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Topological invariants of generalized Kummer manifolds

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1 Generalized Kummer manifolds

Let $L \subseteq \mathbb{C}^{2m}$ be a lattice and consider the compact complex torus $X_L = \mathbb{C}^{2m}/L$. Let $\Gamma \leqslant \mathrm{Aut}(L) \cap \mathrm{SL}_{2m}\mathbb{C}$ be a (finite) group of automorphisms of L whose action on $\mathbb{C}^{2m}$ is semifree (i.e., free away from 0). X_L^Γ consists of finitely many isolated fixed points, so the orbifold X_L/Γ has finitely many isolated singularities. We can resolve these singularities by 'blowing up' each of the singular points. In fact, we can do this once and for all by blowing up the fixed point set X_L^Γ and then passing to a quotient manifold.

Locally, at each $p \in X_L^\Gamma$ choose a holomorphic chart $\mathbf{z} = (z_1, \ldots, z_{2m})\colon U \longrightarrow U' \subseteq \mathbb{C}^{2m}$ with $\mathbf{z}(p) = 0$. Then define

$$\widetilde{U} = \{(u, [w]) \in U \times \mathbb{C}\mathrm{P}^{2m-1} : z_i(u)w_j = z_j(u)w_i\}.$$

This can be extended to define a submanifold of $X_L \times \mathbb{C}\mathrm{P}^{2m-1}$.

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Repeating this for all the fixed points we obtain a map $\pi\colon \widetilde{X}_L \longrightarrow X_L$ which is a biholomorpic equivalence away from X_L^Γ and for each $p \in X_L^\Gamma$ we have $\pi^{-1}p = \mathbb{C}\mathrm{P}_p^{2m-1} \cong \mathbb{C}\mathrm{P}^{2m-1}$.

The normal bundle $\nu(\mathbb{C}\mathrm{P}_p^{2m-1} \longrightarrow \widetilde{X}_L)$ can be identified with the canonical line bundle $\eta \longrightarrow \mathbb{C}\mathrm{P}^{2m-1}$.

Γ may act nontrivially on each *exceptional divisor* $\mathbb{C}\mathrm{P}_p^{2m-1}$. For simplicity, we assume that Γ consists of scalar multiplications, hence the action on the exceptional divisors is trivial. This also means that Γ is cyclic.

The quotient map $\pi\colon \widetilde{X}_L \longrightarrow K_{L,\Gamma} = \widetilde{X}_L/\Gamma$ maps the total space of each $\eta \longrightarrow \mathbb{C}\mathrm{P}_p^{2m-1}$ to $\eta^{d(p)} \longrightarrow \mathbb{C}\mathrm{P}_p^{2m-1}$ where $d(p) = |\operatorname{Stab}_\Gamma(p)|$.

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2 Blow-up formulæ

In order to calculate with blow-ups, a result of I. Porteous [4] is of fundamental importance.

Let $\pi\colon \widetilde{X} \longrightarrow X$ be a blow-up map at the point $p \in X$. Let $j\colon \mathbb{C}\mathrm{P}_p^{n-1} \longrightarrow \widetilde{X}$ be the inclusion of the exceptional divisor and let $\lambda \longrightarrow \mathbb{C}\mathrm{P}_p^{n-1}$ be the normal line bundle of j. Write $u = j_* 1 \in H^2(\widetilde{X})$ and $v = j^* u \in H^2(\mathbb{C}\mathrm{P}_p^{n-1})$. In K-theory we have

$$\pi^*\mathrm{T}X = \mathrm{T}\widetilde{X} + j_!(\pi^*\mathrm{T}X_{|_p} - \lambda),$$

with $j_!$ the K-theory push forward map, and in rational cohomology

$$\begin{aligned}\operatorname{ch}\pi^*\mathrm{T}X &= \operatorname{ch}\mathrm{T}\widetilde{X} + j_*\left((\operatorname{ch}\pi^*\mathrm{T}X_{|_p} - \operatorname{ch}\lambda)\left(\frac{1-e^{-v}}{v}\right)\right)\\ &= \operatorname{ch}\mathrm{T}\widetilde{X} + (n - e^u)(1 - e^{-u}).\end{aligned}$$

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3 Equivariant Index Formulæ

We will consider the calculations of two invariants, namely $\mathrm{sign}(K_{L,\Gamma})$ and $\mathrm{A}(K_{L,\Gamma})$.

Suppose that Γ acts smoothly on a compact closed manifold M.

Theorem 3.1 *We have*

$$\mathrm{sign}(M/\Gamma) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\mathrm{sign}(g,M).$$

where

$$\mathrm{sign}(g,M) = \mathrm{Tr}\, g^*_{|_{H^{2m}(M;\mathbb{R})+}} - \mathrm{Tr}\, g^*_{|_{H^{2m}(M;\mathbb{R})-}}.$$

When Γ acts on a complex manifold by holomorphic maps, this local signature can be calculated in other ways.

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Proposition 3.2 *If $\Gamma = \{1,\tau\}$ where τ is an orientation preserving involution on M, then the self intersection $M^\tau \pitchfork M^\tau$ of M^τ in M has signature*

$$\mathrm{sign}(M^\tau \pitchfork M^\tau) = \mathrm{sign}(\tau, M).$$

Using this we have

Proposition 3.3 *The signature of $K_{L,\{1,\tau\}}$ is*

$$\mathrm{sign}(K_{L,\{1,\tau\}}) = -2^{4m},$$

hence the lattice $H^{2m}(K_{L,\{1,\tau\}};\mathbb{Z})$ equipped with the canonical intersection form is equivalent to

$$\frac{1}{2}\binom{4m}{2m}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \oplus (-2^{4m-3}\mathbb{E}_8).$$

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The A-genus can be determined using another equivariant index formula.

Theorem 3.4 *If Γ acts holomorphically on the compact complex closed manifold M and $\xi \longrightarrow M/\Gamma$ is a holomorphic vector bundle, then*

$$\chi(M/\Gamma, \xi) = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \chi\left(M^g, \frac{\pi^*\xi_{|M^g}}{\lambda_{-1}(\nu(M^g)^*(g)}\right).$$

In particular, if $\xi = \varepsilon_1$ is the trivial 1-dimensional bundle,

$$\mathrm{A}(M/\Gamma) = \chi(M/\Gamma, \varepsilon_1) = \frac{1}{|\Gamma|} \sum_{g\in\Gamma} \chi\left(M^g, \frac{1}{\lambda_{-1}(\nu(M^g)^*)(g)}\right).$$

Proposition 3.5 *The* A*-genus of $K_{L,\{1,\tau\}}$ is*

$$\mathrm{A}(K_{L,\{1,\tau\}}) = -2^{2m-1}.$$

When $m = 1$, this is a well known result on Kummer surfaces.

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4 Brightwell's Suzuki manifold

The Leech lattice $\mathbb{L} \subseteq \mathbb{R}^{24}$ is a module over $\mathbb{Z}[\omega]$ hence has a complex structure and an embedding $\mathbb{L} \subseteq \mathbb{C}^{12}$. Then $\mathrm{Aut}_{\mathbb{Z}[\omega]}(\mathbb{L}) = 6 \cdot \mathrm{Suz}$ and there is a Kummer manifold $K_{\mathbb{L},\mathbb{Z}/6}$ on which the simple group Suz acts. The fixed point data for the action of $\mathbb{Z}/6 = \langle\gamma\rangle$ on $X_{\mathbb{L}}$ is

$$\begin{aligned} \gamma, \gamma^{-1}: &\quad 1 \text{ fixed point } 0 + \mathbb{L}; \\ \gamma^2, \gamma^4: &\quad 3^{12} \text{ fixed points}; \\ \gamma^3: &\quad 2^{24} \text{ fixed points}. \end{aligned}$$

The above methods can be used to determine the following invariants.

$$\begin{aligned} \mathrm{sign}(K_{\mathbb{L},\mathbb{Z}/6}) &= -\frac{1}{6}\left(2^{25} + 4.3^6 - 5 \cdot 3^{12} + 1\right), \\ \mathrm{A}(K_{\mathbb{L},\mathbb{Z}/6}) &= -\frac{1}{3}\left(2^{11} + 3^6 + 1\right). \end{aligned}$$

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References

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