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Artin-Schreier extensions and the Adams spectral sequence for elliptic cohomology

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1 Katz's ring of divided congruences

N. Katz [5] introduced a p-adic *ring of divided congruences* amongst modular forms which is closely related to the topological object $KU_0 E\ell\ell$. This ring is used to determine $E\ell\ell_* E\ell\ell$ in [1] and also proves useful in [2] for calculating the E_2-term of the Adams spectral sequence

$$\mathrm{E}_2^{**} = \mathrm{Ext}^{**}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*) \Longrightarrow \pi_* S_{E\ell\ell}.$$

A more general version also appears in [2, 3, 4].

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For a prime $p > 3$, let

$$\mathrm{DC} = \{\sum_r F_r(q) : F_r \text{ a level 1 mod. form of weight } r\} \subset \mathbb{Z}[1/6]((q)),$$

$$\mathrm{DC}_p = \{\sum_r F_r(q) : F_r \text{ a level 1 mod. form of weight } r\} \subset \mathbb{Z}_{(p)}((q)).$$

Then $\mathrm{DC}/(p) = \mathrm{DC}_p/(p)$. There is an action of $\mathbb{Z}_{(p)}^\times$ on $\mathrm{DC}/(p)$, namely

$$\alpha \cdot \sum_r F_r(q) = \sum_r \alpha^r F_r(q) \quad (\alpha \in \mathbb{Z}_{(p)}^\times).$$

This extends to a continuous action of $\mathbb{Z}_p^\times$ on $\mathrm{DC}/(p)$.

Theorem 1. *There are isomorphisms of rings*

$$KU_0 E\ell\ell \cong \mathrm{DC}, \quad KU_0 E\ell\ell_{(p)} \cong \mathrm{DC}_p, \quad KU_0 E\ell\ell/(p) \cong \mathrm{DC}/(p).$$

Moreover, the actions of $\mathbb{Z}_{(p)}^\times$ on DC_p and $\mathbb{Z}_p^\times$ on $\mathrm{DC}/(p)$ correspond to the actions of the K-theory Adams operations.

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In his work, Katz uses a filtration on $\mathrm{D}_p = \mathrm{DC}/(p)$,

$$\mathrm{D}_p^{(1)} \subset \mathrm{D}_p^{(2)} \subset \cdots \subset \mathrm{D}_p^{(r)} \subset \cdots$$

for which $\mathrm{D}_p^{(r)} = \mathrm{D}_p^{1+p^r\mathbb{Z}_p}$, the ring of invariants with respect to the subgroup $1 + p^r\mathbb{Z}_p \leqslant \mathbb{Z}_p^\times$.

Theorem 2. $\mathrm{D}_p^{(r+1)}$ *is a simple $\mathbb{F}_p$-Artin-Schreier extension of* $\mathrm{D}_p^{(r)}$ *with Galois group* $(1 + p^r\mathbb{Z}_p)/(1 + p^{r+1}\mathbb{Z}_p) \cong \mathbb{Z}/p = \mathbb{F}_p$.

We say that B is a *simple $\mathbb{F}_p$ Artin-Schreier extension* of an $\mathbb{F}_p$-algebra A if $B = A[t]/(t^p - t - a)$ for some $a \in A$; the *Galois group* $\mathrm{Gal}(B/A) \cong \mathbb{F}_p$ consists of the p distinct A-algebra automorphisms

$$f(t) \longmapsto f(t+u) \quad (u \in \mathbb{F}_p).$$

We also have $\mathrm{D}_p^{(1)} = E\ell\ell_*/(p, A-1)$, the ring of mod p modular forms of Swinnerton-Dyer and Serre.

Corollary 3. $E\ell\ell_*/(p)[A^{-1}] \longrightarrow KU_*E\ell\ell/(p)$ *is faithfully flat.*

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2 Cohomology of Artin-Schreier towers

Proposition 4. *If B is a simple $\mathbb{F}_p$-Artin-Schreier extension of A then the p elements $(t+u)^{p-1}$ $(u \in \mathbb{F}_p)$ form a normal basis with respect to the Galois group* $\mathrm{Gal}(B/A)$.

A sequence of such simple $\mathbb{F}_p$-Artin-Schreier extensions

$$A = B_1 \subset B_2 \subset \cdots \subset B_r \subset \cdots$$

is called an $\mathbb{F}_p$-*Artin-Schreier tower* if there is a pro-p group G acting properly on $B = \bigcup_{r \geqslant 1} B_r$ with normal subgroups $G_r \lhd G$ for which $G_1 = G$, $B^{G_r} = B_r$, $G_r/G_{r+1} \cong \mathbb{F}_p$ and $G = \varprojlim_r G_r$.

$\mathrm{H}^*_{\mathrm{c}}$ denotes continuous cohomology.

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Theorem 5. *For such an $\mathbb{F}_p$-Artin-Schreier tower,*

$$\mathrm{H}^*_{\mathrm{c}}(G; B) = \varinjlim_r \mathrm{H}^*(G/G_r; B_r) = A.$$

The proof requires a lemma proved with a simple argument using the Lyndon-Hochschild-Serre spectral sequence.

Lemma 6. *Let $A \subset B \subset C$ be a tower of commutative unital rings. Let $G \leqslant \mathrm{Aut}_A(C)$ be finite and $N \lhd G$ with $C^G = A$, $C^N = B$, and $C \cong B[N]$ as a $B[N]$-module. Then*

$$\mathrm{H}^*(G; C) \cong \mathrm{H}^*(G/N; B).$$

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3 The $E\ell\ell$-theory ASS

We are interested in calculating the elliptic cohomology Adams E_2-term $\mathrm{Ext}^{**}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*)$. At each prime $p > 3$, $E\ell\ell_*$ is v_2-periodic, so we might expect that the only chromatic layers are those for $v_0 = p$, v_1 and v_2. This is really a consequence of a result of Hovey & Sadofsky [4] which shows that the chromatic filtration for a v_n-periodic theory stops at the n-th chromatic layer. We also require some 'change of rings' results.

Let (A, Γ) be a (graded) Hopf algebroid over a ring $\Bbbk$. Given a homomorphism of commutative $\Bbbk$-algebras $f\colon A \longrightarrow B$ we can form $\Sigma_f = B \otimes_A \Gamma \otimes_A B$. Then (B, Σ_f) becomes a Hopf algebroid and there is a natural morphism of Hopf algebroids $f_*\colon (A, \Gamma) \longrightarrow (B, \Sigma_f)$ induced by the evident algebra homomorphism $\Gamma \longrightarrow \Sigma_f$.

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Let M be a left (A, Γ)-comodule. Then $f^*M = B \otimes_A M$ inherits a natural left (B, Σ_f)-comodule structure. In the next result, the key idea of using faithful flatness is due to Würgler and independently by Hopkins [4].

Theorem 7. *If the algebra extension $f\colon A \longrightarrow B$ is faithfully flat, then for any Γ-comodule M there is a natural isomorphism*

$$\mathrm{Ext}^{**}_{\Sigma_f}(B, f^*M) \cong \mathrm{Ext}^{**}_{\Gamma}(A, M).$$

We also need a standard homotopy invariance result.

Proposition 8. *Suppose $f, g\colon A \longrightarrow B$ and $H\colon \Gamma \longrightarrow B$ are $\Bbbk$-algebra homomorphisms s.t. $H \circ \eta_L = f$ and $H \circ \eta_R = g$. Then (B, Σ_f) and (B, Σ_g) are naturally equivalent and there is a natural isomorphism*

$$\mathrm{Ext}^{**}_{\Sigma_f}(B, f^*M) \cong \mathrm{Ext}^{**}_{\Sigma_g}(B, g^*M)$$

for any left Γ-comodule M.

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We can use these results for the case $A = E\ell\ell_*/(p)[A^{-1}]$, $\Gamma = E\ell\ell_* E\ell\ell/(p)[A^{-1}]$ and $f = \eta_R \colon A \longrightarrow B = KU_* E\ell\ell/(p)$ to show that

$$\mathrm{Ext}^{**}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p)[A^{-1}]) \cong \mathrm{Ext}^{**}_{\Sigma_f}(B, B).$$

But

$$E\ell\ell_* E\ell\ell/(p)[A^{-1}] = E\ell\ell_* \underset{MU_*}{\otimes} MU_* MU \underset{MU_*}{\otimes} E\ell\ell_*/(p)[A^{-1}],$$

$$KU_* KU/(p) = KU_* \underset{MU_*}{\otimes} MU_* MU \underset{MU_*}{\otimes} KU_*/(p),$$

and we can obtain Σ_f either from $E\ell\ell_* E\ell\ell/(p)[A^{-1}]$ by tensoring with $KU_* E\ell\ell/(p)$ over $E\ell\ell_*/(p)[A^{-1}]$ or by tensoring $KU_* KU/(p)$ with $KU_* E\ell\ell/(p)[A^{-1}]$ over $KU_*/(p)$.

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Again using 8 and 7, the second interpretation shows that

$$\begin{aligned}\mathrm{Ext}^{**}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p)[A^{-1}]) &\cong \mathrm{Ext}^{**}_{KU_* KU}(KU_*, KU_*/(p)) \\ &\cong \mathrm{Ext}^{**}_{K(1)_* K(1)}(K(1)_*, K(1)_*).\end{aligned}$$

In fact, a similar argument works to show that

$$\mathrm{Ext}^{**}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p)[A^{-1}]) \cong \mathrm{Ext}^{**}_{K(2)_* K(1)}(K(2)_*, K(2)_*).$$

It is perhaps more illuminating to see the general version of such results.

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4 A general version

Theorem 9. *Let R_* be an algebra over $\mathbb{F}_{p^n} \otimes BP_*$ which is annihilated by I_n and in which there exists a unit u satisfying $v_n = u^{(p^n-1)/(p-1)}$. Then the ring*

$$R_*K(n) = R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*$$

is a free R_*-module. Moreover, there is an exhaustive filtration of subalgebras

$$R_* = R_*K(n)^{(1)} \subset R_*K(n)^{(2)} \subset \cdots \subset R_*K(n)^{(k)} \subset \cdots \subset R_*K(n)$$

in which each extension $R_*K(n)^{(k)} \longrightarrow R_*K(n)^{(k+1)}$ is a free $R_*K(n)^{(k)}$-module and a Galois extension of Artin-Schreier type with Galois group $\mathbb{F}_{p^n}$.

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There is an action of the Morava stabiliser group $\mathbb{S}_n$ under which $R_*K(n)^{(k)} = R_*K(n)_*^{\mathbb{S}_n^{[k]}}$, the fixed point set of the closed subgroup

$$\mathbb{S}_n^{[k]} = \{1 + \sum_{k \leqslant r} \alpha_r S^r : \forall r,\, \alpha_r^{p^n} = \alpha_r\} \subseteq \mathbb{S}_n.$$

The continuous cohomology satisfies

$$\mathrm{Ext}^{**}_{K(n)_*K(n)}(K(n)_*, R_*K(n)) \cong \mathrm{H}^*_{\mathrm{c}}(\mathbb{S}_n; R_*K(n)) = R_*.$$

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