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Complex cobordism of Hilbert manifolds

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We will define a notion of cobordism generalizing that of Quillen:

- D. G. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, Adv. in Math. **7** (1971), 29–56.

Details appear in

- C. Özel, On the Complex Cobordism of Flag Varieties Associated to Loop Groups, PhD thesis, University of Glasgow (1998).
- A. Baker & C. Özel, Complex cobordism of Hilbert manifolds with some applications to flag varieties of loop groups, Glasgow University Mathematics Department preprint no. 98/37, `http://www.maths.gla.ac.uk/~andy/dvi-ps.html`

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1 Cobordism of Fredholm maps

A manifold will mean a separable smooth manifold locally modelled on a separable Hilbert space.

Definition 1.1 A smooth map $f\colon X \longrightarrow Y$ between manifolds is *Fredholm* if for each $x \in X$, $df_x\colon T_xX \longrightarrow T_{f(x)}Y$ is a Fredholm operator. The *index* of f at $x \in X$ is then defined by

$$\operatorname{index} f_x = \dim\ker df_x - \dim\operatorname{coker} df_x.$$

The function $X \longrightarrow \mathbb{Z}$ given by $x \longmapsto \operatorname{index} df_x$ is locally constant, hence continuous.

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Definition 1.2 Suppose that $f\colon X \longrightarrow Y$ is a proper Fredholm map with even index at each point. Then f is an *admissible complex orientable map* if there is a smooth factorization

$$f\colon X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y,$$

where $q\colon \xi \longrightarrow Y$ is a finite dimensional smooth complex vector bundle and $\tilde{f}$ is a smooth embedding endowed with a complex structure on its normal bundle $\nu(\tilde{f})$.

A complex orientation for a Fredholm map f of odd index is one for the map $(f,\varepsilon)\colon X \longrightarrow Y \times \mathbb{R}$ given by $(f,\varepsilon)(x) = (f(x),0)$ for every $x \in X$. Then for $x \in X$, $\operatorname{index}(f,\varepsilon)_x = (\operatorname{index} f_x) - 1$ and the finite dimensional complex vector bundle ξ in the smooth factorization will be replaced by $\xi \times \mathbb{R} \longrightarrow Y \times \mathbb{R}$.

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Suppose that f is an admissible complex orientable map with a factorisation $\tilde{f}$. As the map f is Fredholm and ξ is a finite dimensional vector bundle, $\tilde{f}$ is Fredholm. Then

$$\operatorname{index} \tilde{f} = \operatorname{index} f - \dim \xi.$$

There is an obvious notion of isotopy of such factorizations which defines an equivalence relation. There is also a notion of stabilization defining a further equivalence relation.

Definition 1.3 Two factorizations $f\colon X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y$ and $f\colon X \xrightarrow{\tilde{f}'} \xi' \xrightarrow{q'} Y$ are *equivalent* if ξ and ξ' can be embedded as subvector bundles of a vector bundle $\xi'' \longrightarrow Y$ such that $\tilde{f}$ and $\tilde{f}'$ are isotopic in ξ'' with isotopy compatible with the complex structure on the normal bundles.

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Definition 1.4 Let $f_i\colon X_i \longrightarrow Y$ $(i = 0, 1)$ be admissible complex oriented maps. Then f_0 is *cobordant* to f_1 if there is an admissible complex orientable map $h\colon W \longrightarrow Y \times \mathbb{R}$ such that the maps $\varepsilon_i\colon Y \longrightarrow Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i = 0, 1$, are transverse to h and the pull-back map ${\varepsilon_i}^*h$ is equivalent to f_i.

The cobordism class of $f\colon X \longrightarrow Y$ will be denoted $[X, f]$.

This notion of cobordism defines an equivalence relation.

For a separable Hilbert manifold Y, $\mathcal{U}^d(Y)$ denotes the set of cobordism classes of the admissible complex orientable proper Fredholm maps of index $-d$. $\mathcal{U}^*(Y)$ is a graded group under the addition defined by disjoint union. The empty set defines 0. For every Y there is a distinguished element $1 = [Y, \mathrm{Id}_Y] \in \mathcal{U}^0(Y)$.

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If $f\colon X \longrightarrow Y$ is an admissible complex orientable Fredholm map of index r and $g\colon Y \longrightarrow Z$ is an admissible complex orientable Fredholm map with index s, then $g \circ f\colon X \longrightarrow Z$ is an admissible complex orientable map with index $r + s$.

There is *push-forward* (or *Gysin*) map

$$g_*\colon \mathcal{U}^d(Y) \longrightarrow \mathcal{U}^{d-s}(Z)$$
$$g_*[X, f] = [X, g \circ f],$$

which is defines a homomorphism of graded groups.

This gives covariance with respect to proper Fredholm maps to our functor.

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2 Transversality and contravariance

Proposition 2.1 *Let $f\colon X \longrightarrow Y$ be an admissible complex orientable map and $g\colon Z \longrightarrow Y$ a smooth map transverse to f. Then the pull-back map*

$$g^*f : Z \underset{Y}{\sqcap} X \longrightarrow Z$$

is an admissible complex orientable map, where $\underset{Y}{\sqcap}$ denotes the smooth pullback or transverse intersection.

We would like to know that given such a pair of maps (perhaps with g also proper Fredholm) there exists an approximation f' to f transverse to g. It appears that this may not always exist! If Y and Z are finite dimensional then it is well known that such transverse approximations do exist and can be taken to be cobordant to f.

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A similar difficulty applies to the existence of internal cup products $\mathcal{U}^r(Y) \times \mathcal{U}^s(Y) \longrightarrow \mathcal{U}^{r+s}(Y)$ which ought to be defined by

$$[W, g] \cup [Z, h] = \Delta^*[W \times Z, g \times h],$$

where $\Delta\colon Y \longrightarrow Y \times Y$ is the diagonal.

The volume

- Global Analysis, Proc. Symp. Pure Math. **15** (1970), 213–222.

contains many seemingly related results on infinite dimensional transversality but none appear to be suitable for our needs. An example of what is proved there is the following result of F. Quinn.

Theorem 2.2 *Let $f\colon X \longrightarrow Y$ be a Fredholm map and $g\colon M \longrightarrow Y$ an inclusion of a finite-dimensional submanifold. Then there is an approximation g' of g in $C^\infty(M, Y)$ with the fine topology such that g' is transverse to f.*

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Another result following from work therein is the following which establishes contraviance with respect to the class of *Sard functions* of Quinn's paper.

Theorem 2.3 *Let X, Y, Z be infinite dimensional smooth separable Hilbert manifolds, $f\colon X \longrightarrow Y$ an admissible complex orientable map and $g\colon Z \longrightarrow Y$ a Sard function. Then the cobordism class of the pull-back $Z \underset{Y}{\sqcap} X \longrightarrow Z$ only depends on the cobordism class of f. Hence there is a group homomorphism $g^*\colon \mathcal{U}^d(Y) \longrightarrow \mathcal{U}^d(Z)$ given by*

$$g^*[X, f] = [Z \underset{Y}{\sqcap} X, g^*(f)].$$

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3 Euler classes for finite dimensional bundles

It turns out that despite the difficulties with transversality mentioned above, there are Euler classes for finite dimensional complex vector bundles in $\mathcal{U}^*(\ )$. Let $\xi \longrightarrow Y$ be such a bundle of dimension n. Then the zero section $i\colon Y \longrightarrow \xi$ defines a cobordism class $i_*1 = [Y, i] \in \mathcal{U}^{2n}(\xi)$. When Y is finite dimensional we have $\chi(\xi) = i^*i_*1$, however we need to ensure that the right hand side is meaningful.

It turns out that separable Hilbert manifolds possess enough partitions of unity hence many global sections of bundles exist and are Sard functions. The following result of Quinn shows that i^*i_*1 does indeed exist.

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Theorem 3.1 *Let U be an open set in separable infinite dimensional Hilbert space H and let $f\colon X \longrightarrow Y$ be a proper Fredholm map between separable infinite dimensional Hilbert manifolds X and Y. Then the set of maps transverse to f is dense in the closure of Sard function space $\overline{\mathcal{S}(U, Y)}$ in the C^∞ fine topology.*

We also use the Open Embedding Theorem of Eells & Elworthy in proving this.

Theorem 3.2 *Let X be a smooth manifold modelled on the separable infinite dimensional Hilbert space H. Then X is diffeomorphic to an open subset of H.*

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The following projection formula holds.

Theorem 3.3 *Let $f\colon X \longrightarrow Y$ be an admissible complex orientable Fredholm submersion and let $\pi\colon \xi \longrightarrow Y$ be a finite dimensional smooth complex vector bundle. Then*

$$\chi(\xi) \cup [X, f] = f_* \chi(f^* \xi).$$

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4 The relationship between $\mathcal{U}$-theory and MU-theory

Let X be a separable Hilbert manifold and recall Quinn's Theorem 2.3. Then for each proper smooth map $f\colon M \longrightarrow X$ where M is a finite dimensional manifold, there is a pullback homomorphism $f^*\colon \mathcal{U}^*(X) \longrightarrow \mathcal{U}^*(M) = \mathrm{MU}^*(M)$. If we consider all such maps into X, then there is a unique homomorphism $\rho\colon \mathcal{U}^*(X) \longrightarrow \varprojlim_{M \longrightarrow X} \mathrm{MU}^*(M)$, where the limit is taken over all such maps from finite dimensional manifolds. The following conjectures seem reasonable and are consistent with examples.

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Conjecture 4.1 Let X be a separable Hilbert manifold.

A) The natural homomorphism $\rho\colon \mathcal{U}^*(X) \longrightarrow \varprojlim_{M\longrightarrow X} \mathrm{MU}^*(M)$ is surjective.

B) If $\mathcal{U}^{\mathrm{ev}}(X) = 0$ or $\mathcal{U}^{\mathrm{odd}}(X) = 0$, the natural homomorphism $\rho\colon \mathcal{U}^*(X) \longrightarrow \varprojlim_{M\longrightarrow X} \mathrm{MU}^*(M)$ is surjective.

C) If $\mathrm{MU}^{\mathrm{ev}}(X) = 0$ or $\mathrm{MU}^{\mathrm{odd}}(X) = 0$, the natural homomorphism $\rho\colon \mathcal{U}^*(X) \longrightarrow \varprojlim_{M\longrightarrow X} \mathrm{MU}^*(M)$ is surjective.

We hope that surjectivity can be replaced isomorphism, but do not have any examples to support this.

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5 Some examples

Let H be a separable complex Hilbert space and suppose that there is an increasing sequence of finite dimensional subspaces H^n with $\dim H^n = n$ so that $H^\infty = \bigcup_{n\geqslant 1}$ is dense in H. Then the projective space $\mathrm{P}(H)$ is a separable Hilbert manifold containing $\mathrm{P}(H^\infty)$. We have

$$\mathrm{P}(H) = \mathrm{U}(H)/\mathrm{U}(H^1_\perp) \times \mathrm{U}(H')$$

where $H^1_\perp$ is the orthogonal complement of H^1. By Kuiper's Theorem, $\mathrm{U}(H)$ and $\mathrm{U}(H)$ are contractible, hence the inclusion $\mathrm{P}(H^\infty) \longrightarrow \mathrm{P}(H)$ is an equivalence of classifying spaces for $S^1 \cong \mathrm{U}(H^1)$.

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Theorem 5.1 *The restriction map*

$$\rho\colon \mathcal{U}^*(\mathrm{P}(H)) \longrightarrow \mathrm{MU}^*(\mathrm{P}(H))$$

is surjective. More generally, the restriction maps

$$\rho\colon \mathcal{U}^*(\mathrm{Gr}_r(H) \longrightarrow \mathrm{MU}^*(\mathrm{Gr}_r(H))$$

are surjective.

To prove this we may use the fact that for each $n \geqslant 1$, the orthogonal complement $H^n_\perp$ of H^n has a projective space projective space which is a proper Fredholm subspace of $\mathrm{P}(H)$ of index $-2n$ defining a cobordism class $[\mathrm{P}(H^n_\perp), i] \in \mathcal{U}^{2n}(\mathrm{P}(H))$ restricting to $x^n \in \mathrm{MU}^{2n}(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{MU}^{2n}(\mathrm{P}(H^\infty))$.

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6 Schubert calculus in complex cobordism for loop groups

Bressler & Evens generalized ideas of classical Schubert calculus to complex cobordism. For a compact, conected semisimple Lie group G with maximal torus T they described $\mathrm{MU}^*(G/T)$ in terms of certain Bott–Samelson resolutions of the Schubert cells of the complex flag space $G/T \cong G_{\mathbb{C}}/B$ where B is a Borel subgroup containing T. The Schubert cells are indexed by the Weyl group W_G.

For the loop group LG there is a similar description of the cell structure of the complex flag space $LG/T \cong LG_{\mathbb{C}}/\tilde{B}$, where $\tilde{B}$ is a Borel subgroup. This time the cells are indexed by the Affine Weyl group $\hat{W}_G$.

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There are Bott–Samelson resolutions which are iterated $\mathbb{C}\mathrm{P}^1$-bundles over LG/T. Thus for each cell C_w ($w \in \hat{W}_G$) there is a map

$$LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_{\alpha_1} \underset{\widetilde{B}}{\times} P_{\alpha_2} \underset{\widetilde{B}}{\times} \cdots \underset{\widetilde{B}}{\times} P_{\alpha_{\ell(w)}}/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/\tilde{B} \cong LG/T$$

representing an element of $\mathcal{U}^{2\ell(w)}(LG/T)$. Here P_α is the parabolic subgroup containing $\tilde{B}$ associated to the simple root α and $w = r_{\alpha_1} \cdots r_{\alpha_{\ell(w)}}$ is a reduced presentation in terms of fundamental reflections r_α; we have $P_\alpha/\tilde{B} \cong \mathbb{C}\mathrm{P}^1$.

There are also analogues of the Bernstein–Gelfand–Gelfand operators A_α associated to simple roots and defined by

$$A_\alpha = {\pi_\alpha}^* \pi_{\alpha *},$$

defined in terms of the projection in the $\mathbb{C}\mathrm{P}^1$-bundle

$$\pi_\alpha \colon LG/T \cong LG_{\mathbb{C}}/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/P_\alpha.$$

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Using work of Pressley & Segal we also have

Theorem 6.1 *The restriction map*

$$\rho \colon \mathcal{U}^*(LG/T) \longrightarrow \mathrm{MU}^*(LG/T)$$

is surjective.