CONTINUOUS MORAVA K-THEORY AND THE GEOMETRY OF THE $\mathbf{I_n}$-ADIC TOWER

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§1 Introduction

This article concerns completed cohomology theories and the topology of the spaces which represent them. Our principal interest is in those given by the Johnson-Wilson spectra $E(n)$ **[10]** and their I_n-adic completions $\widehat{E(n)}$ (see below), but many of our arguments apply also to a much wider class of theories, such as complex K-theory and its p-adic completion, or elliptic cohomology and its supersingular completion, as defined in **[1]**.

To each standard cohomology theory, $E^*(-)$ say, that we consider here, there are *representing* Ω*-spectra*, *i.e.*, sets of H-spaces $\{\mathbf{E}_r; r \in \mathbb{Z}\}$ with natural isomorphisms (of groups)

$$E^r(X) \cong [X, \mathbf{E}_r],$$

where $[X, \mathbf{E}_r]$ denotes the set of homotopy classes of maps from the space X to $\mathbf{E}_r$. The suspension isomorphism $E^r(X) \cong E^{r+1}(\Sigma X)$ becomes in translation a homotopy equivalence $\mathbf{E}_r \simeq \Omega\mathbf{E}_{r+1}$, and the loop sum operation in $\Omega\mathbf{E}_{r+1}$ defines the H-space product in $\mathbf{E}_r$. Such spaces in an Ω-spectrum are naturally very rich in structure (even more so if the original cohomology theory has products) and this often means that they are of considerable geometric use.

Let us suppose $\mathbf{E}_*$ is an Ω-spectrum representing a cohomology theory $E^*(-)$, and $\widehat{\mathbf{E}}_*$ represents some completed version of the theory, $\widehat{E}^*(-)$ say. Then $\widehat{E}^*(-)$ may have superior properties to $E^*(-)$ by virtue of its completion, but it will almost certainly be harder to understand the topology of the spaces $\widehat{\mathbf{E}}_r$ than that of the corresponding spaces $\mathbf{E}_r$ as the former are likely to be so much larger. Now suppose $L^*(-)$ is some standard cohomological functor; in this paper we set up a notion of *continuous* L-cohomology, $L^*_c(-)$, a variation on $L^*(-)$ which forms a suitable tool for studying completed Ω-spectra. It will allow us to combine the desirable properties of the completed Ω-spectrum with the calculational advantages of the uncomplete one. As an application, we show how our theory in the case of the spectra $\widehat{E(n)}$ can be used to discuss problems on the Morava K-theory of extended powers $K(n)^*(D_p(X))$.

Recall that the Brown-Peterson spectrum BP has coefficient ring

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots, v_r, \ldots],$$

where the element v_r lies in dimension $2(p^r - 1)$. The BP-module ring spectrum $E(n)$ has coefficients

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_{n-1}, v_n, v_n^{-1}].$$

We write I_n for the usual ideal $(p, v_1, \ldots, v_{n-1})$ of $E(n)_*$. In **[3]**, an I_n-adically complete version of this spectrum is constructed. Written $\widehat{E(n)}$, it has as its coefficient ring the inverse limit

$$\widehat{E(n)}_* = \lim_k E(n)_*/I_n^k.$$

In **[2]** and **[4]** a tower of spectra

$$K(n) = E(n)/I_n \longleftarrow \cdots \longleftarrow E(n)/I_n^k \longleftarrow E(n)/I_n^{k+1} \longleftarrow \cdots \tag{1.1}$$

is constructed, with $(E(n)/I_n^k)_* = E(n)_*/I_n^k$. Each $E(n)/I_n^k$ is a module spectrum over both $E(n)$ and $\widehat{E(n)}$. The homotopy inverse limit of this tower is $\widehat{E(n)}$ itself and the fibre of each map $E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$ is a finite product of even suspensions of copies of the Morava K-theory spectrum $K(n)$.

The I_n-adically complete spectrum $\widehat{E(n)}$ has appeared in several different contexts and there are many reasons to view it as a more central object than $E(n)$ itself.

This tower of spectra was of importance in the work of the second author on the Morava K-theory of extended powers $K(n)^*(D_p(X))$ (see **[7]**, **[8]**). If $K(n)^{\mathrm{odd}}(X)$ is trivial then the tower (1.1) shows that all elements of $K(n)^*(X)$ lift to $\widehat{E(n)}^*(X)$ and also, if X is finite, to $E(n)^*(X)$. We used the cohomology of the spaces in the Ω-spectrum for $E(n)$ to describe $K(n)^*(D_p(X))$ for such X. In this paper we extend those results to include many infinite dimensional spaces X by using the analagous terms in the Ω-spectrum for $\widehat{E(n)}$ and our continuous cohomology theories, in this case continuous Morava K-theory, $K(n)_c^*(-)$. In particular, these calculations would conjecturally include the classifying space of a finite group (see **[5]**).

The paper is arranged as follows. In §2 we describe a certain pro-category and set up our concept of continuous cohomology theories. In §3 we compute the continuous Morava K-theory of the pro-spaces corresponding to the spaces in the Ω-spectra of the $E(n)/I_n^k$ and indicate how much of our work would go through on replacing $E(n)$ by an arbitrary Landweber exact spectrum (for example, that representing complex K-theory or elliptic cohomology). This calculation allows us to describe $K(n)^*(D_p(X))$ for any space X with $K(n)^*(X)$ of finite rank over $K(n)_*$ and which satisfies $K(n)^{\mathrm{odd}}(X) = 0$. This is done in §4 where we also explain how in many cases the finite rank assumption can be abandoned. The calculations of §§3 and 4 are all at an odd prime p. In a sense this completes the work of **[7]**; we demonstrate elsewhere **[9]** what happens in the rather different situation where X has both even and odd dimensional Morava K-theory. We conclude by discussing in §5 a general method for constructing cellular models for infinite loop spaces. Applying this to the particular cases we are concerned with here, *i.e.,* the spaces in the Ω-spectra for $E(n)$, $E(n)/I_n^k$ and $\widehat{E(n)}$, we show that much of the cohomological structure we compute in §3 has close underlying geometric analogues.

One moral of the present paper is that there is a close relationship between the (co)homology theories associated to $\widehat{E(n)}$ and $K(n)_c$. We expect that a systematic study of the whole structure associated to the I_n-adic tower and I_n-nil pro-subcategory of the $K(n)$-local category would be of great interest; the case of $n = 1$ (corresponding to p-local K-theory) has been investigated by A. K. Bousfield and others.

It is also possible that the unstable structure associated to pro-systems of $K(n)$-nilpotent spaces (*i.e.,* spaces X with $K(n)^*(X)$ nilpotent) and their continuous Morava K-theory may be investigated from the perspective of Galois cohomology or its generalisations as described in Shatz **[16]**. Indeed, the use of 'continuous' cohomology functors applied to pro-systems seems ubiquitous in studying $K(n)$-local phenomena. We hope to return to this in future work.

§2 Continuous cohomology theories

Let $\mathcal{PS}$ be the category of inverse systems of spaces

$$X^0 \xleftarrow{f^0} X^1 \xleftarrow{f^1} X^2 \xleftarrow{f^2} X^3 \longleftarrow \cdots \tag{2.1}$$

indexed by the natural numbers. A morphism $\psi\colon \{X^k\} \longrightarrow \{Y^k\}$ will be a set of strict maps $\{\psi^k\colon X^k \longrightarrow Y^k\}$ defined for sufficiently large k and compatible with the maps f^i of the inverse systems. Homotopies and cofibres will be defined term by term, as for spaces. Let $\mathcal{PCW}$ be the subcategory of $\mathcal{PS}$ consisting of inverse systems of CW complexes and cellular maps.

Let $L^*(-)$ be a cohomology theory.

Definition 2.2 For $\mathcal{X} = \{X^k\}$ an object in $\mathcal{PS}$, define the *continuous L-cohomology* of $\mathcal{X}$ to be the colimit

$$L_c^*(\mathcal{X}) = \operatorname*{colim}_k L^*(X^k).$$

Proposition 2.3 *This defines a cohomology theory on $\mathcal{PS}$. It does not, however, satisfy the wedge axiom.*

Proof All the Eilenberg-Steenrod axioms follow immediately from the definition. $L_c^*(-)$ is clearly a functor on $\mathcal{PS}$, invariant under homotopy. Colimits over $\mathbb{N}$ preserve exactness and hence we have the long exact sequence of a cofibration; excision holds since it does so for each term in the direct system. The wedge axiom fails as a product of colimits is not in general the same as a colimit of products. □

Remark 2.4 Suppose the cohomology theory $L^*(-)$ is multiplicative. Then our definition of $L_c^*(-)$ means that there is an induced product on $L_c^*(\mathcal{X})$ for $\mathcal{X} \in \mathcal{PS}$.

Remark 2.5 If we identify the space X with the constant diagram, $\overline{X}$ say,

$$X \xleftarrow{1} X \xleftarrow{1} X \xleftarrow{1} X \longleftarrow \cdots,$$

regarded as an object in $\mathcal{PS}$, then $L_c^*(\overline{X})$ will be identical to $L^*(X)$.

Definition 2.6 If $\mathcal{X} = \{X^k\}$ is an object of $\mathcal{PCW}$, write $\mathcal{C}^*(X^k; A)$ for the cellular cochain complex of X^k with coefficients in A. Then define the *continuous cellular cochain complex* of $\mathcal{X}$ as the colimit

$$\mathcal{C}_c^*(\mathcal{X}; A) = \operatorname*{colim}_k \mathcal{C}^*(X^k; A).$$

As homology commutes with colimits, we immediately get

Proposition 2.7 *For $\mathcal{X} \in \mathcal{PCW}$, $H_c^*(\mathcal{X}; A) = H(\mathcal{C}_c^*(\mathcal{X}; A))$.* □

Remark 2.8 Suppose $G = \lim G^k$ is a profinite group. Let A be an abelian group with compatible G^k-action for all k. Then the classifying spaces of the G^k give an object in $\mathcal{PS}$, say $\mathcal{BG}$,

$$BG^0 \longleftarrow BG^1 \longleftarrow BG^2 \longleftarrow BG^3 \longleftarrow \cdots .$$

If we take $L^*(-) = H^*(-;A)$, then $H^*_c(\mathcal{BG};A)$ recovers the profinite group cohomology theory of **[16]**.

Remark 2.9 For $\mathcal{X} = \{X^k\}$ an object in $\mathcal{PCW}$ we have an Atiyah-Hirzebruch spectral sequence (AHSS) for continuous L-cohomology,

$$E_2^{*,*} = H^*_c(\mathcal{X};L^*) \Longrightarrow L^*_c(\mathcal{X}). \tag{2.10}$$

We filter the inverse system (2.1) by the inverse systems of m skeleta

$$(X^0)_m \xleftarrow{f^0} (X^1)_m \xleftarrow{f^1} (X^2)_m \xleftarrow{f^2} (X^3)_m \longleftarrow \cdots$$

and apply the functor $L^*_c(-)$. The resulting E_1-term is given by

$$E_1^{p,q} = \operatorname*{colim}_k \mathcal{C}^p(X^k;L^q) = \mathcal{C}^p_c(\mathcal{X};L^q)$$

as usual. Each E_r-term is then identified as the colimit of the E_r-terms in the spectral sequences

$$E_2 = H^*(X^k;L^*) \Longrightarrow L^*(X^k). \tag{2.11}$$

In particular, the convergence of (2.10) now follows from that of (2.11) for sufficiently large k: we have the identification of the E_2-term as claimed.

Alternatively, the Atiyah-Hirzebruch spectral sequence can be set up for a general object in $\mathcal{PS}$ by mapping the inverse system of spaces into the Postnikov tower for the representing spectrum.

Remark 2.12 We have set up a rather *ad hoc* definition of continuous cohomology, suitable for the items we wish to study. We wonder whether there might be a more systematic approach to this concept which places it in a more orthodox homotopy theoretic framework: for example, how might a corresponding continuous homology theory be defined?

§3 The continuous cohomology of the $\mathbf{I_n}$-adic tower

For a spectrum E let us write $\mathbf{E}_* = \{\mathbf{E}_r; r \in \mathbb{Z}\}$ for the corresponding Ω-spectrum, where the space $\mathbf{E}_r$ represents the cohomology group $E^r(-)$. We write $\widehat{\mathcal{E}(n)}_r$ for the element of $\mathcal{PS}$ given by the spaces in the I_n-adic tower

$$(\mathbf{E(n)}/\mathbf{I_n^1})_r \longleftarrow (\mathbf{E(n)}/\mathbf{I_n^2})_r \longleftarrow (\mathbf{E(n)}/\mathbf{I_n^3})_r \longleftarrow \cdots .$$

In §5 we give an explicit cellular construction of these spaces, and so of $\widehat{\mathcal{E}(n)}_r$ as an element of $\mathcal{PCW}$, but the actual construction is irrelevant to the arguments of this section.

Theorem 3.1 *The algebras $K(n)^*_c(\widehat{\mathcal{E}(n)}_r)$ are nilpotent free for r even, equivalently, for these r, the Frobenius map $x \mapsto x^p$ is a monomorphism.*

We prove this by examining the continuous $\mathbb{F}_p$ cohomology of $\widehat{\mathcal{E}(n)}_r$ and relating it to the cohomology of $\mathbf{E(n)}_r$. Let us first define some notation. As all this section is in characteristic p, we shall suppress writing $\mathbb{F}_p$ whenever possible: thus, $H_*(X)$ denotes $H_*(X;\mathbb{F}_p)$, while $\mathrm{Hom}(A)$ for an $\mathbb{F}_p$-vector space A indicates its linear dual. In the case of A an $\mathbb{F}_p$ Hopf algebra, Tor^A will denote the graded Tor group $\mathrm{Tor}^A(\mathbb{F}_p, \mathbb{F}_p)$. This latter object will arise in our work as follows. For an Ω-spectrum $\mathbf{E}_*$, the homologies of $\mathbf{E}_r$ and $\mathbf{E}_{r+1}$ are linked by the Rothenberg-Steenrod spectral sequence (RSSS)

$$E^2_{**} = \mathrm{Tor}^{H_*(\mathbf{E}_r)} \Longrightarrow H_*(\mathbf{E}'_{r+1}),$$

where $\mathbf{E}'_{r+1}$ denotes the connected component of the basepoint in $\mathbf{E}_{r+1}$. This spectral sequence is induced by filtering $\mathbf{E}'_{r+1} = B\mathbf{E}_r$ by bar degree.

Let us recall some results about $H_*(\mathbf{E}(\mathbf{n})_r)$ from **[8]**. We showed that this is a polynomial or exterior algebra if r is even or odd, respectively, and pointed out that although the cohomology algebras $H^*(\mathbf{E}(\mathbf{n})'_r)$ for even r were not polynomial, they were nilpotent free. To see this, recall that in **[8]** we considered $\mathbf{E}(\mathbf{n})_r$ as a colimit of spaces $\{\mathbf{BP}\langle\mathbf{n}\rangle_{r-2i(p^n-1)}\}$ as $i \to \infty$, the maps being those inducing multiplication by v_n in homotopy. It suffices to note that the Verschiebung $V\colon H_*(\mathbf{BP}\langle\mathbf{n}\rangle'_s) \longrightarrow H_*(\mathbf{BP}\langle\mathbf{n}\rangle'_s)$ (dual to the Frobenius in cohomology) is epimorphic for sufficiently small values of s (since by **[17]** $H_*(\mathbf{BP}\langle\mathbf{n}\rangle'_s)$ is bipolynomial for s small).

We also demonstrated that the Hopf ring map

$$\tau_{E(n)}\colon H^R_*(\mathbf{E}(\mathbf{n})_*) \longrightarrow H_*(\mathbf{E}(\mathbf{n})_*)$$

is an isomorphism (see **[8]** or **[14]** for Hopf ring notation); thus $H_*(\mathbf{E}(\mathbf{n})_*)$ is generated (as a Hopf ring) by $H_0(\mathbf{E}(\mathbf{n})_*)$ plus the homology image of the complex orientation map $\mathbb{C}\mathrm{P}^\infty \longrightarrow \mathbf{E}(\mathbf{n})_2$.

Let us write ϕ^k for the $E(n)$-module map $E(n) \longrightarrow E(n)/I_n^k$ which induces the quotient by I_n^k on coefficients.

Definition 3.2 Say that a homomorphism $H_*(\mathbf{E}(\mathbf{n})_r) \longrightarrow \mathbb{F}_p$ is *continuous* if it factors through the homomorphisms

$$\phi^k_*\colon H_*(\mathbf{E}(\mathbf{n})_r) \longrightarrow H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^\mathbf{k}_\mathbf{n})_r)$$

for all sufficiently large k. Write $\mathrm{Hom_c}(H_*(\mathbf{E}(\mathbf{n})_r))$ for the set of all continuous homomorphisms.

Theorem 3.3 *As Hopf algebras,* $H^*_c(\widehat{\mathcal{E}(n)}_r) \cong \mathrm{Hom_c}(H_*(\mathbf{E}(\mathbf{n})_r))$.

(Compare with the standard idea that the continuous functions on a space are equivalent to the continuous functions on its completion.)

Corollary 3.4 $H^*_c(\widehat{\mathcal{E}(n)}_r)$ *is isomorphic to a subring of* $H^*(\mathbf{E}(\mathbf{n})_r)$. *Hence the ring* $H^*_c(\widehat{\mathcal{E}(n)}_r)$ *has no nilpotent elements if* r *is even.*

Proof of (3.4) We have identified $H^*_c(\widehat{\mathcal{E}(n)}_r)$ as a subring of $\mathrm{Hom}(H_*(\mathbf{E}(\mathbf{n})_r)) = H^*(\mathbf{E}(\mathbf{n})_r)$. The result follows from the above remarks on $H^*(\mathbf{E}(\mathbf{n})_r)$ for r even.□

Proof of (3.1) This is now a simple application of the Atiyah-Hirzebruch spectral sequence

$$H^*_c(\widehat{\mathcal{E}(n)}_r; K(n)_*) \Longrightarrow K(n)^*_c(\widehat{\mathcal{E}(n)}_r).$$

For r even the E_2-term is entirely even dimensional and so the spectral sequence collapses. □

Proof of (3.3) First note that it is sufficient to show that

$$\mathrm{Hom_c}(H_*(\mathbf{E}(\mathbf{n})_r)) = \operatorname*{colim}_k \mathrm{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)) \tag{3.5}$$

since $\operatorname*{colim}_k \mathrm{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)) = \operatorname*{colim}_k H^*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r) = H^*_c(\widehat{\mathcal{E}(n)}_r)$. In fact, we show that the equivalences (3.5) are induced by the maps ϕ^k: the ϕ^k naturally define a map

$$\operatorname*{colim}_k \mathrm{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)) \longrightarrow \mathrm{Hom_c}(H_*(\mathbf{E}(\mathbf{n})_r))$$

and this is to be the isomorphism in (3.5).

We prove (3.5) by induction on the dimension, carrying along also a second inductive hypothesis, namely

$$\bigcap_k \mathrm{Ker}(\phi^k_*) = 0. \tag{3.6}$$

The hypotheses hold in dimension zero by direct calculation: both sides of (3.5) are equal to $\operatorname*{colim}_k \mathrm{Hom}(\mathbb{F}_p[(E(n)/I^k_n)^{*=r}])$. Now suppose that (3.5) and (3.6) both hold for all r and in all dimensions less than m.

Consider the dual RSSS's

$$\mathrm{Hom}(\mathrm{Tor}^{H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)}) \Longrightarrow \mathrm{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})'_{r+1})) \tag{3.7}$$

$$\mathrm{Hom}(\mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)}) \Longrightarrow \mathrm{Hom}(H_*(\mathbf{E}(\mathbf{n})'_{r+1})). \tag{3.8}$$

The colimit over k of (3.7) gives a spectral sequence

$$\operatorname*{colim}_k \mathrm{Hom}(\mathrm{Tor}^{H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)}) \Longrightarrow \operatorname*{colim}_k \mathrm{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})'_{r+1})). \tag{3.9}$$

The maps ϕ^k give us a morphism of spectral sequences, ϕ^* say, from (3.9) to (3.8).

Lemma 3.10 *The spectral sequence homomorphism ϕ^* induces a monomorphism of E_2-terms whose image is*

$$\mathrm{Hom_c}(\mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)}) \subset \mathrm{Hom}(\mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)}),$$

where the topology on $\mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)}$ *is that induced by the kernels of the homomorphisms*

$$(\phi^k_*)_*\colon \mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)} \longrightarrow \mathrm{Tor}^{H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)}.$$

Proof of (3.10) Recall that $H_*(\mathbf{E}(\mathbf{n})_r)$ is polynomial or exterior, depending on whether r is even or odd; hence $\mathrm{Tor}^{H_*(\mathbf{E}(\mathbf{n})_r)}$ is exterior or divided power, respectively, on suspensions of representatives of the $*$-indecomposables of $H_*(\mathbf{E}(\mathbf{n})_r)$. Hypothesis (3.5) shows that in dimensions less than m the set

$$\{\phi^k_*\colon H_*(\mathbf{E}(\mathbf{n})_r) \longrightarrow H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}^{\mathbf{k}}_{\mathbf{n}})_r)\}$$

is a pro-epimorphism, and hence induces a pro-epimorphism of $*$-indecomposable quotients. Meanwhile, (3.6) implies that for r even, if $0 \neq x \in H_*(\mathbf{E}(\mathbf{n})_r)$, $* < m$, then for k sufficiently large, all powers of $(\phi^k)_*(x)$ (of dimensions less than m) will be non-zero and so there are no transpotence terms in the E_2-term of (3.9). The lemma follows by computing Tor groups. $\square$

As (3.8) collapses, the sequence (3.9) must also collapse and we can identify $\operatorname{colim}_k \operatorname{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})'_{r+1}))$ with $\operatorname{Hom}_c(H_*(\mathbf{E}(\mathbf{n})'_{r+1}))$. This proves (3.5) for dimensions less than $m+1$. Hypothesis (3.6) for dimensions less than $m+1$ follows from the calculation of Tor groups and the collapsing of (3.9). This completes the proof of (3.3). $\square$

Remark 3.11 We have used only certain formal properties of the Ω-spectrum $E(n)$, namely that the homology algebras $H_*(\mathbf{E}(\mathbf{n})_r)$ are either polynomial or exterior, depending on the parity of r. By **[6]** this is a property of $H_*(\mathbf{E}_r)$ whenever the spectrum E represents a Landweber exact cohomology theory with coefficients free and of countable rank over some subring of the rationals. Theorem (3.3) thus has an analogue for any completion of a suitable Landweber exact spectrum defined as a homotopy limit like $\widehat{E(n)}$, for example, p-adic K-theory or the first author's completion of elliptic cohomology $\widehat{Ell}_{\mathcal{P}}$, described in **[1]**.

Remark 3.12 We have computed $\operatorname{colim}_k \operatorname{Hom}(H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_r))$ without actually computing much about the individual algebras $H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_r)$. The case of $k=1$, namely Morava K-theory, has already been computed in **[19]**. Strictly speaking, the algebraic object $H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*)$ is not a Hopf ring for $k>1$ as there is no global $\circ$-product, these spectra not being known to be multiplicative. There is however a $\circ$-module action by elements of $H_*(\mathbf{E}(\mathbf{n})_*)$ via ϕ^k_*, and so, following the dubious nomenclature of this area, perhaps these objects ought to be named *Hopf modules,* although it might be more accurate to refer to them as *coalgebraic modules* over *coalgebraic rings.* We see from the proof of (3.3) that the inverse system $\{H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*); k>0\}$ is pro-equivalent to a set of increasing quotients of $H_*(\mathbf{E}(\mathbf{n})_*)$ with trivial intersection of kernels, but for each individual value of k there will be a lot of $*$-torsion in $H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*)$, and this gives rise to elements in the cokernel of $\phi^k_*\colon H_*(\mathbf{E}(\mathbf{n})_*) \longrightarrow H_*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*)$. If pressed for such calculations, use might be made of Wilson's calculations **[19]** for $K(n)$ and the fibrations

$$\prod_{v \in I_n^k/I_n^{k-1}} \left(\Sigma^{|v|}\mathbf{K}(\mathbf{n})_*\right) \longrightarrow (\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^{\mathbf{k+1}})_* \longrightarrow (\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*$$

to work up the tower (1.1), but the method seems tedious and offers, at present, little of interest.

Remark 3.13 As in **[14]** write $H_*^R((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*)$ for the quotient of $H_*^R(\mathbf{E}(\mathbf{n})_*)$ induced by the coefficient homomorphism $E(n)_* \longrightarrow (E(n)/I_n^k)_*$; thus we have quotient homomorphisms

$$\psi^k\colon H_*(\mathbf{E}(\mathbf{n})_*) = H_*^R(\mathbf{E}(\mathbf{n})_*) \longrightarrow H_*^R((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_*).$$

If we give $H_*(\mathbf{E}(\mathbf{n})_*)$ the topology corresponding to the set of neighbourhoods of the identity $\{\operatorname{Ker}(\psi^k)\}$ (this might be named, in Hopf ring terminology, the $[I_n]$-adic topology), then the proof of (3.3) shows that we have an alternative description of $H^*_c(\widehat{\mathcal{E}(n)}_r)$, now as homomorphisms $H_*(\mathbf{E(n)}_r) \longrightarrow \mathbb{F}_p$ continuous with respect to this new topology. (That this topology is strictly different from the one before can be deduced from the description of the Hopf ring $H_*(\mathbf{K(n)}_*)$ in **[19]**: relation (1.1)(j) in that paper does not appear in the Hopf ring $H^R_*(\mathbf{K(n)}_*)$.) Thus we have an equivalence of Hopf corings,

$$H^*_c(\widehat{\mathcal{E}(n)}_*) \cong \operatorname*{colim}_k \operatorname{Hom}(H^R_*((\mathbf{E(n)}/\mathbf{I^k_n})_*)).$$

§4 The Morava K-theory of extended powers.

In **[7]** the second author defined the notion of a *unitary-like embedding* (ULE): at an odd prime the space X is said to have a ULE if there is a space Y with $K(n)^*(Y)$ nilpotent free (*i.e.,* the Frobenius map $x \mapsto x^p$ is a monomorphism) together with a map $e\colon X \longrightarrow Y$ inducing an epimorphism in $K(n)$-cohomology. We showed that if X had a ULE then we could compute $K(n)^*(D_p(X))$, the Morava K-theory of the p^{th} extended power of X. For $X = BG$, the classifying space of a group G, this amounts to a calculation of the wreath product $K(n)^*(B(G \wr \mathbb{Z}/p))$.

The article **[8]** examined the existence of ULE's for finite spaces and proved the following, somewhat stronger, result:

Theorem 4.1 *Given a finite CW complex X with $K(n)^{\text{odd}}(X) = 0$, then there exists a map $e\colon X \longrightarrow Y$, epimorphic in $K(n)^*(-)$, with $K(n)^*(Y)$ a completed polynomial algebra over $K(n)_*$.*

In fact, the $K(n)^{\text{odd}}(X) = 0$ condition is necessary as well as sufficient, owing to the graded commutativity of the multiplication in Morava K-theory. The map e is produced by lifting the elements of $K(n)^*(X)$ to maps into the representing spaces of $E(n)^*(-)$; thus Y is constructed as a product of spaces $\mathbf{E(n)}_r$, with r varying over the even integers. However, the finiteness restriction rules out a number of very interesting spaces, for example, the classifying space of any (non-trivial) finite group is always infinite, even though $K(n)^*(BG)$ is of finite rank over the coefficients, **[13]**.

We show here that those calculations extend to the case of infinite spaces X so long as $K(n)^*(X)$ is finitely generated over $K(n)_*$ and $K(n)^{\text{odd}}(X) = 0$. We use continuous Morava K-theory and the results of §3. Recall, as in (2.5), that a general space X yields the constant inverse system, $\overline{X}$,

$$X \xleftarrow{1} X \xleftarrow{1} X \xleftarrow{1} X \longleftarrow \cdots$$

and that $K(n)^*_c(\overline{X}) = K(n)^*(X)$.

Theorem 4.2 *Given a space X with $K(n)^*(X)$ finitely generated over $K(n)_*$ and $K(n)^{\text{odd}}(X) = 0$, then there exists a morphism in $\mathcal{PS}$, $e\colon \overline{X} \longrightarrow \mathcal{Y}$, epimorphic in $K(n)^*_c(-)$ and with $K(n)^*_c(\mathcal{Y})$ nilpotent free.*

Proof Let $\{x_i\}$ be a finite set of homogeneous $K(n)_*$ generators for $K(n)^*(X)$, *i.e.,* each $x_i \in K(n)^r(X)$ for some $r = r(i)$. As $K(n)^{\text{odd}}(X) = 0$ each x_i, as a map $x_i\colon X \longrightarrow \mathbf{K(n)}_r$, lifts to maps $X \longrightarrow (\mathbf{E(n)}/\mathbf{I^k_n})_r$ for all k and hence gives a map in $\mathcal{PS}$

$$\overline{x_i}\colon \overline{X} \longrightarrow \widehat{\mathcal{E}(n)}_r.$$

This follows from the I_n-adic tower (1.1) since the fibre of each map $E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$ is a finite product of even suspensions of copies of the Morava K-theory spectrum $K(n)$. The map

$$e\colon \overline{X} \longrightarrow \prod_i \widehat{\mathcal{E}(\mathbf{n})}_{r(i)}$$

given by the product of the maps $\overline{x_i}$ now has the required properties by construction and (3.1). (The product of two inverse systems of spaces $\{X^k\}$ and $\{Y^l\}$ is defined as the inverse system of products, $\{X^k \times Y^k\}$.) □

Definition 4.3 Let us say that the spectral sequence

$$H^*(B\mathbb{Z}/p; K(n)^*(F)) \Longrightarrow K(n)^*(E)$$

of the fibration $F \xrightarrow{i} E \xrightarrow{i} B\mathbb{Z}/p$ is *simple* if the only non-trivial differential action involved is that of d_{2p^n-1}, operating as it must via naturality with the AHSS for $K(n)^*(B\mathbb{Z}/p)$.

In [**7**] we showed that X having a ULE implied that the spectral sequence of the extension

$$X^p \longrightarrow D_p(X) \longrightarrow B\mathbb{Z}/p \tag{4.4}$$

was simple. This result gives $K(n)^*(D_p(X))$ in terms of $K(n)^*(X)$. Exactly the same arguments can be made in the continuous case and we achieve a similar description of $K(n)^*_c(\overline{D_p(X)})$ in terms of $K(n)^*_c(\overline{X})$. However, as $K(n)^*(Z) = K(n)^*_c(\overline{Z})$ for $Z = X$ or $D_p(X)$, we have proved the following result.

Corollary 4.5 *Given a space X with $K(n)^*(X)$ finitely generated over $K(n)_*$ and $K(n)^{\mathrm{odd}}(X) = 0$, then the spectral sequence of the fibration (4.4) for $K(n)^*(D_p(X))$ is simple.* □

Of course this result relies on our having a continuous version of the spectral sequence of a fibration, but, as for the continuous AHSS, there are no problems in deriving such a gadget.

Remark 4.6 The condition that the rank of $K(n)^*(X)$ is finite over $K(n)_*$ in the hypotheses of (4.2) and (4.5) is in many cases not really necessary. Suppose $\{x_i; i \in \mathcal{J}\}$ is a set of $K(n)_*$ generators of $K(n)^*(X)$ and that we write $\mathcal{J} = \operatorname{colim}_s \mathcal{J}_s$ where all the $\mathcal{J}_s$ are finite (for example, write $\mathcal{J}$ as the union of its finite subsets). Then the map e in the proof of (4.2) can be replaced by the colimit

$$e\colon \overline{X} \longrightarrow \operatorname*{colim}_s \prod_{i \in \mathcal{J}_s} \widehat{\mathcal{E}(\mathbf{n})}_{r(i)}$$

which will have the required properties provided the derived functors

$$\lim_s{}^t \left\{ \prod_{i \in \mathcal{J}_s} K(n)^*((\mathbf{E}(\mathbf{n})/\mathbf{I}_\mathbf{n}^\mathbf{k})_{r(i)}) \right\}$$

vanish for all $t > 0$ and for all k sufficiently large. As the maps in the inverse system are all epimorphisms, it seems likely that this will be the case in many instances.

Remark 4.7 Our work shows that for spaces X satisfying certain hypotheses (most notably the condition that $K(n)^{\text{odd}}(X) = 0$) we can describe the object $K(n)^*(D_p(X))$ in terms of the ring $K(n)^*(X)$. The corresponding problem when $K(n)^*(X)$ has both even and odd dimensional elements is different. The case of $n = 1$, *i.e.,* the mod p K-theory of extended powers, is studied in **[11]**. Another non-trivial differential is identified, related to the modulo p Bockstein in $K\mathbb{F}_p^*(X)$. In the case of Morava K-theory, there is a whole pyramid of non-trivial differentials related to $v_m^{(r)}$ Bocksteins, $m < n$ and r in a certain range, depending on m. These Bocksteins are, of course, related to those described in **[4]** as connecting maps in the tower of fibrations (1.1). We discuss this in **[9]**.

§5 The geometry of infinite loop spaces

In this section we discuss a method for constructing explicit cellular models for spaces in an Ω-spectrum. In particular, we shall be concerned with those representing $E(n)$ and $E(n)/I_n^k$. These constructions rely on Milgram's work **[12]** on the classifying space of a topological group and are basically the geometric point of view behind the construction of the cup product map in **[15]**. Although somewhat theoretical, we believe they shed light on the nature of the relationships between the infinite loop spaces we have been considering.

As before, we write $\mathbf{E}_* = \{\mathbf{E}_r; r \in \mathbb{Z}\}$ for an arbitrary Ω-spectrum. We begin by describing the method of construction for such $\mathbf{E}_*$.

First note that by the very definition of an Ω-spectrum we have relationships $\mathbf{E}_r = \Omega\mathbf{E}_{r+1}$. Hence $B\mathbf{E}_r = \mathbf{E}'_{r+1}$ where $B\mathbf{E}_r$ is the classifying space of $\mathbf{E}_r$ and $\mathbf{E}'_{r+1}$ denotes the connected component of the basepoint in $\mathbf{E}_{r+1}$. The whole space $\mathbf{E}_{r+1}$ can thus be recovered as

$$\mathbf{E}_{r+1} = B\mathbf{E}_r \times \pi_0(\mathbf{E}_{r+1}).$$

Recall that $\pi_0(\mathbf{E}_{r+1}) = E^{r+1}(\text{point}) = E_{-r-1}$.

In **[12]**, Milgram shows that given an H-space X with a CW structure, with respect to which the product is cellular, then a cellular construction of the classifying space BX is given by the bar construction on the cells of X. Thus BX is composed out of cells labelled $[e_1|\dots|e_u]$, where e_i is a cell of X and

$$\dim[e_1|\dots|e_u] = u + \sum_{i=1}^{u} \dim e_i. \tag{5.1}$$

The cell $[e_1|\dots|e_u]$ is said to have *bar degree* u.

Construct models for $\mathbf{E}_r$ inductively, as follows. Take the 0-skeleton for each $\mathbf{E}_r$ as a set of points in one to one correspondence with the group $\pi_0(\mathbf{E}_r)$, the product structure corresponding to the loop sum operation in $[S^0; \Omega\mathbf{E}_{r+1}]$. In general construct the t-skeleton for $\mathbf{E}_r$ from the $(t-1)$-skeleton of $\mathbf{E}_{r-1}$ by using Milgram's model: this is possible since (5.1) shows that all the t-cells of $B\mathbf{E}_r$ are given by elements $[e_1|\dots|e_u]$ with dim $e_i < t$. The cellular approximation theorem allows us to arrange the H-space product in $\mathbf{E}_r$ to be cellular.

This process not only builds models for the spaces $\mathbf{E}_r$, but also cellular representations for maps $\phi\colon \mathbf{E}_* \longrightarrow \mathbf{F}_*$ out of the morphisms $\phi_*\colon E_* \longrightarrow F_*$ on coefficients.

Of course this is just making explicit a technique used homologically in many Hopf ring calculations to date. It is this construction that lies behind the success of the iterated Rothenberg-Steenrod spectral sequences in such work as **[6]**, **[14]**, **[15]**, **[19]**, *etc.* as well as in our §3 above.

Digressing for a moment, in the case where the H-space product on $\mathbf{E}_r$ is abelian for each r (that is, when $\mathbf{E}_*$ is a generalised Eilenberg-MacLane spectrum), the product on $\mathbf{E}'_{r+1} = B\mathbf{E}_r$ is determined by that on $\mathbf{E}_r$ and is easily read off as the shuffle product on the bar complex. This is pointed out in **[12]**, where the ideas above are used to sketch an inductive construction of Eilenberg-MacLane spaces. For a general Ω-spectrum, however, the product is not abelian and fails to coincide with the shuffle product. We are making real use of the fact that we know there to be a product on these spaces (since they are all infinite loop spaces), and of the cellular approximation theorem to make it cellular. In fact, the divergence of the product on $\mathbf{E}_{r+1}$ from the shuffle product on $B\mathbf{E}_r$ appears as non-trivial extension problems in the E^∞-term of the RSSS, familiar to all who have thought about the $*$-multiplicative properties of Hopf rings via this spectral sequence (see, for example, **[14]**). On the other hand, observe the lack of non-trivial extension problems in the the work of **[18]**, §8, where the $\mathbb{F}_p$ homology of the modulo p Eilenberg-MacLane spaces is calculated by Hopf ring methods. See **[15]** for more discussion of the modulo p Eilenberg-MacLane spaces from this perspective.

Now apply this means of construction to the spectra $E(n)$ and $E(n)/I_n^k$ and to the maps $\phi^k \colon E(n) \longrightarrow E(n)/I_n^k$ inducing the quotient by I_n^k on coefficients. The following is immediate from the construction (compare with the inductive hypothesis (3.6) and the remarks at the end of §3).

Proposition 5.2 *With these constructions each map* $\phi^k \colon \mathbf{E(n)}_r \longrightarrow (\mathbf{E(n)}/\mathbf{I_n^k})_r$ *is a quotient map in the category of CW complexes. Moreover, on any given cell in* $\mathbf{E(n)}_r$*,* ϕ^k *is a monomorphism for sufficiently large* k*.*

Proof The cells in our constructions of $\mathbf{E(n)}_*$ and $(\mathbf{E(n)}/\mathbf{I_n^k})_*$ are labelled by all possible (finite) iterated bar terms of elements of the respective coefficient rings. The maps ϕ^k are given on these cells by the corresponding coefficient homomorphisms. However, any element of $E(n)_*$ is not in the kernel of the quotient map $E(n)_* \longrightarrow (E(n)/I_n^k)_*$ for all k sufficiently large. $\square$

For coefficients A write $\mathcal{C}_*(X;A)$ for the cellular chain complex of a CW complex X. Give $\mathcal{C}_*(\mathbf{E(n)}_r;A)$ a topology corresponding to the I_n-adic topology on the coefficient ring: a basis of open neighbourhoods of the identity is given by the kernels of the induced homomorphism $\phi^k_* \colon \mathcal{C}_*(\mathbf{E(n)}_r;A) \longrightarrow \mathcal{C}_*((\mathbf{E(n)}/\mathbf{I_n^k})_r;A)$. In fact, each element of this basis can be written in the form $A\{U_k\}$, $k = 1,2,\ldots$ where the U_k is a set of formal differences of cells in $\mathbf{E(n)}_r$. Note that the topology is Hausdorff by (5.2). Our CW construction gives an equivalence between the continuous cochain complex $\mathcal{C}^*_c(\mathbf{E(n)}_r;A) = \operatorname*{colim}_k \mathcal{C}^*((\mathbf{E(n)}/\mathbf{I_n^k})_r;A)$ and the set of continuous maps $\mathcal{C}_*(\mathbf{E(n)}_r;\mathbb{Z}) \longrightarrow A$, where A is given the discrete topology.

Corollary 5.3 $H^*_c(\widehat{\mathcal{E}(n)}_r;\mathbb{F}_p) = H(\mathrm{Hom_c}(\mathcal{C}_*(\mathbf{E(n)}_r;\mathbb{F}_p)))$. $\square$

This result gives the following perspective on the results of §3. Firstly, we have a continuous universal coefficient theorem for $\mathbf{E(n)}_r$,

$$\mathrm{Hom_c}(H_*(\mathbf{E(n)}_r;\mathbb{F}_p)) = H(\mathrm{Hom_c}(\mathcal{C}_*(\mathbf{E(n)}_r;\mathbb{F}_p))).$$

and secondly, it explains the relationship between this topology on $H_*(\mathbf{E(n)}_r;\mathbb{F}_p)$ and the $[I_n]$-adic topology defined in the remark (3.13): the iterated RSSS calculation for $H_*(\mathbf{E(n)}_*;\mathbb{F}_p)$ and standard identification of Hopf ring elements in the spectral sequence (analagous to the one in **[14]** for $H_*(\mathbf{MU}_*;\mathbb{F}_p)$) shows that every $x \in \operatorname{Ker}(\psi^k)$ has a representative in the open set $\mathbb{F}_p\{U_k\} \subset \mathcal{C}_*(\mathbf{E(n)}_*;\mathbb{F}_p)$ (see (3.13) for the definition of ψ^k).

Remark 5.4 The same iterated bar construction gives a CW model for each space $\widehat{\mathbf{E(n)}}_r$. This is a CW complex whose set of n cells and attaching maps are given by the inverse limit of the sets of n cells and attaching maps of the models for $(\mathbf{E(n)}/\mathbf{I_n^k})_r$. The completion map $E(n)_* \longrightarrow \widehat{E(n)}_*$ induces an embedding of our model of $\mathbf{E(n)}_r$ in our model of $\widehat{\mathbf{E(n)}}_r$.

As far as the application of our work in §4 goes, it would have sufficed to compute the Hopf ring for $\widehat{E(n)}$ and, provided $K(n)^*(\widehat{\mathbf{E(n)}}_{2r})$ was nilpotent free, proceed along the lines of **[8]** without any need for $K(n)^*_c(-)$. Unfortunately, it is not clear how to set about computing the cohomology of a general homotopy inverse limit.

In **[6]** we showed that if the Ω-spectrum $\mathbf{E}_*$ represented a Landweber exact theory with homotopy free over some subring R of the rationals, then each space $\mathbf{E}_r$, after p-localisation, was the direct limit of products of Wilson spaces $B(p,n)$, the irreducible H-spaces that the spaces in the Ω-spectrum for BP factor into, **[17]**, the maps in the direct system being inclusions of subproducts. One might hope that as $\widehat{E(n)}$ also gives rise to a Landweber exact theory (see **[3]**), the spaces $\widehat{\mathbf{E(n)}}_r$ could be expressed perhaps as a simple inverse limit of finite products of Wilson spaces, the maps in the inverse system being projections. An inspection of the requirements of the homotopy of such spaces seems to indicate that no such nice description could exist: examining $\pi_0(-)$, for example, it would require that the p-adic integers, $\mathbb{Z}_p$, be written as an inverse limit of copies of the p-local integers $\mathbb{Z}_{(p)}$.

However, combining the remark (5.4) and the description of $\mathbf{E(n)}_r$ as a colimit of finite products of Wilson spaces, we arrive at the following result:

Proposition 5.5 *The spaces $\widehat{\mathbf{E(n)}}_r$ in the Ω-spectrum for $\widehat{E(n)}$ are the inverse limits of finite products of cellular quotients of Wilson spaces.* □

Using the iterated bar construction of this section for the Wilson spaces (they all arise as terms in certain Ω-spectra), this decomposition of $\widehat{\mathcal{E}(n)}_r$ can be made more explicit.

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