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Invariants for finite dimensional groups in vertex operator algebras associated to basic representations of affine algebras

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Abstract. We investigate the invariant vertex operator subalgebras of the vertex operator algebras associated with the $\mathbb{A}, \mathbb{D}, \mathbb{E}$ series of simply laced root lattices and the related affine algebras. We also discuss certain generalized Casimir operators which may be related to the action of a central extension of the Lie algebra of differential operators on the circle introduced by Kač, Radul et al. One motivation for this work lies in work on elliptic genera by the second author, while work of Dong and Mason provides a more algebraic setting for such calculations.

Introduction.

Vertex operator algebras have been the focus of considerable attention from various viewpoints and for numerous reasons relating to their appearance in diverse parts of mathematics. Even the most familiar examples may still generate interesting questions. One developing application of vertex operator algebras is to the theory of elliptic genera as described by the second author [**9,10**], where attention is focused on certain vertex operator (super)-algebras on which compact Lie groups act by automorphisms, the objects of geometric interest being the invariant vertex operator subalgebras. This suggests the problem of computing such invariant vertex operator subalgebras and understanding the decomposition of the original vertex operator algebra as a module over this.

Our original goal in this work was to carry out such calculations for the familiar cases of vertex operator algebras associated with the affinizations of finite dimensional simply laced Lie algebras. However, after circulating a draft of some of our work we became aware of work of Kač and Radul [**7**] in which some of our results had already appeared; also in [**2**], Dong, Li and Mason proved a conjecture that we had made, thus confirming that decompositions over the invariant vertex operator subalgebras have interesting algebraic properties in very general circumstances.

In this paper we give calculational details for the vertex operator algebras associated with the $\mathbb{A}, \mathbb{D}, \mathbb{E}$ series of simply laced root lattices and the related affine

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algebras. In future work, we hope to relate the 'generalized Casimir operators' to the (projective) action of the differential operators on the circle discussed in [**5,7**].

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§1 Some general recollections.

Let $\mathbb{L}$ denote a finite rank lattice with symmetric, even integral valued positive definite inner product $(\ ,\ )\colon \mathbb{L}\times\mathbb{L}\to\mathbb{Z}$. According to [**4**], there is a vertex operator algebra $\mathrm{V}(\mathbb{L})=(\mathrm{V}(\mathbb{L})_*,\mathrm{Y},\mathbf{1},\omega)$ which provides a model for the *basic representation* of a certain untwisted affine Kač–Moody algebra $\widehat{\mathfrak{g}}$ which is the affinization of a finite dimensional real Lie algebra $\mathfrak{g}$ whose roots may be viewed as elements of $\mathbb{L}$. We record some facts about this vertex operator algebra, most of which can be found in [**4**] (see in particular 8.7.13, 8.7.20 and corollary 8.6.3). Recall that as part of the structure of $\mathrm{V}(\mathbb{L})$ we have an action of the Virasoro algebra $\mathfrak{vir}$, with generators L_n $(n\in\mathbb{Z})$ together with the central generator C (acting by scalar multiplication). The following omnibus result contains useful basic properties of the action of $\mathfrak{g}$ on $\mathrm{V}(\mathbb{L})$.

Proposition 1.1.

1. *The action of $\mathfrak{g}\subseteq\widehat{\mathfrak{g}}$ on $V(\mathbb{L})$ commutes with the action of $\mathfrak{vir}$, i.e.,*
$$[L_n,x]=0\quad(n\in\mathbb{Z},\ x\in\mathfrak{g}).$$
In particular, we have
$$x\omega=xL_{-2}\mathbf{1}=L_{-2}x\mathbf{1}=0\quad(x\in\mathfrak{g}).$$
2. *The action of $\mathfrak{g}$ is compatible with the natural Hermitian structure on $\mathrm{V}(\mathbb{L})$, in the sense that its elements act as Hermitian operators.*
3. *The action of $\mathfrak{g}$ on $\mathrm{V}(\mathbb{L})$ integrates to an action of a compact connected Lie group G with Lie algebra $\mathfrak{g}$.*
4. *The action of each $g\in G$ on $\mathrm{V}(\mathbb{L})$ commutes with the vertex operator Y in the sense that*
$$\mathrm{Y}(gv,z)=g\,\mathrm{Y}(v,z)g^{-1}\quad g\in G,\ v\in V(\mathbb{L})$$
and g fixes the vacuum and conformal vectors. Hence, G acts by automorphisms of the vertex operator algebra $\mathrm{V}(\mathbb{L})$.

Corollary 1.2. *The invariant subspace $\mathrm{V}(\mathbb{L})^G$ supports the structure of a vertex operator subalgebra of $\mathrm{V}(\mathbb{L})$,*
$$\mathrm{V}(\mathbb{L})^G=(\mathrm{V}(\mathbb{L})^G_*,\mathrm{Y},\mathbf{1},\omega).$$

§2 The case of $\mathbb{A}_1$.

When $\mathbb{L}$ is the root lattice of $\mathrm{SU}(2)$,
$$\mathbb{A}_1\cong\{n\sqrt{2}:n\in\mathbb{Z}\},$$
the basic representation of $\widehat{\mathfrak{su}(2)}$ gives a particularly important example of a vertex operator algebra, $\mathrm{V}=\mathrm{V}(\mathbb{A}_1)$. This time, the Lie group acting is of course $\mathrm{SU}(2)$, which acts unitarily on V. We will describe the invariant vertex operator algebra $\mathrm{V}^{\mathrm{SU}(2)}$.

There are several ways to determine the action of SU(2) on V. We proceed as follows. Let h, e, f be generators for the complexification $\mathfrak{sl}(2)$ of $\mathfrak{su}(2)$, satisfying the usual relations

$$[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h.$$

The Cartan subalgebra of the affinization $\widehat{\mathfrak{su}(2)}$ is the 3-dimensional abelian subalgebra $\hat{\mathfrak{h}} = \mathbb{C}\{h, c, d\}$. We could determine the character of the basic representation on $\hat{\mathfrak{h}}$ using the *Kač character formula,* however, it is perhaps more instructive to use a direct approach. This involves explicitly describing the action of h on V; from [**4**] we have

$$\mathrm{V} = \mathbb{C}\{\mathbb{A}_1\} \otimes \mathrm{S}(\mathfrak{h}^-)$$

where

$$\mathfrak{h} = \mathbb{R}\{h\}, \quad \mathfrak{h}^- = \mathbb{C} \underset{\mathbb{R}}{\otimes} \mathfrak{h}[t^{-1}]$$

and $\mathrm{S}(W)$ denotes the symmetric algebra on the $\mathbb{C}$-vector space W. The action of h is given by

$$h \cdot (e^{k\alpha} \otimes v) = k\alpha(h) e^{k\alpha} \otimes v = 2k e^{k\alpha} \otimes v$$

where α denotes the generator of $\mathbb{A}_1$ such that $\alpha(h) = 2$. If $T \subseteq \mathrm{SU}(2)$ denotes the torus generated by exponentiating $i\mathfrak{h} = \mathbb{R}ih$, we have for $\exp(2\pi i t h) \in T$,

$$\exp(2\pi i t h) \cdot (e^{k\alpha} \otimes v) = e^{2\pi i t k\alpha(h)} e^{k\alpha} \otimes v = e^{4\pi i t k} e^{k\alpha} \otimes v.$$

We can ignore the generator c which acts by multiplication by 1. The action of d is

$$d \cdot (e^{k\alpha} \otimes v) = (k^2 + \deg v)(e^{k\alpha} \otimes v),$$

where the later term refers to the natural grading on $\mathrm{S}(\mathfrak{h}^-)$ induced from decreeing the grading of $\mathfrak{h}t^{-k}$ to be k. On the subalgebra $\mathbb{C}\{h, d\}$, this gives

$$\dim_q \mathrm{V} = \frac{\sum_{m \in \mathbb{Z}} q^{m^2} z^{2m}}{\varphi(q)},$$

where $\varphi(q) = \prod_{n \geqslant 1}(1 - q^n)$, q^k indexes the occurrences of the character $(h, d) \longmapsto (0, k)$ and z^ℓ indexes the number of occurrences of $(h, d) \longmapsto (\ell, 0)$.

Now the irreducible finite dimensional representations of $\mathfrak{su}(2)$ are the W_m ($m \geqslant 0$), of dimension $m + 1$. These have characters

$$\operatorname{char} W_m = z^m + z^{m-2} + \cdots + z^{2-m} + z^{-m}.$$

For $m \geqslant 1$, we have the formula

$$\operatorname{char} W_{2m} = \operatorname{char} W_{2m-2} + z^{2m} + z^{-2m},$$

and hence,

$$\operatorname{char}_q \mathrm{V} = \left(\sum_{m \geqslant 0} q^{m^2} (\operatorname{char} W_{2m} - \operatorname{char} W_{2m-2}) \right) \varphi(q)^{-1}.$$

Rearranging this, we have

$$\operatorname{char}_q \mathrm{V} = \left(\sum_{m \geqslant 0} \operatorname{char} W_{2m} (1 - q^{2m+1}) q^{m^2} \right) \varphi(q)^{-1}.$$

Thus, for each irreducible W_{2m} there is a subspace $\mathrm{V}^{[2m]}$ with graded dimension

$$\dim_q \mathrm{V}^{[2m]} = \frac{(2m+1)(1-q^{2m+1})q^{m^2}}{\varphi(q)}.$$

Since the actions of $\mathfrak{vir}$ and SU(2) commute, each $\mathrm{V}^{[2m]}$ is a highest weight module for $\mathfrak{vir}$. In fact, writing $v^{[2m]}$ for an element of $\mathrm{V}^{[2m]}$ satisfying the two conditions

$$\begin{aligned} ev^{[2m]} &= 0, \\ L_k v^{[2m]} &= 0 \quad k \geqslant 1, \end{aligned}$$

we can consider the $\mathfrak{vir}$-submodule V^{2m} generated by $v^{[2m]}$. Then by [**8**] (end of Chap. 6), V^{2m} is irreducible and

$$\begin{aligned} \mathrm{V}^{[2m]} &\cong W_{2m} \otimes \mathrm{V}^{2m}, \\ \dim_q \mathrm{V}^{2m} &= \frac{q^{m^2}(1-q^{2m+1})}{\varphi(q)}. \end{aligned}$$

Notice that the invariant vertex operator algebra $\mathrm{V}^{\mathrm{SU}(2)} = \mathrm{V}^{[0]} = \mathrm{V}^0$ has graded dimension

$$\dim_q \mathrm{V}^{\mathrm{SU}(2)} = \frac{(1-q)}{\varphi(q)} \equiv 1 \pmod{q^2},$$

hence $\dim \mathrm{V}_1^{\mathrm{SU}(2)} = 0$. Moreover, each V^{2m} is an irreducible module over $\mathrm{V}^{\mathrm{SU}(2)}$ (since it is irreducible over $\mathfrak{vir}$).

To construct highest weight vectors for SU(2) which are also highest weight vectors with respect to the Virasoro algebra $\mathfrak{vir}$, hence generators over $\mathrm{V}^{\mathrm{SU}(2)}$, we proceed as follows. It is easily checked that for each $k \geqslant 0$, the element $e^{k\alpha} = e^{k\alpha} \otimes 1$ satisfies

$$e \cdot e^{k\alpha} = 0 \quad \text{and} \quad h \cdot e^{k\alpha} = 2k e^{k\alpha}.$$

Hence, $e^{k\alpha}$ is a highest weight vector for a copy of the irreducible W_{2k}. Since this irreducible first occurs in weight k^2 with multiplicity 1, $e^{k\alpha}$ must be a highest weight vector for $\mathfrak{vir}$. But then all of the elements

$$f^r \cdot e^{k\alpha} \quad (r = 0, \ldots, k)$$

are weight vectors for SU(2) and highest weight vectors for $\mathfrak{vir}$. The totality of vectors obtained this way accounts for the 'singular vectors' with respect to $\mathfrak{vir}$ described in [**8**], lemma 6.1, at least for even values of m. Later, we will show that this procedure also works for other cases.

§3 The case of SU($\ell + 1$).

In this section we will generalize the results for SU(2) to SU($\ell+1$), using the following result found in [**6**].

PROPOSITION 3.1. (see [**6**], exercise 12.17). *Let $\mathbb{L}$ be a root lattice of type $\mathbb{A}_\ell$ ($\ell \geqslant 1$), $\mathbb{D}_\ell$ ($\ell \geqslant 4$) or $\mathbb{E}_\ell$ ($\ell = 6, 7, 8$). Then if $\mathfrak{g}$ is the associated simple Lie algebra, as a $\mathfrak{g}$-module, the graded vector space $\mathrm{V}(\mathbb{L})$ has occurrences of the representation $W(\lambda)$ for $\lambda \in \mathbb{L}_+$ with multiplicity given by the q-series*

$$\sum_{m \geqslant 0} (\mathrm{mult}_\lambda \mathrm{V}(\mathbb{L})_m) q^m = \varphi(q)^{-\ell} q^{(\lambda|\lambda)/2} \prod_{\alpha \in \mathring{\Delta}_+} \left(1 - q^{(\lambda+\bar{\rho}|\alpha)}\right).$$

Here $\mathbb{L}_+$ is the set of roots which are also dominant weights for $\mathfrak{g}$ and $\bar\rho$ is half the sum of all the positive roots, also characterised by the requirement that $(\bar\rho \mid \alpha_i) = 1$ for each positive simple root α_i.

In this section we use the case of $\mathbb{A}_\ell^{(1)}$. The result we require says the following for the basic representation of $\widehat{\mathfrak{su}(\ell+1)}$, whose highest weight is Λ_0 satisfying

$$\Lambda_0(h) = \Lambda_0(d) = 0 \quad \text{and} \quad \Lambda_0(c) = 1.$$

Then for any weight λ viewed as a character of the standard Cartan subalgebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \underset{\mathbb{R}}{\otimes} \mathbb{C} \oplus \mathbb{C}c \oplus \mathbb{C}d \cong \mathbb{C}^{\ell+2},$$

if $\lambda(c) = \lambda(d) = 0$ and $\lambda(h) \in \mathbb{R}$, there is a unique irreducible module for $\mathfrak{su}(\ell+1)$ with highest weight λ. Provided $0 \leqslant \lambda(\mathbb{A}_\ell) \subseteq \mathbb{Z}$, this integrates to a unitary representation of $\mathrm{SU}(\ell+1)$; we label this representation $W(\lambda)$. It is easy to see that for the basic representation $W(\Lambda_0)$, constructed as a vertex operator algebra in [**4**], the only such weights that can occur in $W(\Lambda_0)$ viewed as an $\mathfrak{su}(\ell+1)$-module are also in the root lattice $\mathbb{A}_\ell$. We denote the set of all *dominant roots* by $\mathbb{A}_\ell^+$.

In the case under consideration, recall that $\mathbb{A}_\ell$ is the free $\mathbb{Z}$-module with basis $\{\alpha_1, \ldots, \alpha_\ell\}$ where the α_i are the positive simple roots. Thus we are interested in evaluating the series in Proposition 3.1 for weights of the form

$$\lambda = \sum_{1 \leqslant i \leqslant n} r_i \alpha_i \quad (r_i \geqslant 0).$$

We also have

$$\begin{aligned}
\mathring{\Delta}_+ &= \{\alpha_r + \alpha_{r+1} + \cdots + \alpha_{r+s-1} : 1 \leqslant r \leqslant r+s-1 \leqslant \ell\}, \\
\bar\rho &= \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha, \\
(\bar\rho \mid \alpha_i) &= 1,
\end{aligned}$$

and

$$(\alpha_i \mid \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

To determine the invariants under the action of $\mathrm{SU}(\ell+1)$, we take $\lambda = 0$ and obtain

$$\begin{aligned}
\dim_q \mathrm{V}(\mathbb{A}_\ell)^{\mathrm{SU}(\ell+1)} &= \sum_{m \geqslant 0} (\operatorname{mult}_0 \mathrm{V}(\mathbb{A}_\ell)_m) q^m \\
&= \varphi(q)^{-\ell} \prod_{\alpha \in \mathring{\Delta}_+} \left(1 - q^{(\bar\rho \mid \alpha)}\right) \\
&= \varphi(q)^{-\ell} \prod_{\mathbf{s} \cdot \boldsymbol{\alpha} \in \mathring{\Delta}_+} \left(1 - q^{\operatorname{len} \mathbf{s}}\right) \\
&= \varphi(q)^{-\ell} \prod_{1 \leqslant s \leqslant \ell} (1 - q^s)^{\ell - s + 1},
\end{aligned}$$

where we use the notation

$$\begin{aligned}
\mathbf{s} &= (s_1, \dots, s_\ell),\\
\operatorname{len}\mathbf{s} &= \sum_{1\leqslant i\leqslant \ell} s_i,\\
\boldsymbol{\alpha} &= (\alpha_1, \dots, \alpha_\ell),\\
\mathbf{s}\cdot\boldsymbol{\alpha} &= \sum_{1\leqslant i\leqslant \ell} s_i\alpha_i.
\end{aligned}$$

In the case where $\ell = 1$, we recover our earlier result. Notice that

$$\dim_q \mathrm{V}(\mathbb{A}_\ell)^{\mathrm{SU}(\ell+1)} \equiv 1 + q^2 \pmod{q^3},$$

hence $\dim \mathrm{V}(\mathbb{A}_\ell)_1^{\mathrm{SU}(\ell+1)} = 0$ and $\dim \mathrm{V}(\mathbb{A}_\ell)_2^{\mathrm{SU}(\ell+1)} = 1$, implying that $\mathrm{V}(\mathbb{A}_\ell)_2^{\mathrm{SU}(\ell+1)}$ is spanned by the conformal vector ω.

For a general positive weight λ, we have

$$\sum_{m\geqslant 0} (\operatorname{mult}_\lambda \mathrm{V}(\mathbb{A}_\ell)_m) q^m = \varphi(q)^{-\ell} q^{(\lambda|\lambda)/2} \prod_{\alpha\in\mathring{\Delta}_+} \left(1 - q^{(\lambda+\bar{\rho}|\alpha)}\right).$$

For a general positive weight λ we have the following:

THEOREM 3.2. *For the positive weight $\lambda = \sum_{1\leqslant i\leqslant\ell} r_i\alpha_i$, the multiplicity of the $W(\lambda)$ in $\mathrm{V}(\mathbb{A}_\ell)$ is given by*

$$\sum_{m\geqslant 0} (\operatorname{mult}_\lambda \mathrm{V}(\mathbb{A}_\ell)_m) q^m = \varphi(q)^{-\ell} q^{(\lambda|\lambda)/2} \prod_{\substack{1\leqslant s\leqslant \ell\\ 1\leqslant t\leqslant \ell-s+1}}\prod \left(1 - q^{t+r_s+r_{s+t-1}-r_{s-1}-r_{s+t}}\right),$$

where

$$\frac{1}{2}(\lambda \mid \lambda) = \sum_{1\leqslant i\leqslant\ell} r_i^2 - \sum_{1\leqslant i\leqslant \ell-1} r_i r_{i+1}.$$

PROOF. For each $\alpha \in \mathring{\Delta}_+$ with

$$\alpha = \alpha_s + \alpha_{s+1} \cdots + \alpha_{s+t-1} \quad (1 \leqslant s \leqslant \ell,\ 1 \leqslant t \leqslant \ell - s + 1),$$

we have

$$\begin{aligned}
(\lambda + \bar{\rho} \mid \alpha) &= (\lambda \mid \alpha) + t\\
&= t + 2(r_s + \cdots + r_{s+t-1}) - (r_{s-1} + \cdots + r_{s+t-2}) - (r_{s+1} + \cdots + r_{s+t})\\
&= t + r_s + r_{s+t-1} - r_{s-1} - r_{s+t},
\end{aligned}$$

where we set $r_0 = 0 = r_{\ell+1}$. The calculation of $(\lambda \mid \lambda)/2$ is straightforward. □

We may write $\mathrm{V}(\mathbb{A}_\ell)_*^{[\lambda]}$ for the summand of $\mathrm{V}(\mathbb{A}_\ell)$ corresponding to the irreducible $W(\lambda)$; this decomposes as

$$\mathrm{V}(\mathbb{A}_\ell)_*^{[\lambda]} = W(\lambda) \underset{\mathbb{C}}{\otimes} \mathrm{V}(\mathbb{A}_\ell)_*^\lambda$$

where $\mathrm{V}(\mathbb{A}_\ell)_*^\lambda$ denotes the highest weight $\mathrm{V}(\mathbb{A}_\ell)^{\mathrm{SU}(\ell+1)}$-submodule generated by a highest weight vector for $\mathrm{SU}(\ell+1)$.

Now we exhibit a highest weight vector for $\mathfrak{su}(\ell+1)$ in the lowest occurrence of the irreducible $W(\lambda)$ with $\lambda \in \mathbb{A}_\ell^+$, i.e., in weight $(\lambda \mid \lambda)/2$. Consider $e^\lambda = e^\lambda \otimes 1$ which has this weight; it is certainly a weight vector since for $h \in \mathfrak{h}$,

$$h \cdot e^\lambda = \lambda(h) e^\lambda.$$

Now for any $\alpha \in \overset{\circ}{\Delta}_+$, $e_\alpha = e^\alpha(-1)$ and

$$\mathrm{Y}(e^\alpha, z) = E^-(-\alpha, z)E^+(-\alpha, z)z^\alpha e^\alpha = \sum_{r\in\mathbb{Z}} e^\alpha(r) z^{-r-1},$$

which implies that

$$\mathrm{Y}(e^\alpha, z)e^\lambda = E^-(-\alpha, z) z^{(\alpha|\lambda)} e^{\lambda+\alpha},$$

which has no terms in negative powers of z. Hence $e_\alpha \cdot e^\lambda = 0$ which means that e^λ is a highest weight vector. From e^λ we can produce a basis of weight vectors for a copy of the irreducible $W(\lambda)$, each of which generates a module over $\mathrm{V}(\mathbb{A}_\ell)^{\mathrm{SU}(\ell+1)}$.

§4 **The case of $\mathbb{D}_\ell$.**

In this section, we give the character of the invariant vertex operator subalgebra for the case of $\mathbb{D}_\ell$ ($\ell \geqslant 3$) for which the associated compact group is $\mathrm{Spin}(2\ell)$. Actually, we could equally well use $\mathrm{SO}(2\ell)$ since all weights occurring are again in the root lattice. Again we use Proposition 3.1. First we recall some relevant facts.

Viewing $\mathbb{D}_\ell$ as a subgroup of $\mathbb{R}^\ell$, equipped with the standard orthonormal basis $\{e_1, \ldots, e_\ell\}$, the simple roots are

$$\alpha_i = e_i - e_{i+1} \quad (1 \leqslant i \leqslant \ell-1), \quad \alpha_\ell = e_{\ell-1} + e_\ell.$$

The set of positive roots is

$$\Delta_+ = \{e_i - e_j, e_i + e_j : 1 \leqslant i < j \leqslant \ell\}.$$

The following identities hold for elements of Δ_+:

$$\begin{aligned} e_i - e_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}, \\ e_i + e_j &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \alpha_{\ell-2}) + \alpha_{\ell-1} + \alpha_\ell. \end{aligned}$$

Also,

$$(\alpha_i \mid \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ and } \{i,j\} \neq \{\ell-1, \ell\}, \\ -1 & \text{if } \{i,j\} = \{\ell-2, \ell\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence for positive roots $\lambda = \sum_{1\leqslant k\leqslant\ell} r_k\alpha_k$ in $\mathbb{D}_\ell$ and $\alpha = \sum_{1\leqslant k\leqslant\ell} s_k\alpha_k$, we have

$$\begin{aligned} \frac{1}{2}(\lambda \mid \lambda) &= \sum_{1\leqslant k\leqslant\ell} r_k^2 - \sum_{1\leqslant k\leqslant\ell-2} r_k r_{k+1} - r_{\ell-2} r_\ell, \\ (\lambda \mid \alpha) &= 2\sum_{1\leqslant k\leqslant\ell} r_k s_k - \sum_{1\leqslant k\leqslant\ell} (r_k s_{k+1} + r_{k+1} s_k) - (r_{\ell-2} s_\ell + r_\ell s_{\ell-2}), \\ (\bar\rho \mid \alpha) &= \sum_{1\leqslant k\leqslant\ell} s_k. \end{aligned}$$

Also, if $1 \leqslant i < j \leqslant \ell$, then

$$\begin{aligned}
(\bar{\rho} \mid e_i - e_j) &= j - i, \\
(\bar{\rho} \mid e_i + e_j) &= j - i + 2(\ell - 1 - j) + 2 = 2\ell - i - j.
\end{aligned}$$

For s in the range $1 \leqslant s \leqslant 2\ell - 3$, the solutions of $2\ell - i - j = s$ in this range are determined by the values of i in the range $1 \leqslant i < 2\ell - s - i \leqslant \ell$ which is equivalent to

$$\begin{cases} \max\{\ell - 2t + 1, 1\} \leqslant i \leqslant \ell - t & \text{if } s = 2t - 1 \text{ is odd,} \\ \max\{\ell - 2t, 1\} \leqslant i \leqslant \ell - t - 1 & \text{if } s = 2t \text{ is even.} \end{cases}$$

In the first case, there are $\ell - t$ solutions when $t \geqslant \ell/2$, and t solutions when $t \leqslant (\ell - 1)/2$. In the second case, there are $\ell - t - 1$ solutions when $t \geqslant \ell/2$ and t solutions when $t \leqslant (\ell - 1)/2$. Notice that the ranges of values of t are

$$\begin{cases} 1 \leqslant t \leqslant \ell - 1 & \text{if } s = 2t - 1 \text{ is odd,} \\ 1 \leqslant t \leqslant \ell - 2 & \text{if } s = 2t \text{ is even.} \end{cases}$$

From all of this, we deduce the following.

Theorem 4.1. *The graded dimension of the* $\mathrm{Spin}(2\ell)$*-invariants in* $\mathrm{V}(\mathbb{D}_\ell)$ *is given by*

$$\begin{aligned}
\dim_q \mathrm{V}(\mathbb{D}_\ell)^{\mathrm{Spin}(2\ell)} &= \varphi(q)^{-\ell}(1 - q^{2\ell-3}) \prod_{1 \leqslant r \leqslant \ell-1} (1 - q^r)^{\ell - r} \\
\times \prod_{1 \leqslant r \leqslant (\ell-1)/2} &(1 - q^{2r-1})^r (1 - q^{2r})^r \prod_{\ell/2 \leqslant r \leqslant \ell-2} (1 - q^{2r-1})^{\ell-r}(1 - q^{2r})^{\ell-r-1}.
\end{aligned}$$

In particular,

$$\dim_q \mathrm{V}(\mathbb{D}_\ell)^{\mathrm{Spin}(2\ell)} \equiv 1 + q^2 \pmod{q^3},$$

implying $\dim_q \mathrm{V}(\mathbb{D}_\ell)_1^{\mathrm{Spin}(2\ell)} = 0$ *and* $\dim_q \mathrm{V}(\mathbb{D}_\ell)_2^{\mathrm{Spin}(2\ell)} = 1$.

§5 The case of $\mathbb{E}_\ell$.

Similar considerations apply to the case of the root lattices $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$. We record only the dimensions of the invariant vertex operator subalgebras for each $\mathbb{E}_\ell$, obtained using the symbolic algebra programme Maple with the aid of J. Stembridge's package *Root Systems and Finite Coxeter Groups.*

THEOREM 5.1. *For $\ell = 6, 7, 8$, the graded dimensions of the* $\mathrm{E}(\ell)$*-invariants in* $\mathrm{V}(\mathbb{E}_\ell)$ *are given by*

$$\begin{aligned}
\dim_q \mathrm{V}(\mathbb{E}_6)^{\mathrm{E}(6)} &= (1-q)^6(1-q^2)^5(1-q^3)^5(1-q^4)^5(1-q^5)^4(1-q^6)^3\\
&\qquad (1-q^7)^3(1-q^8)^2(1-q^9)(1-q^{10})(1-q^{11})\varphi(q)^{-6}\\
&\equiv 1+q^2+q^3+2q^4+3q^5+6q^6+7q^7+13q^8 \pmod{q^9},\\
\dim_q \mathrm{V}(\mathbb{E}_7)^{\mathrm{E}(7)} &= (1-q)^7(1-q^2)^6(1-q^3)^6(1-q^4)^6(1-q^5)^6(1-q^6)^5\\
&\qquad (1-q^7)^5(1-q^8)^4(1-q^9)^4(1-q^{10})^3(1-q^{11})^3\\
&\qquad (1-q^{12})^2(1-q^{13})^2(1-q^{14})(1-q^{15})\\
&\qquad (1-q^{16})(1-q^{17})\varphi(q)^{-7}\\
&\equiv 1+q^2+q^3+2q^4+2q^5+5q^6+5q^7+10q^8 \pmod{q^9},\\
\dim_q \mathrm{V}(\mathbb{E}_8)^{\mathrm{E}(8)} &= (1-q)^8(1-q^2)^7(1-q^3)^7(1-q^4)^7(1-q^5)^7(1-q^6)^7\\
&\qquad (1-q^7)^7(1-q^8)^6(1-q^9)^6(1-q^{10})^6(1-q^{11})^6\\
&\qquad (1-q^{12})^5(1-q^{13})^5(1-q^{14})^4(1-q^{15})^4(1-q^{16})^4\\
&\qquad (1-q^{17})^4(1-q^{18})^3(1-q^{19})^3(1-q^{20})^2(1-q^{21})^2\\
&\qquad (1-q^{22})^2(1-q^{23})^2(1-q^{24})(1-q^{25})(1-q^{26})\\
&\qquad (1-q^{27})(1-q^{28})(1-q^{29})\varphi(q)^{-8}\\
&\equiv 1+q^2+q^3+2q^4+2q^5+4q^6+4q^7+8q^8 \pmod{q^9}.
\end{aligned}$$

§6 Weight spaces in abelian intertwining algebras associated to dual lattices.

In [**1**], the notion of vertex operator algebra was extended to that of an abelian intertwining algebra. In particular, associated to a positive definite integral lattice $\mathbb{L}$ with dual lattice $\mathbb{L}^0$ is an abelian intertwining algebra $\tilde{\mathrm{V}}(\mathbb{L})$ which is the direct sum

$$\tilde{\mathrm{V}}(\mathbb{L}) = \bigoplus_{\alpha\in\mathbb{L}^0/\mathbb{L}} \tilde{\mathrm{V}}(\alpha)$$

of the irreducible modules $\tilde{\mathrm{V}}(\alpha)$ where $\tilde{\mathrm{V}}(0) = \mathrm{V}(\mathbb{L})$. The formula of Proposition 3.1 can then be generalized.

PROPOSITION 6.1. *Let* $\mathbb{L}$ *be a root lattice of type* $\mathbb{A}_\ell$ $(\ell \geqslant 1)$, $\mathbb{D}_\ell$ $(\ell \geqslant 4)$ *or* $\mathbb{E}_\ell$ $(\ell = 6, 7, 8)$. *If* $\mathfrak{g}$ *is the associated simple Lie algebra, then the graded vector space* $\tilde{\mathrm{V}}(\mathbb{L})$ *is naturally a graded* $\mathfrak{g}$*-module, and for each* $\lambda \in {\mathbb{L}^0}^+$ (*the set of dominant weights for* $\mathfrak{g}$) *the associated finite dimensional highest weight representation* $W(\lambda)$ *occurs with multiplicity given by the q-series*

$$\sum_{m\geqslant 0} (\mathrm{mult}_\lambda \tilde{\mathrm{V}}(\mathbb{L})_m) q^m = \varphi(q)^{-\ell} q^{(\lambda|\lambda)/2} \prod_{\alpha\in\mathring{\Delta}_+} \left(1 - q^{(\lambda+\bar{\rho}|\alpha)}\right).$$

For $\mathbb{L} = \mathbb{A}_1$, we have $\mathbb{L}^0 = (1/2)\mathbb{A}_1$ and so for each dominant weight $\lambda = (2k-1)\omega$ $(1 \leqslant k \in \mathbb{Z})$ where ω is the fundamental weight satisfying $(\omega \mid \alpha) = 1$, the SU(2)-irreducible with highest weight λ occurs only in the summand $\tilde{\mathrm{V}}((1/2)\alpha + \mathbb{A}_1)$ and

$$\sum_{m\geqslant 0} (\mathrm{mult}_{(2k-1)\omega} \tilde{\mathrm{V}}(\mathbb{A}_1)_m) q^m = \frac{q^{(2k-1)^2/4}(1-q^{2k})}{\varphi(q)}.$$

We leave the details of other cases to the interested reader.

§7 The structure of the invariants as a module over the Virasoro algebra.

By the commutativity of the action of G with the Virasoro algebra $\mathfrak{vir}$ generated by the conformal vector ω, the invariant subvertex operator algebra $\mathrm{V}(\mathbb{L})^G$ is a $\mathfrak{vir}$-submodule. In the case $\mathbb{L} = \mathbb{A}_1$, it is even cyclic over $\mathfrak{vir}$, but for $\mathbb{L} = \mathbb{A}_\ell, \mathbb{D}_\ell, \mathbb{E}_\ell$ in general this is not so. It may be that under the action of the universal central extension of the Lie algebra of differential operators on the circle discussed in [**5,7**], these modules are irreducible or cyclic. In §8 we discuss certain operators which may be related to this action. We hope to return to this question in future work.

§8 Constructing invariant elements using generalized Casimir operators.

We assume that $\mathfrak{g}$ is a finite dimensional simple Lie algebra over $\mathbb{C}$. Then the affinization $\widehat{\mathfrak{g}}$ has the following description. As a complex vector space,

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

We set $\mathfrak{g}(n) = \mathfrak{g}t^n \subseteq \widehat{\mathfrak{g}}$ and $x(n) = xt^n$ for $x \in \mathfrak{g}$. The bracket for elements $x(m), y(n) \in \mathfrak{g}[t, t^{-1}]$ is given by

$$[x(m), y(n)] = [x, y](m + n) + m(x \mid y)\delta_{m+n,0}c,$$

where $(\ \mid\ )$ is a non-degenerate invariant symmetric bilinear form for $\mathfrak{g}$ (for example, the Killing form). If $x = x(0) \in \mathfrak{g}$, then $[x(0), y(n)] = [x, y](n)$. Also, c is central and $[d, x(n)] = nx(n)$.

Now recall that for two irreducible modules P, Q of $\mathfrak{g}$, we have

$$\mathrm{Hom}_{\mathfrak{g}}(P, Q) \cong (Q \otimes P^*)^{\mathfrak{g}} \cong \begin{cases} \mathbb{C} & \text{if } P \cong Q, \\ 0 & \text{otherwise,} \end{cases}$$

where P^* denotes the $\mathbb{C}$-dual of P. In particular, for $m, n \in \mathbb{Z}$, the two $\mathfrak{g}$-modules $\mathfrak{g}(m), \mathfrak{g}(-n)$ are irreducible and indeed canonically isomorphic to the adjoint module $\mathfrak{g}$ itself. Hence,

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}(m), \mathfrak{g}(-n)) \cong (\mathfrak{g}(-n) \otimes \mathfrak{g}(m)^*)^{\mathfrak{g}} \cong \mathbb{C},$$

where the 'identity' map $x(m) \longmapsto x(-n)$ is the natural choice of generator. Using the canonical invariant inner product $\langle\ ,\ \rangle$ on $\widehat{\mathfrak{g}}$ for which

$$\langle x(r), y(s)\rangle = (x \mid y)\delta_{r+s,0},$$

we have an identification

$$\mathfrak{g}(m)^* \cong \mathfrak{g}(-m); \quad \langle x(m),\ \rangle \longleftrightarrow x^*(-m),$$

where we use the isomorphism induced by $(\ \mid\ )$,

$$\mathfrak{g}^* \cong \mathfrak{g}; \quad (x \mid\ ) \longleftrightarrow x^*.$$

Choosing a basis $\{x_i\}$ for $\mathfrak{g}$, we may express the element of $\mathfrak{g}(-n) \otimes \mathfrak{g}(-m)$ corresponding to the identity in $\mathbb{C}$ as

$$\Omega(m, n) = \sum_i x_i(-n) \otimes x_i^*(-m).$$

If $\{x_i\}$ is an orthonormal basis with respect to $(\ |\ )$, then

$$\Omega(m,n) = \sum_i x_i(-n) \otimes x_i(-m).$$

For a $\widehat{\mathfrak{g}}$-module M, we may view $\Omega(m,n)$ as determining an operator

$$z \longmapsto \sum_i x_i(-n)x_i(-m)z.$$

The $\Omega(m,n)$ are 'generalized Casimir operators' in the sense of the following result which is verified by a modification of the usual proof for Casimir operators.

PROPOSITION 8.1. *For a $\widehat{\mathfrak{g}}$-module M, $v \in M$, $0 \leqslant m,n \in \mathbb{Z}$ and $x \in \mathfrak{g}$, we have*

$$x\Omega(m,n) \cdot v = \Omega(m,n)x \cdot v.$$

If module M is the basic representation of $\widehat{\mathfrak{g}}$, then there is an associated vertex operator algebra; moreover, when $\mathfrak{g}$ is *simply laced* (i.e., $\mathfrak{g}$ is in one of the series $\mathbb{A}_\ell$, $\mathbb{D}_\ell$ or $\mathbb{E}_\ell$) we are in one of the situations considered earlier and the basic representation can be constructed as a Fock space as described in [**4**]. In this case we set

$$\omega(m,n) = \Omega(m,n) \cdot \mathbf{1}.$$

It turns out that

$$\frac{1}{4}\omega(1,1) = \omega,$$

the conformal vector. In general, the elements $\omega(m,n)$ will be referred to as 'generalized conformal elements'.

We recall some standard facts about the invariant bilinear form $(\ |\ )$ for a real simple Lie algebra $\mathfrak{g}$. For a Cartan subalgebra $\mathfrak{h}$, the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathring{\Delta}_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

is in fact a decomposition into mutually orthogonal subspaces of the form $\mathfrak{h}$ and $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Furthermore, the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dually paired and conjugate to each other, i.e.,

$$\mathfrak{g}_{-\alpha} \cong \mathfrak{g}_\alpha^* \cong \bar{\mathfrak{g}}_\alpha.$$

By choosing $(\ |\ )$ suitably, we can ensure that for each root α there are elements $h_\alpha \in \mathfrak{h}$ and $e_\alpha \in \mathfrak{g}_\alpha$ such that

$$[e_\alpha, e_{-\alpha}] = ih_\alpha, \quad (e_\alpha \mid e_{-\alpha}) = 1.$$

If we choose an orthonormal basis $\{h_i\}$ for $\mathfrak{h}$ (on which $(\ |\ )$ can be assumed positive definite), then we obtain a basis for $\mathfrak{g}$ of the form

$$\mathcal{B} = \{h_i\} \cup \{e_\alpha, e_{-\alpha} : \alpha \in \mathring{\Delta}_+\}$$

Using this basis, we obtain the following expression for the operator $\Omega(m,n)$ when $m,n \geqslant 1$,

$$\Omega(m,n) = \sum_i h_i(-n)h_i(-m) + \sum_{\alpha \in \mathring{\Delta}_+} (e_\alpha(-n)e_{-\alpha}(-m) + e_{-\alpha}(-n)e_\alpha(-m))$$

We will use vertex operators to calculate the generalized conformal elements $\omega(m,n)$. Recall from [**4**] that in $\mathrm{V}(\mathbb{L})$,

$$\mathrm{Y}(\alpha, z) = E^-(-\alpha, z)E^+(-\alpha, z)z^\alpha e^\alpha,$$

for $\alpha \in \mathbb{L}$. Here

$$\begin{aligned} E^-(\gamma, z) &= \exp\left(-\sum_{k\geqslant 1}\gamma(-k)z^k/k\right), \\ E^+(\gamma, z) &= \exp\left(\sum_{k\geqslant 1}\gamma(k)z^{-k}/k\right), \end{aligned}$$

and for $m \geqslant 1$, $\gamma(m)$ is the derivation satisfying

$$\gamma(m)\cdot(e^{\beta_1}\beta_2(-n)) = \begin{cases} m(\gamma \mid \beta_2)e^{\beta_1} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

To calculate $e_\alpha(k)$, we recall that

$$\mathrm{Y}(\alpha, z) = \sum_{k\in\mathbb{Z}} e_\alpha(k)z^{-k-1}.$$

Consider the generating functions (as series in z and w)

$$\begin{aligned} \mathrm{Y}(\alpha, z)\,\mathrm{Y}(-\alpha, w) &= \sum_{r,s\in\mathbb{Z}} e_\alpha(-r)e_{-\alpha}(-s)z^{r-1}w^{s-1} \\ \mathrm{Y}(-\alpha, z)\,\mathrm{Y}(\alpha, w) &= \sum_{r,s\in\mathbb{Z}} e_{-\alpha}(-r)e_\alpha(-s)z^{r-1}w^{s-1}. \end{aligned}$$

Summing these and applying the result to the vacuum vector $\mathbf{1}$ gives

$$[\mathrm{Y}(\alpha, z)\,\mathrm{Y}(-\alpha, w) + \mathrm{Y}(-\alpha, z)\,\mathrm{Y}(\alpha, w)]\cdot\mathbf{1} = \\ \sum_{\substack{r\in\mathbb{Z}\\ 1\leqslant s}} [e_\alpha(-r)e_{-\alpha}(-s) + e_{-\alpha}(-r)e_\alpha(-s)]\cdot\mathbf{1}\; z^{r-1}w^{s-1},$$

since $\mathbf{1}$ is a highest weight vector for $\widehat{\mathfrak{g}}$. We may calculate the left hand side of the last equation using the formal expansion

$$(1 - wz^{-1})^{-n} = \sum_{k\geqslant 0}\binom{-n}{k}w^kz^{-k}$$

for $n \in \mathbb{Z}$, together with the following identity which holds for $\lambda, \mu \in \mathbb{C}$,

$$\exp(\lambda\alpha(k)z^{-k}/k)\cdot\exp\left(\mu\alpha(-k)w^k/k\right) = \exp\left(\lambda\mu(\alpha \mid \alpha)w^kz^{-k}/k + \mu\alpha(-k)w^k/k\right).$$

We obtain

$$
\begin{aligned}
&[\mathrm{Y}(\alpha,z)\,\mathrm{Y}(-\alpha,w)+\mathrm{Y}(-\alpha,z)\,\mathrm{Y}(\alpha,w)]\cdot\mathbf{1}= \\
&\qquad z^{-(\alpha|\alpha)}\left[E^-(-\alpha,z)E^+(-\alpha,z)\right]\cdot\exp\left(-\sum_{k\geqslant 1}\alpha(-k)w^k/k\right) \\
&\qquad\quad + z^{-(\alpha|\alpha)}\left[E^-(\alpha,z)E^+(\alpha,z)\right]\cdot\exp\left(\sum_{k\geqslant 1}\alpha(-k)w^k/k\right) \\
&= z^{-(\alpha|\alpha)}(1-wz^{-1})^{-(\alpha|\alpha)}\left[\exp\left(\sum_{m\geqslant 1}\alpha(-m)z^m/m\right)\exp\left(-\sum_{k\geqslant 1}\alpha(-k)w^k/k\right)\right. \\
&\qquad\qquad \left. +\exp\left(-\sum_{m\geqslant 1}\alpha(-m)z^m/m\right)\exp\left(\sum_{k\geqslant 1}\alpha(-k)w^k/k\right)\right]
\end{aligned}
$$

Subtracting the term $2z^{-(\alpha|\alpha)}(1-wz^{-1})^{-(\alpha|\alpha)}$, we obtain an expression involving terms of non-zero degree from the symmetric algebra

$$\mathrm{S}(\mathfrak{h}^-)\cong e^0\otimes\mathrm{S}(\mathfrak{h}^-)\subseteq\mathbb{C}\{\mathbb{L}\}\otimes\mathrm{S}(\mathfrak{h}^-),$$

which is easily verified to be symmetric in z and w. On summing over $\alpha\in\mathring{\Delta}_+$ and using the fact that the action of $h(-k)$ $(k\geqslant 1)$ is by multiplication, we obtain an expression for

$$\sum_{m,n\geqslant 1}(\omega(m,n)-h_i(-n)h_i(-m))z^{n-1}w^{m-1}.$$

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