CLASSIFYING SPACES, VIRASORO EQUIVARIANT BUNDLES, ELLIPTIC COHOMOLOGY AND MOONSHINE

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Introduction

This work explores some connections between the elliptic cohomology of classifying spaces for finite groups, Virasoro equivariant bundles over their loop spaces and Moonshine for finite groups. Our motivation is as follows: up to homotopy we can replace the loop group $\mathcal{L}\mathrm{B}G$ by the disjoint union $\coprod_{[\gamma]} \mathrm{B}\,\mathrm{C}_G(\gamma)$ of classifying spaces of centralizers of elements γ representing conjugacy classes of elements in G. An elliptic object over $\mathcal{L}\mathrm{B}G$ then becomes a compatible family of graded infinite dimensional representations of the subgroups $\mathrm{C}_G(\gamma)$, which in turn defines an element in J. Devoto's equivariant elliptic cohomology ring $\mathcal{E}\ell\ell^*_G$. Up to localization (inversion of the order of G) and completion with respect to powers of the kernel of the homomorphism $\mathcal{E}\ell\ell^*_G \longrightarrow \mathcal{E}\ell\ell^*_{\{1\}}$, this ring is isomorphic to $E\ell\ell^*(\mathrm{B}G)$, see [5, 6]. Moreover, elliptic objects of this kind are already known: for example the 2-variable Thompson series forming part of the Moonshine associated with the simple groups M_{24} and the Monster $\mathbb{M}$. Indeed the compatibility condition between the characters of the representations of $\mathrm{C}_G(\gamma)$ mentioned above was originally formulated by S. Norton in an Appendix to [21], independently of the work on elliptic genera by various authors leading to the definition of elliptic cohomology (see the various contributions to [17]). In a slightly different direction, J-L. Brylinski [2] introduced the group of Virasoro equivariant bundles over the loop space $\mathcal{L}M$ of a *simply connected* manifold M as part of his investigation of a Dirac operator on $\mathcal{L}M$ with coefficients in a suitable vector bundle. Our suggested definition of an elliptic object (= equivariant bundle over a not necessarily simply-connected space) starts from this, and builds in additional structure suggested by Moonshine. We propose it as only provisional, for while it fits in well with Devoto's construction, the localization which this requires suggests that, even in the very special case of $X = \mathrm{B}G$, further refinement will be necessary to obtain a geometric definition of an elliptic-like cohomology theory.

The following example may help the reader to follow our general construction. Accepting for the moment that up to completion and localization a class in $E\ell\ell^{\mathrm{even}}(\mathrm{B}G)$ is represented by an infinite dimensional bundle over the loop space $\mathcal{L}\mathrm{B}G$, restriction to the subspace of constant loops defines a map $E\ell\ell^*(\mathrm{B}G) \longrightarrow K^*(\mathrm{B}G)((q))$. The image of the representation ring $R(G)$ in the coefficients of the power series ring on the right is dense, giving a privileged position to representations whose characters satisfy a modularity condition. The 1-dimensional Thompson series of [4, 21, 32] are certainly of this type. For the Mathieu group M_{24} we have a particularly

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simple construction, starting with the Dedekind η-function,

$$\eta(\tau) = q^{1/24} \prod_{r=1}^{\infty} (1 - q^r).$$

For an arbitrary group element $g \in M_{24} \subseteq S_{24}$ representing a conjugacy class $[g]$, decompose g as $(1)^{j_1} \cdots (r)^{j_r}$ with $j_1 + 2_{j2} + \cdots + rj_r = 24$, a product of disjoint cycles. Write

$$\eta_g(\tau) = \eta(\tau)^{j_1} \eta(2\tau)^{j_2} \cdots \eta(r\tau)^{j_r} = \sum_n a_g(n) q^n,$$

where $q = e^{2\pi i \tau}$. In particular,

$$\eta_1(\tau) = \eta(\tau)^{24} = \Delta.$$

As $[g]$ runs through all possible conjugacy classes, and writing $\omega_n(g) = a_g(n)$, we see that

$$\Omega_n(\tau) = \sum_{n=1}^{\infty} \omega_n q^n$$

becomes a Thompson series. That the class functions $\omega_n(g)$ are actually characters follows from the identification of the product $\prod_{r=1}^{\infty}(1 - q^r)^{-1}$ with the formal character of the symmetric algebra on $Vq + Vq^2 + Vq^3 + \cdots$, where V is the natural permutation representation module. This is a special case of the construction in §4 below; its defect is that the representation Ω is inhomogeneous, i.e., the characters are modular forms of varying weight. For the record,

$$\text{the weight of } \eta_g(\tau) = \frac{1}{2}(\text{number of cycles in } g),$$

while the level equals the product of the lengths of the longest and shortest cycles in the decomposition. The variation in the level is to be expected, and arises naturally in Devoto's description of $\mathcal{E}\ell\ell^*_{\mathbb{Z}/p}(\text{point})$, see [6], section 2.3. However, the variation of the weight is more serious. We avoid it by replacing Ω by another Thompson series Θ/Ω, for which the components of the graded character all have weight zero, that is are modular functions. Here we have

$$\Theta(\tau) = \sum_n \theta_n q^n,$$

where θ_n is a permutation representation associated with an action on a suitable lattice L. This extension of the construction is explained below in §5. For the special case of a finite simple group such as M_{24} see [21], section 6.

In terms of centralizers the construction outlined above yields representations (or flatly graded bundles) for pairs $(1, g)$. The extension to pairs (h, g), at least for the group M_{24}, is contained in [24, 25], and we return to it in the last section. We restrict ourselves to elements of odd order, for which the representation module Θ/Ω does not have to be modified in order to satisfy the 'genus zero' condition required by Moonshine. This is very much in the spirit of elliptic cohomology, which in the form in which we discuss it, is localized away from the prime 2. Indeed it is interesting to ask whether the construction of the 'correct' modular forms for all conjugacy classes of commuting pairs of elements has anything to do with the extension of the definition of the homology theory $E\ell\ell_*$ to the prime 2 by means of a spectrum closely related to that of connective K-theory, see [15].

For the reader's convenience, we summarise the content of each section of this paper as follows.

§1 Recollection of Devoto's definition of equivariant elliptic cohomology.

§2 $E\ell\ell^*(\mathrm{B}G)$ and families of flat bundles.

§3 Modification of J. Brylinski's definition of an equivariant Virasoro bundle to allow for non-trivial fundamental groups.

§4 Examples drawn from [2] following a suggestion of G. Segal [29].

§5 Combination of our Brylinski-motivated construction with lattices. This can be thought of as the abstract formulation of some of the work of Mason.

§6 We show how the construction of §5 ties in with the work of Devoto, that is how an elliptic system in the sense of G. Mason is really an element in the zero-th elliptic cohomology group of the classifying space $\mathrm{B}G$.

We refer to the excellent works of Hirzebruch et al. [9], Landweber [17, 18], Landweber, Ravenel & Stong [19] and Segal [29] for background details on elliptic genera and elliptic cohomology. For work on the elliptic cohomology of classifying spaces of finite groups and equivariant elliptic cohomology, see Hopkins, Kuhn & Ravenel [11, 12, 10, 16] and Devoto [5, 6].

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1. Equivariant elliptic cohomology and classifying spaces for finite groups

Recent work of Devoto [5, 6] provides equivariant versions of elliptic cohomology (of level 2) for finite groups of odd order. This work includes a completion theorem identifying the elliptic cohomology of $\mathrm{B}G$. It seems plausible that, even without a geometric model for elliptic cohomology, Devoto's work can be extended to cover the case of a level 1 version for arbitrary finite groups. Because of this we restate his results in a form anticipating such an extension. We could however restrict attention to groups of odd order and replace the full modular group by the congruence subgroup $\Gamma_0(2)$. We therefore denote by Γ whichever of these groups $\mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(2)$ is being used. We will give a brief description of Devoto's results in a form suitable for our use, *making the assumption that the extensions above are valid.* We may return to the validity of this assumption in future work. For details of the level 1 version of elliptic cohomology see Landweber, Ravenel & Stong [19] and Baker [1]; although still unpublished, the influential preprint of Hopkins, Kuhn & Ravenel [11] also provides an important background to our work.

Throughout, G will denote a finite group and we set

$$\|G\| = \begin{cases} 3|G| & \text{if } \Gamma = \mathrm{SL}_2(\mathbb{Z}), \\ |G| & \text{if } \Gamma = \Gamma_0(2). \end{cases}$$

There is a G-equivariant cohomology theory $\mathcal{E}\ell\ell_G(\ )$ defined on the category of finite G-CW complexes, and possessing the following properties.

Let

$$\mathrm{T}G = \{(\gamma_1, \gamma_2) \in G^2 : \gamma_1\gamma_2 = \gamma_2\gamma_1\},$$

and

$$\mathfrak{h} = \{\tau \in \mathbb{C} : \operatorname{im} \tau > 0\}.$$

There is an action of $\Gamma \times G$ on the product space $\mathrm{T}G \times \mathfrak{h}$ given by

$$(A, \gamma) \cdot (\gamma_1, \gamma_2, \tau) = \left(\gamma\gamma_1^d\gamma_2^{-b}\gamma^{-1}, \gamma\gamma_1^{-c}\gamma_2^a\gamma^{-1}, \frac{a\tau + b}{c\tau + d}\right),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\gamma \in G$.

The coefficient ring $\mathcal{E}\ell\ell_G^* = \mathcal{E}\ell\ell_G^*(\text{point})$ is concentrated in even degrees and $\mathcal{E}\ell\ell_G^{2k}$ ($k \in \mathbb{Z}$) is the group of all holomorphic functions $F\colon \mathrm{T}G \times \mathfrak{h} \longrightarrow \mathbb{C}$ satisfying the following four conditions.

Dev-1 For $A \in \Gamma$ and $(\gamma_1, \gamma_2) \in \mathrm{T}G$,

$$F(A \cdot ((\gamma_1, \gamma_2), \tau)) = (c\tau + d)^{-2k} F((\gamma_1, \gamma_2), \tau).$$

Dev-2 For $(\gamma_1, \gamma_2) \in \mathrm{T}G$, the function $F((\gamma_1, \gamma_2),\ )\colon \mathfrak{h} \longrightarrow \mathbb{C}$ is a modular form for some congruence subgroup $\Gamma' \subseteq \Gamma$ (we can assume that $\Gamma_0(|G|) \subseteq \Gamma'$), with a q-expansion at each cusp of the form

$$\sum_{-\infty \ll r} a_{r/|G|}(\gamma_1, \gamma_2) q^{r/d_0|G|},$$

where $q^{r/N} = e^{2\pi i r/N}$, $a_{r/|G|}(\gamma_1, \gamma_2) \in \mathbb{Z}[1/||G||, \zeta_{|G|}]$ and $d_0 \in \mathbb{N}$ is a constant depending only on the function F.

Dev-3 The functions $a_{r/|G|}$ are class functions on $\mathrm{T}G$ in the sense that for $(\gamma_1, \gamma_2) \in \mathrm{T}G$ and $\gamma \in G$,

$$a_{r/|G|}(\gamma\gamma_1\gamma^{-1}, \gamma\gamma_2\gamma^{-1}) = a_{r/|G|}(\gamma_1, \gamma_2).$$

Dev-4 the functions $a_{r/|G|}$ are 'Galois invariant' in the sense that for any $t \in \mathbb{Z}$ prime to $|\gamma_2|$, we have

$$a_{r/|G|}(\gamma_1, \gamma_2^t) = \sigma_t a_{r/|G|}(\gamma_1, \gamma_2),$$

where σ_t is the unique automorphism of the ring $\mathbb{Z}[1/||G||, \zeta_{|\gamma_2|}]$ for which

$$\sigma_t(\zeta_{|\gamma_2|}) = \zeta_{|\gamma_2|}^t.$$

For each γ_1, the q-expansion coefficient functions $a_{r/|G|}(\gamma_1,\ )$ are actually rational virtual characters on the centralizer $\mathrm{C}_G(\gamma_1)$, and thus lie in the rationalized representation ring $\mathbb{Q} \otimes \mathrm{R}(\mathrm{C}_G(\gamma_1))$. Integrality results of Katz [14] imply that there is a non-zero integer N for which $Na_{r/|G|}(\gamma_1,\ ) \in \mathrm{R}(\mathrm{C}_G(\gamma_1))$.

Devoto also proves a completion result that gives a description of $E\ell\ell^*(\mathrm{B}G)$ in terms of the equivariant theory $\mathcal{E}\ell\ell_G^*(\ )$, namely

$$E\ell\ell^*(\mathrm{B}G)[1/|G|] \cong \widehat{\mathcal{E}\ell\ell_G^*}, \tag{1.1}$$

where the right hand side is the completion with respect to $I_G = \ker \varepsilon \colon \mathcal{E}\ell\ell_G^* \longrightarrow E\ell\ell^*[1/|G|]$, the kernel of the augmentation map corresponding to the inclusion of the trivial group into G.

In §6 we will make use of Devoto's work to identify certain objects as elements of $E\ell\ell^*(\mathrm{B}G)$.

2. Some free loop spaces of the classifying space of a finite group

Let G be a finite group. It is well known that $\mathrm{B}G$ and $\mathrm{E}G$ can be expressed as unions of finite dimensional smooth manifolds. For example, by considering the permutation action of G on itself, G can be identified as a subgroup of either $\mathrm{O}(|G|)$ or $\mathrm{U}(|G|)$; thus we can take $\mathrm{E}G$ to be a union of finite dimensional Stiefel manifolds with smooth free G action and $\mathrm{B}G$ as the union of the orbit spaces. We will now consider the topology of the space $\mathcal{L}\mathrm{B}G$ of free smooth loops in $\mathrm{B}G$ in detail. The following is probably well known, but still seems worth describing in detail.

We will begin by identifying the homotopy type of $\mathcal{L}\mathrm{B}G$. Let $b_0 \in \mathrm{B}G$ be a chosen basepoint. We will write $\mathbb{T} \xrightarrow{f} \mathrm{B}G$ if $f \in \mathcal{L}\mathrm{B}G$, and $\mathbb{T} \xrightarrow[0]{f} \mathrm{B}G$ if $f \in \Omega\mathrm{B}G = \Omega_{b_0}\mathrm{B}G$, the subspace of loops based at b_0.

For each $\alpha \in G$, define a subspace of the space of all smooth paths in a universal free contractible G space $\mathrm{E}G$,

$$\mathcal{L}_\alpha\mathrm{E}G = \{p \colon \mathbb{R} \longrightarrow \mathrm{E}G : p(t+1) = \alpha p(t) \,\forall t \in \mathbb{R}\},$$

and let

$$\mathcal{L}_G\mathrm{E}G = \coprod_{\alpha \in G} \mathcal{L}_\alpha\mathrm{E}G.$$

An element $p \in \mathcal{L}_\alpha\mathrm{E}G$ is determined by its values on the unit interval $[0,1]$, and we will sometimes identify $\mathcal{L}_\alpha\mathrm{E}G$ with the space of suitably differentiable maps

$$\mathcal{L}_\alpha\mathrm{E}G = \{p \colon [0,1] \longrightarrow \mathrm{E}G : p(1) = \alpha p(0)\}.$$

There is a free left action of G on $\mathcal{L}_G\mathrm{E}G$ given by

$$(\gamma \cdot p)(t) = \gamma p(t).$$

Under this action, $\gamma \in G$ maps $\mathcal{L}_\alpha\mathrm{E}G$ into $\mathcal{L}_{\gamma\alpha\gamma^{-1}}\mathrm{E}G$; hence the centralizer $\mathrm{C}_G(\alpha)$ acts upon $\mathcal{L}_\alpha\mathrm{E}G$. The quotient space $\mathcal{L}_G\mathrm{E}G/G$ may be identified with the free loop space $\mathcal{L}\mathrm{B}G$. In fact, each space $\mathcal{L}_\alpha\mathrm{E}G$ can be seen to be contractible using the following explicit contraction which was pointed out to us by Richard Steiner. Given a based contraction $H \colon [0,1] \times \mathrm{E}G \longrightarrow \mathrm{E}G$ for which $H(0,x) = x$ and $H(1,x) = e_0$ (the base point of $\mathrm{E}G$),

$$H_\alpha \colon [0,1] \times \mathcal{L}_\alpha\mathrm{E}G \longrightarrow \mathcal{L}_\alpha\mathrm{E}G;$$

$$H_\alpha(t,p)(s) = \begin{cases} H(t, p(s)) & \text{if } 0 \leqslant s+t \leqslant 1, \\ H\left(1-s, \alpha H\left(\dfrac{s+t-1}{s}, \alpha^{-1} \cdot p(s)\right)\right) & \text{if } 1 \leqslant s+t \leqslant 2. \end{cases}$$

By the universality of $\mathrm{E}G$, there are G-equivariant maps $\mathcal{L}_\alpha\mathrm{E}G \longrightarrow \mathrm{E}G \times \{\alpha\}$, and thus a G-equivariant map $\mathrm{P} \longrightarrow \mathrm{E}G \times G$, where the left G-action on the codomain is given by

$$(e,\alpha) \cdot \gamma = (\gamma e, \gamma\alpha\gamma^{-1}).$$

On passing to quotients this gives

$$\mathcal{L}\mathrm{B}G \cong \mathcal{L}_G \mathrm{E}G/G \simeq \mathrm{E}G \underset{G}{\times} G_c. \tag{2.1}$$

It follows from this description that the connected components of $\mathcal{L}\mathrm{B}G$ are indexed on the set of conjugacy classes of G, and the component corresponding to the class $[\alpha]$ containing $\alpha \in G$ has the homotopy type of

$$\mathrm{E}G \underset{G}{\times} [\alpha] \cong \mathrm{E}G \underset{G}{\times} G/\,\mathrm{C}_G(\alpha) \cong \mathrm{B}\,\mathrm{C}_G(\alpha).$$

We will write $\mathcal{L}_{[\alpha]}\mathrm{B}G$ for this component. Given a choice of representative α for a conjugacy class $[\alpha]$, the space $\mathcal{L}_\alpha \mathrm{E}G$ is a universal cover of $\mathcal{L}_{[\alpha]}\mathrm{B}G$, and the projection $\mathcal{L}_\alpha \mathrm{E}G \longrightarrow \mathcal{L}_{[\alpha]}\mathrm{B}G$ is a principal $\mathrm{C}_G(\alpha)$-bundle. We associate to the latter the bundle

$$\mathcal{L}_\alpha \mathrm{E}G \underset{\mathrm{C}_G(\alpha)}{\times} G \longrightarrow \mathcal{L}_{[\alpha]}\mathrm{B}G$$

with fibre G and equivalent to the pullback of $\mathrm{E}G \longrightarrow \mathrm{B}G$ along the projection $\mathcal{L}_{[\alpha]}\mathrm{B}G \longrightarrow \mathrm{B}G$. Thus for any G-space F, the associated bundle $\mathrm{E}G \times_G F \longrightarrow \mathrm{B}G$ pulls back to the bundle

$$\mathcal{L}_\alpha \mathrm{E}G \underset{\mathrm{C}_G(\alpha)}{\times} F \longrightarrow \mathcal{L}_{[\alpha]}\mathrm{B}G,$$

with structure group $\mathrm{C}_G(\alpha)$. All of this depends upon the choice of element α in the conjugacy class $[\alpha]$.

The double loop space $\mathcal{L}^2\mathrm{B}G = \mathrm{Map}(\mathbb{T}^2, \mathrm{B}G)$ can be analyzed in a similar fashion. We identify $\mathcal{L}^2\mathrm{B}G$ with the iterated mapping space $\mathcal{L}(\mathcal{L}\mathrm{B}G)$, and write its elements in the form $F(\ ,\ )$, from which we may derive the functions $\mathbb{T} \longrightarrow \mathcal{L}\mathrm{B}G$ given by $z \longmapsto F(z,\ )$ and $z \longmapsto F(\ , z)$. The analogues of the $\mathcal{L}_\alpha \mathrm{E}G$ are the spaces

$$\mathcal{L}^2_{\alpha,\beta}\mathrm{E}G = \big\{F\colon \mathbb{R}^2 \longrightarrow \mathrm{E}G : F(x+1,y) = \alpha F(x,y),\ F(x,y+1) = \beta F(x,y),\ \forall x,y \in \mathbb{R}\big\}$$

defined for all pairs $(\alpha,\beta) \in G^2$ with $\beta \in \mathrm{C}_G(\alpha)$. These spaces can be combined into the disjoint union over all such commuting pairs,

$$\mathcal{L}^2_G \mathrm{E}G = \coprod_{\alpha,\beta} \mathcal{L}^2_{\alpha,\beta}\mathrm{E}G.$$

Under the evident action of G on the latter space, $\gamma \in G$ maps $\mathcal{L}^2_{\alpha,\beta}\mathrm{E}G$ onto $\mathcal{L}^2_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}\mathrm{E}G$. Also the contractible space $\mathcal{L}^2_{\alpha,\beta}\mathrm{E}G$ is preserved by the group $\mathrm{C}_G(\alpha,\beta) = \mathrm{C}_G(\alpha) \cap \mathrm{C}_G(\beta)$ which acts freely on it, hence the quotient $\mathcal{L}^2_{\alpha,\beta}\mathrm{E}G/\,\mathrm{C}_G(\alpha,\beta)$ is homotopy equivalent to the classifying space $\mathrm{B}\,\mathrm{C}_G(\alpha,\beta)$.

There is a natural action of $(\mathrm{Diff}^+)^2 = \mathrm{Diff}^+ \times \mathrm{Diff}^+$ on $\mathcal{L}^2\mathrm{B}G$ given by

$$((\Phi_1, \Phi_2) \cdot F)(z,w) = F(\Phi_1^{-1}(z), \Phi_2^{-1}(w)).$$

We will modify this by defining a 2-parameter family of actions of the diffeomorphism group Diff^+.

For each pair of coprime integers r, s, there is a copy of the circle

$$\mathbb{T}_{r,s} = \big\{(u,v) \in \mathbb{T}^2 : u^s = v^r\big\} \subseteq \mathbb{T}^2.$$

For each $(z,w) \in \mathbb{T}^2$, there is also the translate

$$(z,w)\mathbb{T}_{r,s} = \left\{(u,v) \in \mathbb{T}^2 : u^s z^{-s} = v^r w^{-r}\right\} \subseteq \mathbb{T}^2.$$

Now choose any pair of integers r', s' so that $\det \begin{pmatrix} r & s \\ r' & s' \end{pmatrix} = 1$. If we base the circle $(z,w)\mathbb{T}_{r,s}$ at its unique point (z_0, w_0) which also lies on the circle $\mathbb{T}_{r',s'}$, we can identify the standard circle $\mathbb{T}$ with $(z_0,w_0)\mathbb{T}_{r,s}$ by using the map (depending on r,s,r',s')

$$\Psi \colon \mathbb{T} \xrightarrow{\cong} (z_0,w_0)\mathbb{T}_{r,s}; \quad t \longmapsto (z_0 t^r, w_0 t^s).$$

For a given pair r,s, the circles of the form $(z_0,w_0)\mathbb{T}_{r,s}$ (where (z_0,w_0) runs through the circle $\mathbb{T}_{r',s'}$) partition $\mathbb{T}^2$.

Each $\Phi \in \mathrm{Diff}^+$ acts on $\mathbb{T}^2$ via its action on each circle $(z_0,w_0)\mathbb{T}_{r,s}$ using the identification with $\mathbb{T}$; thus $\Phi(\Psi(t)) = \Psi(\Phi(t))$. This yields an action of Diff^+ on $\mathbb{T}^2$ which respects the partition above. The choices in this definition turn out to be unimportant, since for any other complementary circle through $(1,1)$ used to base the circles parallel to $\mathbb{T}_{r,s}$, the associated actions of Diff^+ are conjugate in the full diffeomorphism group by a Dehn twist generated by the pair associated to a homology cycle of the circle $\mathbb{T}_{r,s}$. More generally, any element $A \in \mathrm{SL}_2(\mathbb{Z})$ gives rise to an $\mathbb{R}$-linear isomorphism on $\mathbb{R}^2$ which induces an orientation preserving diffeomorphism Θ_A of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then we find that the images under A, $A\mathbb{T}_{r,s}$ and $A\mathbb{T}_{r',s'}$ are also circles yielding a partition of $\mathbb{T}^2$; associated to this is a conjugate action of Diff^+ and a fixed point set in $\mathcal{L}^2\mathrm{B}G$, diffeomorphic to $\mathcal{L}\mathrm{B}G$.

Associated to this action of Diff^+ is another action on the space $\mathcal{L}^2\mathrm{B}G$ for which

$$(\Phi \cdot F)(u,v) = F(\Phi^{-1}(u,v)).$$

This action has fixed point set

$$\mathcal{L}^2_{r,s}\mathrm{B}G = \{F : F(t^r z, t^s w) = F(z,w)\ \forall\, t,z,w \in \mathbb{T}\}.$$

An element $F \in \mathcal{L}^2_{r,s}\mathrm{B}G$ is determined by its values on the set $z = 1$, and moreover we have $F(1, \zeta_r^s w) = F(1,w)$ for every w. Hence the function defined on $\mathbb{R}$ by the formula

$$x \longmapsto F(1, e^{2\pi i x})$$

has period $1/r$. By evaluating the first variable at 1, we can identify this fixed point set with the single loop space $\mathcal{L}\mathrm{B}G$. Explicitly, we have

$$\mathcal{L}^2_{r,s}\mathrm{B}G \cong \mathcal{L}\mathrm{B}G; \quad F \longleftrightarrow (z \longmapsto F(1, z^{1/r})).$$

Similarly, the function

$$x \longmapsto F(e^{2\pi i x}, 1)$$

has period $1/s$, and by evaluating the second variable at 1, we can make a further identification with the space $\mathcal{L}\mathrm{B}G$ given by

$$\mathcal{L}^2_{r,s}\mathrm{B}G \cong \mathcal{L}\mathrm{B}G; \quad F \longleftrightarrow (z \longmapsto F(z^{1/s}, 1)).$$

We will work with the first of these from now on.

Notice that any $F \in \mathcal{L}^2_{r,s}\mathrm{B}G$ satisfies the functional equation

$$F(u, v\zeta_r^{-s}) = F(u, v),$$

where $\zeta_r = \exp(2\pi i/r)$. On choosing an integer s' for which $s's \equiv -1 \pmod r$, we have

$$F(u, v\zeta_r) = F(u, v(\zeta_r^{-s'})^s) = F(u, v).$$

Similarly,

$$F(u\zeta_s, v) = F(u, v).$$

Thus we can view elements of $\mathcal{L}^2_{r,s}\mathrm{B}G$ as G-equivalence classes of maps $\widetilde{F}\colon \mathbb{R}^2 \longrightarrow \mathrm{E}G$ for which there exist elements $\alpha, \beta \in G$ satisfying the equations

$$\widetilde{F}(x + 1/s, y) = \alpha\widetilde{F}(x, y), \quad \widetilde{F}(x, y + 1/r) = \beta\widetilde{F}(x, y).$$

Notice that for such a map $\widetilde{F}$ together with α, β as above, the relation $\alpha\beta = \beta\alpha$ must hold, since after factoring out by the action of G we obtain a map into $\mathrm{B}G$ from the torus formed by identifying opposite sides of the rectangle $[0, 1/s] \times [0, 1/r]$. The formulæ

$$\widetilde{F}(x + 1, y) = \widetilde{F}(x + r(1/s), y) = \alpha^s\widetilde{F}(x, y), \tag{2.2}$$

$$\widetilde{F}(x, y + 1) = \widetilde{F}(x, y + s(1/r)) = \beta^s\widetilde{F}(x, y), \tag{2.3}$$

show that the holonomy along each of the unit lengths parallel to the two axes is α^s and β^r respectively.

3. Virasoro equivariant vector bundles over non-simply connected spaces

In this section we extend work of Brylinski [2] to a notion of Virasoro equivariant bundle which includes the case of a loop space $\mathcal{L}X$ where X is path connected but not necessarily simply connected. Our primary motivation is to give a definition which applies to the case of a classifying space for a finite group $\mathrm{B}G$, and we are not sure whether this is 'correct' for more general cases. However, there are connections with Segal's notion of 'elliptic object' described in [29], section 6. The following discussion is modelled on that for the case of $X = \mathrm{B}G$ in §2.

The free loop space $\mathcal{L}X$ has components indexed on the conjugacy classes in the fundamental group $\Pi = \pi_1(X, x_0)$, where x_0 is a chosen base point. If $\widetilde{X}$ denotes the universal cover of X, for each $\alpha \in \Pi$ we have the space

$$\mathcal{L}_\alpha X = \left\{p\colon \mathbb{R} \longrightarrow \widetilde{X} : p(t+1) = \alpha p(t)\ \forall t \in \mathbb{R}\right\}$$

and the disjoint union

$$\mathcal{L}_\Pi X = \coprod_\alpha \mathcal{L}_\alpha X.$$

There is a free action of the fundamental group Π upon $\mathcal{L}_\Pi X$ given by $(\gamma \cdot p)(t) = \gamma p(t)$ for $\gamma \in \Pi$, under which $\mathcal{L}_\alpha X$ is mapped into $\mathcal{L}_{\gamma\alpha\gamma^{-1}}X$ by γ. From this we deduce that $\mathrm{C}_\Pi(\alpha)$ acts freely upon the space $\mathcal{L}_\alpha X$ whose quotient $\mathcal{L}_\alpha X/\mathrm{C}_\Pi(\alpha)$ may be identified with the component $\mathcal{L}_{[\alpha]}X$ of $\mathcal{L}X$ consisting of loops which lift to paths in $\widetilde{X}$ having holonomy in the conjugacy class of α (which we denote by α).

Similar constructions work for the double loop space, giving spaces $\mathrm{P}^2_{\alpha,\beta}(X)$ defined for all $\alpha, \beta \in \Pi$ with $\beta \in \mathrm{C}_\Pi(\alpha)$, and also their disjoint union $\mathrm{P}^2(X)$. Here the centralizer $\mathrm{C}_\Pi(\alpha,\beta) = \mathrm{C}_\Pi(\alpha) \cap \mathrm{C}_\Pi(\beta)$ acts freely upon $\mathcal{L}^2_{\alpha,\beta}X$.

Now let $\zeta \longrightarrow \mathcal{L}X$ be a $\Bbbk$-Hilbert bundle, i.e., a vector bundle locally modelled on a Hilbert space $\mathcal{H}$ over the field $\Bbbk$. If $\mathcal{H}$ is infinite dimensional then we may need to insist on further conditions, for example to ensure the existence of partitions of unity, as is done in [2]. We assume that this bundle has an associated principal bundle, say $\mathcal{Q} \longrightarrow \mathcal{L}X$, with structure group $\mathcal{G}$ acting on $\mathcal{H}$ by isometries. We will require some further conditions on ζ.

Conditions 3.1.
VB-1 There is an action of the Virasoro algebra $\mathfrak{vir}_\Bbbk$ on ζ covering the action of $\mathfrak{diff}_\Bbbk$ associated to the natural action of Diff^+ on $\mathcal{L}X$.
VB-2 On each component $\mathcal{L}_{[\alpha]}X$ of $\mathcal{L}X$ there is a bundle decomposition

$$\zeta_{|_{\mathcal{L}_{[\alpha]}X}} \cong \widehat{\bigoplus_r} \xi_{[\alpha],r},$$

where the sum is over rational numbers r and each $\xi_{[\alpha],r} \longrightarrow \mathcal{L}_{[\alpha]}X$ is the eigenbundle for L_0 for the eigenvalue $-r$, and is the finite dimensional flat bundle vector bundle associated to a finite dimensional representation $W^\xi_{\alpha,r}$ of $\mathrm{C}_\Pi(\alpha)$.
VB-3 There are lower bounds $r_{[\alpha]} \in \mathbb{Q}$ and $d_{[\alpha]} \in \mathbb{N}$ for the indexing r and the denominators appearing in Condition 3.1 VB-2.

We will say that ζ is *admissible* if all the requirements of Condition 3.1 are satisfied; if all of these hold except for Condition 3.1 VB-3, then we will say that ζ is *unboundedly admissible.*

Given an admissible bundle as above, we can define the *character* of ξ by

$$\mathrm{char}_q\, \xi = \sum_{r\in\mathbb{Z}} \chi_{\mathrm{C}_\Pi(\alpha)} W^\xi_{\alpha,r} q^r,$$

where $\chi_G W$ denotes the character of the G-representation W and $q^r = e^{2\pi i r\tau}$ is viewed as a function on $\mathfrak{h}$. This generalizes the standard notion of graded dimension for graded vector spaces [7] which we will also use.

We might specify similar conditions on a Hilbert bundle over the double loop space of X using the full diffeomorphism group of the torus $\mathbb{T}^2$. However, we will restrict attention to conditions on the action of Diff^+ and $\mathfrak{diff}_\Bbbk$ related to the fixed point spaces $\mathcal{L}^2_{r,s}X$ obtained as fixed point sets of the 'diagonal' actions of the diffeomorphism group Diff^+ obtained by acting on the circles of rational slope s/r suitably based. This is described in detail for the case of $X = \mathrm{B}G$ in §2. In particular we recall that $\mathcal{L}^2_{r,s}X \cong \mathcal{L}X$ under the pairing

$$F \longleftrightarrow (z \longmapsto F(1, z^{1/r})).$$

Thus for any Hilbert bundle $\zeta \longrightarrow \mathcal{L}^2X$, there is a family of bundles

$$\left\{\zeta_{r,s} \longrightarrow \mathcal{L}^2_{r,s}X\right\},$$

where r, s range over coprime pairs, obtained by restricting to the sets $\mathcal{L}^2_{r,s}X$ and then transporting across to $\mathcal{L}X$. Notice that each restriction of ζ to a set of the form $\mathcal{L}^2_{r,s}X$ admits an action of the diffeomorphism group Diff^+ and hence of the Lie algebras $\mathfrak{diff}_{\mathbb{R}}$ and $\mathfrak{vir}_{\mathbb{R}}$.

A Hilbert bundle $\zeta \longrightarrow \mathcal{L}^2X$ is *admissible* (respectively *unboundedly admissible*) if for each pair of coprime integers r, s, the bundle $\zeta_{r,s}$ is admissible (respectively unboundedly admissible) over $\mathcal{L}X$.

Now let M be a connected oriented smooth manifold, together with an oriented d-dimensional bundle $\xi \longrightarrow M$ equipped with a Riemannian structure, a Spin structure and a compatible connection ∇. We also assume that the characteristic class $\dfrac{1}{2}p_1(\xi)$ in $H^4(\mathrm{B}G;\mathbb{Z})$ is 0. For the significance of this in terms of Spin-like structures on the loop space see [26].

If $\Pi = \pi_1(M)$ and $\widetilde{M}$ is the universal cover of M (viewed as a free left Π-space), then for $\alpha \in \Pi$ we define a set of smooth maps by

$$\mathcal{L}_\alpha\widetilde{M} = \{f\colon \mathbb{R} \longrightarrow \widetilde{M} : f(t+1) = \alpha f(t)\ \forall t \in \mathbb{R}\}.$$

The fundamental group Π acts freely on the disjoint union $\coprod_\alpha \mathcal{L}_\alpha\widetilde{M}$, and

$$\mathcal{L}M = \left(\coprod_\alpha \mathcal{L}_\alpha\widetilde{M}\right)/\Pi = \coprod_{[\alpha]\in A} \mathcal{L}_\alpha\widetilde{M}/\,\mathrm{C}_\Pi(\alpha),$$

where A is the set of conjugacy classes of Π and the coproduct is taken over a complete set of representatives for elements of A.

Let $q\colon P \longrightarrow M$ denote the given Spin bundle of ξ, with structure group $\mathrm{Spin}(d)$. We denote the holonomy group of q with respect to the connection ∇ and a chosen base point $x_0 \in M$, by

$$\mathrm{Hol}(M) = \mathrm{Hol}_\nabla(M) \subseteq \mathrm{Spin}(d).$$

In general, $\mathrm{Hol}(M)$ is neither connected nor closed (see the recent work of Wilking [33]). The closure $H = \overline{\mathrm{Hol}(M)}$ has identity component which will be denoted H_0. There is a surjection of groups $\Pi \longrightarrow \overline{\Pi} = H/H_0$, with finite image since $\mathrm{Spin}(d)$ is compact.

Let $\widetilde{H}$ be the pullback group in the diagram

$$\begin{array}{ccc} \widetilde{H} & \longrightarrow & \Pi \\ \downarrow & & \downarrow \\ H & \longrightarrow & \overline{\Pi} \end{array}$$

whose vertical arrows have kernel H_0. If $h \in \widetilde{H}$ (respectively $h \in H$) we denote the image of h in Π (respectively $\overline{\Pi}$) by $[h]$.

For each $\alpha \in \Pi$, we can use the connection ∇ to define the holonomy of $f \in \mathcal{L}_\alpha\widetilde{M}$, $\mathrm{hol}(f) \in H$, for which

$$\mathrm{hol}(f)H_0 = \overline{\alpha} \in \overline{\Pi}.$$

Let

$$\mathcal{P}_\alpha = \{F\colon \mathbb{R} \longrightarrow P : qF \in \mathcal{L}_\alpha M \text{ and } F(t+1) = \mathrm{hol}(qF)F(t)\},$$

which is the total space of a principal fibration $\widetilde{q}\colon \mathcal{P}_\alpha \longrightarrow \mathcal{L}_\alpha M$ whose fibre over f is

$$\big\{g(t)F(t) :\ g\colon \mathbb{R} \longrightarrow H \text{ and } g(t+1) = \mathrm{hol}(f)g(t)\,\mathrm{hol}(f)^{-1}\big\}$$

for any given element $F \in \widetilde{q}^{-1}(f)$, and thus the structure group is

$$\mathcal{L}_f H = \left\{ g \colon \mathbb{R} \longrightarrow H : g(t+1) = \mathrm{hol}(f) g(t) \,\mathrm{hol}(f)^{-1} \right\}.$$

In particular, this contains the constant functions

$$H \cap \mathcal{L}_f H = \{ h \in H : \mathrm{hol}(f) h = h \,\mathrm{hol}(f) \}.$$

For $f, g \in \mathcal{L}_\alpha M$, the groups $\mathcal{L}_f H$ and $\mathcal{L}_g H$ are of course isomorphic.

Now suppose that we have a projective representation V of the affine Lie algebra $\widehat{\mathfrak{spin}(d)}$ which we assume integrates to a completion $\widehat{V}$ carrying a projective representation of the loop group $\mathcal{L}\,\mathrm{Spin}(d)$ with some level ℓ, say. We may follow Brylinski [2] in defining a bundle ξ_V^α over each space $\mathcal{L}_\alpha M$. For an open set $U \subseteq \mathcal{L}_\alpha M$, the sections on U are taken to be

$$\Gamma(\xi_V^\alpha, U) = \mathrm{Map}_{\mathcal{L}_f H \rtimes \mathrm{Diff}_{\mathbb{R}}^+}(\mathcal{P}_\alpha, V),$$

the space of $\mathcal{L}_f H \rtimes \mathrm{Diff}_{\mathbb{R}}^+$-equivariant maps. Here the group is the semi-direct product of $\mathcal{L}_f H$ with the group of orientation preserving, quasi-periodic diffeomorphisms of the line,

$$\mathrm{Diff}_{\mathbb{R}}^+ = \{ \varphi \colon \mathbb{R} \longrightarrow \mathbb{R} : \varphi \text{ invertible}, \varphi(0) = 0, \varphi(t+1) = \varphi(t) + 1 \text{ for all } t \in \mathbb{R} \}.$$

By construction, such a bundle is automatically $\mathfrak{vir}$-equivariant in the sense of Brylinski, since we can now interpret the Virasoro algebra $\mathfrak{vir}$ as densely contained in the Lie algebra of a central extension of $\mathrm{Diff}_{\mathbb{R}}^+$. The L_0-eigenspaces give rise to the natural grading on V and these are dense in $\widehat{V}$. Moreover, each grading V_r gives rise to finite dimensional bundle with structure group $\mathrm{Spin}(d)$. Such bundles can be fitted together over all the components $\mathcal{L}_\alpha M$.

A modified version $\xi_V^{H_0,\alpha}$ of this is obtained by replacing V with the invariant subspace V^{H_0}. As the action of $\mathrm{Spin}(d)$ commutes with the action of the Virasoro algebra, the grading on V restricts to one on V^{H_0}. The resulting bundle has sections

$$\Gamma(\xi_V^{H_0,\alpha}, U) = \mathrm{Map}_{\mathcal{L}_f H \rtimes \mathrm{Diff}_{\mathbb{R}}^+}(\mathcal{P}_\alpha, V^{H_0}),$$

is still $\mathfrak{vir}$-equivariant and has fibre V^{H_0}. The bundles associated to the spaces $V_r^{H_0}$ are all flat since the structure group is H/H_0. In particular, if the original bundle is flat, this gives an admissible bundle over $\mathcal{L}\mathrm{B}G$ in the above sense. Over a component $\mathcal{L}_{[\alpha]}\mathrm{B}H/H_0$ there is a flat bundle with structure group which is a projective representation of $\mathrm{C}_{H/H_0}(\alpha)$. In fact, the projective representations that occur are all associated to a fixed central extension of H/H_0 by the circle $\mathbb{T}$. As such extensions are classified by the finite group $H^2(\mathrm{B}H/H_0, \mathbb{Z})$, they all come from central finite coverings of H/H_0. When this cohomology vanishes, we have honest representations of these centralizers. This happens when H/H_0 is a non-abelian simple group.

4. Constructing Moonshine-like Virasoro equivariant vector bundles

In this section we will construct some examples of Virasoro equivariant vector bundles over loop space $\mathcal{L}\mathrm{B}G$ for a finite group G. These appear very naturally in terms of the framework discussed above. We will also make use of constructions from [7]. These bundles are also related to the two variable Moonshine-like constructions of Mason [20, 25], generalizing the single variable Thompson series of Thompson [32], Conway and Norton [4] and Norton [27].

We will make three standing assumptions on G:

Gp-1 The cohomology group $H^2(\mathrm{B}G;\mathbb{Z}/2)$ is trivial.

Gp-2 The cohomology group $H^3(\mathrm{B}G;\mathbb{Z})$ is trivial.

Gp-3 The cohomology group $H^4(\mathrm{B}G;\mathbb{Z})$ is trivial.

The first of these conditions guarantees that any orientable representation lifts to Spin, while the second and third force all projective representations of G to be genuine representations and certain bundles over $\mathcal{L}\mathrm{B}G$ to be admissible in the sense of §3. We could try to impose different conditions in the examples studied, but the above are convenient as they are satisfied by groups of particular interest, for example, the Mathieu group M_{23}. In contrast, the integral cohomology in dimension 4 of M_{24} contains both 2 and 3-torsion, although there is no 3-torsion in dimension 3, and Gp-1 is satisfied.

Let V be a finite dimensional real representation of G, and suppose that for each $\gamma \in G$, the trace of the action of γ, $\mathrm{Tr}_V(\gamma)$, is rational (hence an integer).

Consider the bundle $\zeta_V \longrightarrow \mathrm{B}G$ whose total space is $\mathrm{E}G \times_G V$. For any loop $f\colon \mathbb{T} \longrightarrow \mathrm{B}G$ or double loop $F\colon \mathbb{T}^2 \longrightarrow \mathrm{B}G$, the pullback bundle $f^*\zeta_V \longrightarrow \mathbb{T}$ (or $F^*\zeta_V \longrightarrow \mathbb{T}^2$) gives rise to a space of smooth sections $\Gamma(f^*\zeta_V, \mathbb{T})$ (or $\Gamma(F^*\zeta_V, \mathbb{T}^2)$).

Let us consider in detail such a section for a loop f whose holonomy lies in a conjugacy class $[\alpha]$ say. The total space of $f^*\zeta_V$ consist of G-equivalences classes of pairs $(\widetilde{f}, v)$ where $\widetilde{f}\colon \mathbb{R} \longrightarrow \mathrm{E}G$ covers the loop f and has $f(t+1) = \alpha' \in [\alpha]$, and $v \in V$. The action of G is given by

$$\gamma \cdot (\widetilde{f}, v) = (\gamma \cdot \widetilde{f}, \gamma v).$$

Thus a section is given by a G-equivalence class of maps defined on $\mathbb{R}$ of the form

$$t \longmapsto (\widetilde{f}(t), v(t)),$$

where $v\colon \mathbb{R} \longrightarrow V$ is smooth, and necessarily satisfies the condition

$$[\widetilde{f}(t+1), v(t+1)]_G = [\widetilde{f}(t), v(t)]_G.$$

Hence

$$v(t+1) = \alpha v(t).$$

Since α has finite order $|\alpha|$, $v(t+|\alpha|) = v(t)$, thus the function v has a Fourier expansion

$$v(t) = \sum_{k\in\mathbb{Z}} v_{k/|\alpha|} e^{2\pi i k t/|\alpha|},$$

for $v_{k/|\alpha|} \in V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\alpha v_{k/|\alpha|} = \zeta_{|\alpha|}^k v_{k/|\alpha|}$, and so

$$v_{k/|\alpha|} \in V^{\alpha}_{k/|\alpha|} = \{u \in V : \alpha u = \zeta_{|\alpha|}^k u\}.$$

For a double loop $F\colon \mathbb{T}^2 \longrightarrow \mathrm{B}G$, we have a similar analysis, where this time a section is a function

$$(x,y) \longmapsto (\widetilde{F}(x,y), v(x,y)),$$

for a smooth map $v\colon \mathbb{R}^2 \longrightarrow V$ and

$$[\widetilde{F}(x+1,y), v(x+1,y)]_G = [\widetilde{F}(x,y), v(x,y)]_G,$$
$$[\widetilde{F}(x,y+1), v(x,y+1)]_G = [\widetilde{F}(x,y), v(x,y)]_G.$$

Thus, if $\widetilde{F}(x+1,y) = \alpha\widetilde{F}(x,y)$ and $\widetilde{F}(x,y+1) = \beta\widetilde{F}(x,y)$, we have

$$v(x+1,y) = \alpha v(x,y), \quad v(x,y+1) = \beta v(x,y).$$

In terms of Fourier expansions, we have

$$v(x,y) = \sum_{k,l\in\mathbb{Z}} v_{k/|\alpha|,l/|\beta|} e^{2\pi i(kx/|\alpha|+ly/|\beta|)}$$

where $v_{k/|\alpha|,l/|\beta|} \in V \otimes_{\mathbb{R}} \mathbb{C}$ satisfy

$$\alpha v_{k/|\alpha|,l/|\beta|} = \zeta_{|\alpha|}^{k} v_{k/|\alpha|,l/|\beta|}, \tag{4.1}$$
$$\beta v_{k/|\alpha|,l/|\beta|} = \zeta_{|\beta|}^{l} v_{k/|\alpha|,l/|\beta|}. \tag{4.2}$$

Now consider the restriction of the latter section functor to the fixed point set $\mathcal{L}^2_{r,s}\mathrm{B}G$. As a double loop $F \in \mathcal{L}^2_{r,s}\mathrm{B}G$ has period $1/s$ in the first factor and $1/r$ in the second, we obtain a Fourier series of the form

$$v(x,y) = \sum_{k,l\in\mathbb{Z}} v_{ks/|\alpha|,lr/|\beta|} e^{2\pi i(ksx/|\alpha|+lry/|\beta|)}$$

for $v_{ks/|\alpha|,lr/|\beta|} \in V \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$\alpha v_{k/|\alpha|,l/|\beta|} = \zeta_{|\alpha|}^{ks} v_{ks/|\alpha|,lr/|\beta|}, \tag{4.3}$$
$$\beta v_{k/|\alpha|,l/|\beta|} = \zeta_{|\beta|}^{lr} v_{ks/|\alpha|,lr/|\beta|}. \tag{4.4}$$

If $\widetilde{F}$ is a lift of F having holonomy α along the interval of length $1/s$ in the x-direction and β along the interval of length $1/r$ in the y-direction, then the section must satisfy the conditions

$$v(x+1/s,y) = \alpha v(x,y), \tag{4.5}$$
$$v(x,y+1/r) = \beta v(x,y), \tag{4.6}$$

which are equivalent to

$$\alpha v_{k,l} = \zeta_s^{k} v_{k,l}, \tag{4.7}$$
$$\beta v_{k,l} = \zeta_r^{l} v_{k,l}. \tag{4.8}$$

Of course, by travelling along a unit interval in either the x or y-direction, we would get back to the situation of Equations (4.1), (4.2), but now the holonomy would be α^s or β^r in place of α or β.

We also identified the space $\mathcal{L}^2_{r,s}\mathrm{B}G$ with $\mathcal{L}\mathrm{B}G$ by the assignment

$$F \longleftrightarrow (z \longmapsto F(1, z^{1/r}))$$

which has inverse

$$f \longleftrightarrow ((z,w) \longmapsto f(z^{-s/r}w)).$$

Over a loop $f\colon \mathbb{T} \longrightarrow \mathrm{B}G$, the fibre of the corresponding bundle consists of all maps $\mathbb{R} \longrightarrow \mathrm{E}G \times_G V$ of the form

$$t \longmapsto \left[\widetilde{F}(0, t/r), \sum_{j,k} v_{j,k} e^{2\pi ikt/r} \right]_G ,$$

where $\widetilde{F}\colon \mathbb{R}^2 \longrightarrow \mathrm{E}G$ is a lift of F and the Fourier expansion satisfies the conditions of Equations (4.5), (4.6). Writing $v_k = \sum_j v_{j,k}$, for a lift of f with holonomy γ,

$$\gamma \cdot v_k = \zeta_r^k v_k.$$

There is an action of the Lie algebra of vector fields $\mathfrak{diff}_{\mathbb{k}}$ on the bundle of sections over $\mathbb{T}$ arising from the action of Diff^+ on $\mathcal{L}^2\mathrm{B}G$ which has fixed point set $\mathcal{L}^2_{r,s}\mathrm{B}G$. In particular, the infinitesimal rotation generator L_0 acts on a section of the form described above by the rule

$$L_0 \left(\left[\widetilde{F}(x,y), \sum_{j,k\in\mathbb{Z}} v_{j,k} e^{2\pi i(jrx+ksy)/rs} \right] \right) = \left[\widetilde{F}(x,y), \sum_{j,k\in\mathbb{Z}} \frac{-(jr+ks)}{rs} v_{j,k} e^{2\pi i(jrx+ksy)/rs} \right],$$

where

$$(x,y) \longmapsto [\widetilde{F}(x,y), v(x,y)],$$

with

$$v(x,y) = \sum_{j,k\in\mathbb{Z}} v_{j,k} e^{2\pi i(jrx+ksy)/rs}.$$

Thus the eigenspaces of L_0 correspond to the rational numbers of the form $(jr+ks)/rs$. Notice that the requirements of Condition 3.1 are satisfied with the possible exception of VB-3.

For simplicity, we now consider only the case where $s = 1$ (this case appears to cover all the interesting situations that have appeared in algebraic settings). Equation (4.7) ensures that sections over $f \in \mathcal{L}\mathrm{B}G$ with a lift $\widetilde{f}$ of holonomy $\alpha \in G$ have the form

$$\sum_{k\in\mathbb{Z}} v_{k/|\alpha|} z^{kr}$$

with

$$v_{k/|\alpha|} \in V^{\alpha}_{k/r} = \left\{ v \in V : \alpha v = \zeta_r^k v \right\}.$$

Since $V_{k/r}$ affords a representation of $\mathrm{C}_G(\alpha)$, we may follow the prescription of §3, viewing this space of sections as a bundle over $\mathcal{L}\mathrm{B}G$ which on the component of loops with holonomy conjugate to α restricts to a completed sum of flat bundles having structure group reducible to $\mathrm{C}_G(\alpha)$. Moreover, these are eigenbundles for L_0.

We will find it useful to assign the term appearing as coefficient of $z^{k/r}$ a rational grading of k/r. We can keep track of this by a writing it as the coefficient of a power of an indeterminate q, thought of as having grading 1. Thus our space of sections restricted to the component $\mathcal{L}_{[\alpha]}\mathrm{B}G$ is

$$V_q^{[\alpha]} = \widehat{\bigoplus}_{k\in\mathbb{Z}} V_{k/r}^{[\alpha]} q^{k/r}$$

and by Equation (2.1), this is naturally thought of as lying in $\mathrm{KU}^0(\mathrm{B}\,\mathrm{C}_G(\alpha))[[q,q^{-1}]]$.

We can also form the Fock space

$$\bigotimes_{\substack{0\leqslant k\leqslant r-1\\ 0\leqslant l\in\mathbb{Z}}} S\left(V_{k/r}^{[\alpha]} q^{(lr+k)/r}\right)$$

which should be viewed as the restriction to the component $\mathcal{L}_{[\alpha]}\mathrm{B}G$ of a Fock bundle over $\mathcal{L}\mathrm{B}G$. This Fock bundle is related to constructions of Mason [20] and Frenkel, Lepowsky & Meurman [7], see also Kač & Raina [13] for a detailed discussion of bosonic Fock spaces with Virasoro action.

5. Moonshine-like Virasoro equivariant vector bundles associated to a lattice

In this section we construct more Moonshine-like bundles, this time basing our constructions on a lattice L on which the finite group G acts. We follow in some detail the notation, terminology and results of the book of Frenkel, Lepowsky & Meurman [7]. We are also motivated by work of Mason [20, 25].

Let L be a lattice (i.e., a free abelian group with $\operatorname{rank} L$ finite) equipped with a symmetric positive definite integer valued inner product $\langle\ ,\ \rangle : L\times L \longrightarrow \mathbb{Z}$ which is *even* in the sense that $\langle \ell,\ell\rangle \in 2\mathbb{Z}$ for $\ell\in L$. Suppose that G acts on L by orientation preserving linear isometries, hence for $\ell_1,\ell_2\in L$ and $\gamma\in G$,

$$\langle \gamma\ell_1, \gamma\ell_2\rangle = \langle \ell_1,\ell_2\rangle\,.$$

We set $L_{2n} = \{\ell\in L : \langle\ell,\ell\rangle = 2n$ and give this set grading n; thus we have $L = \coprod_{n\geqslant 0} L_{2n}$ as sets with G action.

Following [7], we consider a central extension

$$1 \longrightarrow \langle\kappa\rangle \longrightarrow \widehat{L} \xrightarrow{\pi} L \longrightarrow 1$$

where $\langle\kappa\rangle \cong \mathbb{Z}/2$ and the associated 2-cocycle $\varepsilon_0\colon L\times L\longrightarrow\langle\kappa\rangle$ satisfies

$$\varepsilon_0(\ell_1,\ell_2) - \varepsilon_0(\ell_2,\ell_1) = c_0(\ell_1,\ell_2)$$

where c_0 is the commutator map

$$c_0(\ell_1,\ell_2) = \langle\ell_1,\ell_2\rangle \pmod 2.$$

It will be useful to have a fixed choice of section for π, say $s\colon L\longrightarrow\widehat{L}$ which we assume satisfies $s(0)=e$ (the identity element in $\widehat{L}$).

Now we can define a grading on $\widehat{L}$ by setting

$$\widehat{L}_{2n} = \{\widehat{\ell}\in\widehat{L} : \pi(\widehat{\ell})\in L_{2n}\}.$$

Thus $\left|\widehat{L}_{2n}\right| = 2\,|L_{2n}|$. Now for each $n\geqslant 0$ we can form the free $\Bbbk$-module on the elements of L_{2n} (respectively $\widehat{L}_{2n}$) namely $\Bbbk[L_{2n}]$ (respectively $\Bbbk[\widehat{L}_{2n}]$); combining these, we obtain graded group rings $\Bbbk[L]_*$ (resp. $\Bbbk[\widehat{L}]_*$). Now let N_* be a graded $\Bbbk[\langle\kappa\rangle]$-module; then we can form the left

$\Bbbk[\widehat{L}]_*$-module $\Bbbk[\widehat{L}]_* \otimes_{\Bbbk[\langle\kappa\rangle]} N_*$, given the usual tensor product grading. For example, we could take $\Bbbk$ itself with the action $\kappa \cdot x = -x$ (given grading 0); we will denote this $\Bbbk$-module by $\Bbbk^-$ and also set

$$\Bbbk\{L\} = \Bbbk[\widehat{L}]_* \underset{\Bbbk[\langle\kappa\rangle]}{\otimes} \Bbbk^-.$$

There is an isomorphism of $\Bbbk$-modules

$$\Bbbk[L]_* \cong \Bbbk\{L\}; \quad \ell \longleftrightarrow s(\ell) \otimes 1 = e_\ell,$$

where we use a chosen section s as above. Notice that we have the relation $\kappa e_\ell = -e_\ell$ for any $\ell \in L$. Now we obtain action of G on $\Bbbk\{L\}$ from that on L by using this isomorphism.

Now let $\mathfrak{h} = \Bbbk \otimes_{\mathbb{Z}} L$. Then as in [7], we define $S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-)$ to be the symmetric algebra on $\widehat{\mathfrak{h}}$, where we give a homogeneous symmetric tensor of degree n grading n (this is not quite the definition of [7] but amounts to the same thing). Of course, both of these are $\Bbbk[G]$-modules in a natural way. Finally, we can form the graded $\Bbbk[G]$-module

$$V_{L*} = \Bbbk\{L\}_* \underset{\Bbbk}{\otimes} S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-).$$

Notice that as $\Bbbk[G]$-modules, there is an isomorphism

$$V_{L*} \cong \Bbbk[L]_* \underset{\Bbbk}{\otimes} S(\widehat{\mathfrak{h}}_{\mathbb{Z}}^-),$$

and the reader may well ask why we did not simply make this the definition. The point is that there is further structure associated to V_{L*} as described in [7], and in general this is not compatible with action of G. However, there may be conditions on the action of G which force such compatibility, and we wish to stress this possibility.

We recall from [7] the following facts about the module V_{L*}.

Theorem 5.1. *The module V_{L*} possesses an action of the Virasoro algebra $\mathfrak{vir}_\Bbbk$.*

We will refrain from giving precise formulæ here but refer the reader to [7] for details.

Next we will describe some Hilbert bundles having as their fibres such modules over the Virasoro algebra. We will give a general construction related to work of G. Mason [20]. Let X be a G-space and let $\boldsymbol{V} \longrightarrow X$ be a G-equivariant vector bundle. Thus for each $x \in X$ there is a vector space V_x and for each $\gamma \in G$ a linear map

$$V_x \xrightarrow{\gamma\cdot} V_{\gamma \cdot x}.$$

Notice that V_x provides a linear representation of $\mathrm{Stab}_G(x)$, the stabilizer of x. If we now form the vector bundle

$$\mathrm{E}G \underset{G}{\times} \boldsymbol{V} \longrightarrow \mathrm{E}G \underset{G}{\times} X,$$

we can pull back over a loop $f\colon \mathbb{T} \longrightarrow \mathrm{B}G$ and take sections. Now such a section must be a G-equivalence class of maps of the form

$$t \longmapsto (\widetilde{f}(t), v(t))$$

for $\widetilde{f}\colon \mathbb{R} \longrightarrow \mathrm{E}G$ a lift of f, a map $v\colon \mathbb{R} \longrightarrow \boldsymbol{V}$, and which satisfy the conditions

$$v(t+1) = \alpha \cdot v(t), \quad f(t+1) = \alpha \cdot f(t).$$

Notice that if we choose a lift $\widetilde{f}$ then the section takes values in the restriction of $\boldsymbol{V}$ to the fixed point set X^α. In practise we are only interested in one case which we will describe below.

As above let L be a finite rank even lattice with G acting by orientation preserving isometries. Consider L as a G-set and form the covering space $\mathrm{E}G \times_G L \longrightarrow \mathrm{B}G$. Pulling back over a loop f and taking sections we obtain G-equivalence classes of maps $\mathbb{R} \longrightarrow \mathrm{E}G \times L$ of the form

$$t \longmapsto (\widetilde{f}(t), \ell),$$

where $\widetilde{f}(t+1) = \alpha \widetilde{f}(t)$ and $\alpha\ell = \ell$.

Over the component consisting of loops whose holonomy lies in a given conjugacy class $[\alpha]$, we obtain a covering with fibre L^α and structure group $\mathrm{C}_G(\alpha)$. As each fibre is still an even lattice we can form the fibrewise construction of the realification $\mathfrak{h} = \mathbb{R} \otimes L^\alpha$ and so obtain a vector bundle over this component with fibre $\mathfrak{h}$. We may also form the Fock bundles with fibres $S(\widehat{\mathfrak{h}}^-_{\mathbb{Z}})$ and the tensor product with $\mathbb{C}\{L\}$, $\mathbb{C}\{L\} \otimes_{\mathbb{R}} S(\widehat{\mathfrak{h}}^-_{\mathbb{Z}})$. On each fibre this bundle agrees with the module $V_{L^\alpha *}$ constructed earlier, this time with $\mathrm{C}_G(\alpha)$ in place of G. We have thus constructed a Virasoro equivariant bundle over $\mathcal{L}\mathrm{B}G$ with character

$$q^{\operatorname{rank} L^\alpha/24} \Theta_{L^\alpha}/\eta_\alpha(\mathbb{R} \otimes L^\alpha).$$

The normalization factor of $q^{\operatorname{rank} L/24}$ appears here since we have graded our bundle so that its non-trivial degrees begin at 0, and the q-series $\Theta_{L^\alpha}/\eta_\alpha(\mathbb{R} \otimes L^\alpha)$ of [20] possesses the modularity property of Condition Dev-1.

6. Recognizing bundles in elliptic cohomology

In this section we explain how our constructions of bundles over the loop space $\mathcal{L}\mathrm{B}G$ give rise to elements of $E\ell\ell^*(\mathrm{B}G)$ using the work of Devoto outlined in §1. The basic idea is to note that our bundles have characters which in effect lie in the equivariant elliptic cohomology $\mathcal{E}\ell\ell^*_G$ and thus in its completion $\widehat{\mathcal{E}\ell\ell^*_G}$ which by Equation 1.1 agrees with $E\ell\ell^*(\mathrm{B}G)$.

Given an admissible bundle $\xi \longrightarrow \mathcal{L}\mathrm{B}G$, i.e., one satisfying the restrictions of Condition 3.1, we may consider its character

$$\operatorname{char}_q \xi = \sum_{r \in \mathbb{Z}} \chi_{\mathrm{C}_G(\alpha)} W^\xi_{[\alpha],r}.$$

In general we will need to multiply this by a suitable power of q to obtain a q-expansion with the modularity requirement of Condition Dev-1, even when the remaining Conditions Dev-2, Dev-3 and Dev-4 all hold.

Let us consider the bundle constructed in §5. This has character

$$q^{\operatorname{rank} L^\alpha/24} \theta_{L^\alpha}/\eta_\alpha(\mathbb{R} \otimes L^\alpha)$$

and the series $\Theta_{L^\alpha}/\eta_\alpha(\mathbb{R} \otimes L^\alpha)$ is a modular form for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Here we have the representation modules

$$\Theta_{L^\alpha} = \sum_{0 \leqslant n} \mathbb{C}[L_{2n}] q^n,$$

and

$$\Omega(\mathbb{R}\otimes L^\alpha) = q^{\operatorname{rank} L^\alpha/24}\sum_{0\leqslant n}\Lambda^n\mathfrak{h}^\alpha q^n,$$

where $\Lambda^n V$ denotes the nth exterior power of the representation V. When evaluated on the identity element, such a character yields the graded dimension of the underlying vector space, given by a series of the form

$$\left(\theta_{L^\alpha}/\eta_\alpha(\mathbb{R}\otimes L^\alpha)\right)(1) = \left(\sum_{0\leqslant n}|L_{2n}|q^n\right)\left(q^{1/24}\prod_{1\leqslant n}(1-q^n)\right)^{-\operatorname{rank} L^\alpha}$$

which was shown in [20] to have the required modularity properties. The remaining Conditions Dev-2, Dev-3 and Dev-4 also hold, and thus viewed as a function on $\mathrm{T}G\times\mathfrak{h}$, the assignment

$$((\alpha,\beta),\tau)\longmapsto\left(\Theta_{L^\alpha}/\eta_\alpha(\mathbb{R}\otimes L^\alpha)\right)(\beta)$$

(with $q^r = e^{2\pi i r\tau}$) corresponds to an element in $\mathcal{E}\ell\ell_G$, hence on passing to the completion we obtain an element of $E\ell\ell^*(\mathrm{B}G)$.

By way of illustration, recall that the $+1$-eigenspace of a natural involution on the quotient module Θ/Ω contributes one summand to the original monstrous moonshine module while the other depends more on the 2-local structure of the Monster $\mathbb{M}$. The restriction of Θ/Ω to the subgroup M_{24} was introduced in the Introduction. Note that in this special case the lattice L has rank 24.

In the terminology of Devoto, we have

$$\omega((1,g),\tau) = \eta_g(\tau)$$

for elements g belonging to the centralizer of the identity, that is for an arbitrary conjugacy class $[g]$. The generalization to an arbitrary commuting pair (h,g) is given by a rather more complicated formula, see [20], (3.7) and (4.20). Inspection of the subgroup structure of M_{24} shows that up to conjugacy such commuting pairs generate subgroups of the following forms:

$$C_2\times C_2,\ C_2\times C_4,\ C_4\times C_4,\ C_2\times C_8,\ C_2\times C_6,\ C_2\times C_{10},\ C_3\times C_3.$$

It is not hard to write down the modular forms associated to each of these. For example, associated to $C_3\times C_3$ are the conjugacy types

$$C_3\times C_3\,A: \qquad \omega(3A,3A,\tau) = \eta(3\tau)^8,$$
$$C_3\times C_3\,B: \qquad \omega(3A,3B,\tau) = \eta(3\tau)^2\eta(9\tau)^2,$$

with $(\text{weight},\text{level})$ being $(4,9)$ and $(2,27)$ respectively.

As we have already pointed out, the quotient Θ/Ω is needed in order to obtain a homogeneous element associated with a class in $E\ell\ell^*(\mathrm{B}G)$, and this needs further modification if the ‘genus zero’ condition for moonshine is to be satisfied. Since this last step only affects elements of even order, it is beyond the scope of this paper.

Similar considerations using methods of [20] show that the bundles of §4 also correspond to elliptic cohomology classes. Indeed the whole thrust of this paper has been to show that the two variable characters introduced by Mason and Norton fit into a theory of bundles over a loop space $\mathcal{L}\mathrm{B}G$, and that elliptic cohomology provides the best framework in which to discuss them. The next task will be to describe these moonshine classes in terms of the algebraic structure of $E\ell\ell^*(\mathrm{B}G)$. For the group M_{24} this is done in chapter 6 of [31], while partial information is available for the groups Co_0 and $\mathbb{M}$. It may well be that such calculations will help explain why moonshine is only associated with certain groups and off which properties of elliptic cohomology it is reflected.

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