathrm{Spec}_{\Bbbk}(R, -) = \mathscr{C}(R, -)$$

for $R \in \mathscr{C}$, defined on $\mathscr{C}$ and taking values in the category of sets. If such a functor takes values in the category of groupoids, then there are commutative rings A and Γ and morphisms

$$A \xleftarrow{\varepsilon} \Gamma \underset{\eta_R}{\overset{\eta_L}{\leftleftarrows}} A \qquad \Gamma \otimes_A \Gamma \xleftarrow{\psi} \Gamma \qquad \Gamma \xleftarrow{\chi} \Gamma$$

inducing the dual groupoid structure. This structure makes the pair (A, Γ) into a *Hopf algebroid* in $\mathscr{C}$. The tensor product is to be interpreted as the A-bimodule tensor product ${}_{\eta_R}\otimes_{\eta_L}$.
If $\eta_R = \eta_L$ then (A, Γ) is a *Hopf algebra*, which is a cogroup object in $\mathscr{C}$, and the dual object is a group valued functor.

L-complete Hopf algebroids

We will consider objects in $\widehat{\mathscr{M}}$ which have the required structure to be cogroupoid objects. These actually lie in a subcategory of *commutative bi-ring objects*.

A *commutative bi-ring object* Γ is a non-unital monoid equipped with two units $\eta_L, \eta_R \colon R \longrightarrow \Gamma$ which we view as giving left and right R-module structures.

An R-biunital ring object Γ is *L-complete* if Γ is L-complete as both a left and a right R-module.

Suppose that Γ is an L-complete commutative R-bi-unital ring object which has the following additional structure:

- a *counit*: a ring homomorphism $\varepsilon \colon \Gamma \longrightarrow R$;
- a *coproduct*: a ring homomorphism $\psi \colon \Gamma \longrightarrow \Gamma \widehat{\otimes} \Gamma = \Gamma_R \widehat{\otimes}_R \Gamma$;
- an *antipode*: a ring homomorphism $\chi \colon \Gamma \longrightarrow \Gamma$.

Then Γ is an *L-complete Hopf algebroid* if

- with this structure, Γ becomes a cogroupoid object,
- if Γ is pro-free as a left (or equivalently as a right) R-module,
- the ideal $\mathfrak{m} \lhd R$ is *invariant*, *i.e.*, $\mathfrak{m}\Gamma = \Gamma\mathfrak{m}$.

We often write such a pair by (R, Γ) if the structure maps are clear.

Example 1 Take an honest Hopf algebroid (R, Γ) which is flat as an R-module and $\mathfrak{m}\Gamma = \Gamma\mathfrak{m}$. Then $(R, L_0\Gamma)$ is an L-complete Hopf algebroid.

Example 2 Let G be a profinite group acting continuously on R. Then $(R, \mathrm{Map}^c(G, R))$ can be given the structure of an L-complete Hopf algebroid.

Comodules

Let (R, Γ) be an L-complete Hopf algebroid. A (left) *comodule* over (R, Γ) is an L-complete R-module M with a module homomorphism map $\rho\colon M \longrightarrow \Gamma \widehat{\otimes} M$ which is coassociative and counital. The category of all such comodules $\widehat{\mathscr{M}}$ need not be abelian, but restricting to comodules which are finitely generated or have bounded torsion then we obtain abelian categories ${}^{\Gamma}\widehat{\mathscr{M}}_{\mathrm{bt}}$ and ${}^{\Gamma}\widehat{\mathscr{M}}_{\mathrm{fg}}$.

Goal Would like some kind of Jordan-Hölder type filtration for finitely generated comodules.
Prehistory (ca 1993) Did something like this for certain types of Hopf algebroids coming from pro-p-groups acting on power series algebras $R = \mathbb{Z}_p[[v_1, \ldots, v_{n-1}]]$. But also showed they couldn't be generalised to *Landweber filtrations* corresponding to the invariant ideals $(p, v_1, \ldots, v_{r-1}) \lhd R$.

Jordan-Hölder filtrations

A comodule M is *discrete* if each element is annihilated by some $\mathfrak{m}^k$; if M is finitely generated then this implies that some $\mathfrak{m}^\ell$ annihilates M.

For a finitely generated discrete module, there is a filtration by subcomodules

$$M \supseteq \mathfrak{m}M \supseteq \mathfrak{m}^2 M \supseteq \cdots$$

which eventually reaches 0. The filtration quotients are comodules over the Hopf algebra $(R/\mathfrak{m}, \Gamma/\mathfrak{m}\Gamma)$ with $R/\mathfrak{m}$ the residue field.

Theorem

If $(R/\mathfrak{m}, \Gamma/\mathfrak{m}\Gamma)$ is a unipotent Hopf algebra, then every finitely generated discrete (R, Γ)-comodule M admits a filtration by subcomodules

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{r-1} \supseteq M_r = 0,$$

where $M_k/M_{k+1} \cong R/\mathfrak{m}$. Hence $R/\mathfrak{m}$ is the only finitely generated simple comodule.

Some examples

- $(R, \mathrm{Map}^c(G, R))$ where G is a pro-p-group and char $R/\mathfrak{m} = p$. This comes from the standard fact that the Hopf algebra is unipotent $(R/\mathfrak{m}, \mathrm{Map}^c(G, R/\mathfrak{m}))$ – equivalently, the pro-group ring $R/\mathfrak{m}[[G]]$ is a limit of finite group rings which Artinian local rings.
- Coming from Algebraic topology: The Lubin-Tate Hopf algebroids $((E_n)_0, {E_n}_0^\vee E_n)$. These are built up from extensions of examples with pro-p-groups and unicursal Hopf algebroids. The reduced unicursal Hopf algebroids can be described in terms of Galois extensions.