A_∞ STRUCTURES ON SOME SPECTRA RELATED TO MORAVA K-THEORIES

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Abstract. Let p denote an odd prime. We show that the spectrum $\widehat{E(n)}$, the I_n-adic completion of Johnson and Wilson's $E(n)$, admits a unique topological A_∞ structure compatible with its canonical ring spectrum structure. Furthermore, the canonical morphism of ring spectra $\widehat{E(n)} \longrightarrow K(n)$ admits an A_∞ structure whichever of the uncountably many A_∞ structures of A. Robinson is imposed upon $K(n)$, the nth Morava K-theory at the prime p.

We construct an inverse system of A_∞ module spectra over $\widehat{E(n)}$

$$\cdots \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k \longrightarrow \cdots \longrightarrow E(n)/I_n = K(n)$$

for which

$$\underset{\underset{k}{\longleftarrow}}{\operatorname{holim}}\, E(n)/I_n^k \simeq \widehat{E(n)}.$$

§0 Introduction.

Recently, A. Robinson has described a theory of A_∞ ring spectra, their module spectra and the associated derived categories (see [9], [10], [11], [12]). As a special case, in [12] he showed that at an odd prime p the nth Morava K-theory spectrum $K(n)$ admits uncountably many distinct A_∞ structures compatible with its canonical multiplication.

The principal result of the present work is to show that $\widehat{E(n)}$, the (Noetherian) I_n-adic completion of the spectrum $E(n)$ defined by D. C. Johnson and W. S. Wilson, admits a unique *topological* A_∞ structure compatible with its canonical ring spectrum structure; moreover, the canonical morphism of ring spectra $\widehat{E(n)} \longrightarrow K(n)$ can be given the structure of an A_∞ morphism *whichever* of Robinson's A_∞ structures we take.

As an application, we construct an inverse system of A_∞ module spectra over $\widehat{E(n)}$

$$\cdots \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k \longrightarrow \cdots \longrightarrow E(n)/I_n = K(n)$$

Key words and phrases. A_∞ structure, Morava K-theory.

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for which

$$\underset{\underset{k}{\longleftarrow}}{\operatorname{holim}}\, E(n)/I_n^k \simeq \widehat{E(n)}.$$

For each $k \geq 0$ we have $\pi_* \left(E(n)/I_n^k\right) \cong E(n)_*/I_n^k$ as a module over $\widehat{E(n)}_*$. Associated to this is a spectral sequence for each spectrum X, converging to $\widehat{E(n)}^*(X)$ and whose E_1 term is a direct sum of copies of $K(n)^*(X)$ with differentials constructed from certain "Bockstein" operations (this was first observed to the author by U. Würgler).

It is the author's contention that these results support the view that Morava K-theory $K(n)^*(\ )$ is best thought of as "$\widehat{E(n)}$ modulo I_n", and that $\widehat{E(n)}$ is in many regards more fundamental. This is analogous to the classical case of p-local and mod p cohomology. For more on this see §4 of [3].

I would like to thank Alan Robinson and Urs Würgler for sharing their insights and enthusiasm, and commenting on versions of this paper.

We refer the reader to the books of Adams [1] and Ravenel [8] for all background material and otherwise unexplained notation.

§1 A_∞ structures on topological spectra.

Let $\mathbf{C}$ be category. Then an object $T \in \mathbf{C}$ is said to be a *topological (or topologised) object in* $\mathbf{C}$ if the functor $\mathbf{C}(\ \ ,T)$ takes values in the category of topological spaces **TopSp**. If T_1, T_2 are such topological objects, we say that a morphism $\varphi \in \mathbf{C}(T_1, T_2)$ is *continuous* if the induced natural transformation

$$\overline{\varphi} : \mathbf{C}(\ \ ,T_1) \longrightarrow \mathbf{C}(\ \ ,T_2)$$

is a natural transformation of **TopSp** valued functors. Clearly, the collection of topological objects and continuous morphisms forms an overcategory $\mathbf{C}^{\mathbf{Top}}$ of $\mathbf{C}$.

For example, if $\mathbf{C} = \mathbf{Groups}$ is the category of groups, then a topological object G is a topological group, as can be seen by considering the morphism set $\mathbf{Groups}(\mathbb{Z}, G)$. An even more basic example is provided by **Sets**, the category of sets, in which the topological objects are the topological spaces! This can be seen by making use of the one point set.

Now consider the homotopy category of spectra, **hSpectra**. Of course, we must here fix on a particular version for this and we prefer to use that one constructed in [13] for technical reasons. Thus we obtain the homotopy category of *topological spectra*, $\mathbf{hSpectra}^{\mathbf{Top}}$. Now for a topological spectrum $T \in \mathbf{hSpectra}$ we can also view T as an object of the category of spectra **Spectra**; given two such topological spectra T_1 and T_2 we say that a map of spectra $\theta : T_1 \longrightarrow T_2 \in \mathbf{Spectra}$ is *a continuous map of spectra* if the homotopy class of θ is a continuous morphism of spectra, i.e. is in $\mathbf{hSpectra}^{\mathbf{Top}}(T_1, T_2)$. We can form the category of such topological spectra and continuous maps as an overcategory $\mathbf{Spectra}^{\mathbf{Top}}$ of **Spectra**. Notice that the canonical functor $\mathbf{Spectra} \longrightarrow \mathbf{hSpectra}$ maps $\mathbf{Spectra}^{\mathbf{Top}}$ onto $\mathbf{hSpectra}^{\mathbf{Top}}$. We will often use the notation

$$(T_2)^*_{\mathrm{c}}(T_1) = \mathbf{hSpectra}^{\mathbf{Top}}(T_1, T_2),$$

where $*$ indicates the usual grading on morphisms in **hSpectra**.

Now suppose that E is a ring spectrum, which is also a topological spectrum. Then we say that E is a *topological ring spectrum* if the structure maps are continuous maps of spectra. We clearly have a related notion of topological module spectra over such topological ring spectra.

In [9], [10], [11] and [12], a theory of A_∞ ring spectra and their module spectra was described. We claim that the whole of that work can be applied to the category of topological ring spectra to give a theory of A_∞ topological ring spectra and topological module spectra. We leave the details to the reader. We observe however that the definitions of the A_n structure maps make use of maps of spectra of form

$$K_n \ltimes E^{(n)} \longrightarrow E$$

where we take the Stasheff cell K_n as a discretely topological object in **Spectra** and similarly for the sphere S^n.

Assuming that such a theory works satisfactorily, we obtain the following results based upon [12,§1].

Let E be a topological ring spectrum and assume that $E_*(E)$ is flat as a left or right E_* module. Then there are two natural equivalences of cohomology theories

$$(E \wedge E)^*(\ ) \cong E_*(E) \otimes_{E_*} E^*(\ ) \tag{1.1}$$

defined using the two E_* module structures on $E_*(E)$. Now we can topologise $E_*(E)$ in two ways as a topological E_* module, by decreeing that under either of the left or right units $E_* \xrightarrow{\eta_L} E_*(E) \xleftarrow{\eta_R} E_*$, the images of open ideals in E_* generate the open ideals in $E_*(E)$. This gives the theory $(E \wedge E)^*(\ )$ the structure of a topologically valued cohomology theory in two distinct but canonically isomorphic ways; similarly, $E \wedge E$ inherits two distinct but isomorphic topological structures. In the case of $E = \widehat{E(n)}$ which we will consider later, these are actually equal, but this is false in general.

Now assume further that

$$E^*_{\mathrm{c}}\left(E^{(k)}\right) \cong \mathcal{H}om_{E_*}\left(E_*(E)^{\otimes k}, E_*\right) \tag{1.2}$$

where $\mathcal{H}om$ denotes E_* homomorphisms continuous with respect to the right hand topology on $E_*(E)^{\otimes k}$. We have the following result obtained by modifying details in [12], in particular [12,Theorem 1.11].

Theorem (1.3). *If E has a topological A_{n-1} structure for $n \geq 4$, then the obstruction to extending the underlying topological A_{n-2} structure to an A_n structure is a certain element of the continuous Hochschild cohomology group*

$$\mathcal{HH}^{n,3-n}\left(E_*(E), E_*\right)$$

which vanishes if and only if such an extension exists.

The following related result is obtained from a modification of [12,Theorem 1.14] applied to the identity map $E \longrightarrow E$; unfortunately, the published version of that result is incorrect and care needs to be taken in using it. Following discussions with the referee of an earlier version of the present paper and Alan Robinson, we have

Theorem (1.4). *Suppose that E has a given topological A_∞ structure. If all the continuous Hochschild cohomology groups*

$$\mathcal{HH}^{m,2-m}\left(E_*(E), E_*\right)$$

vanish for $m \geq n$, then there is a unique topological A_∞ structure extending the underlying A_{n-2} structure.

In the examples which we will consider in this paper, the relevant Hochschild groups in (1.3) and (1.4) vanish, hence we will have no obstructions to obtaining A_∞ structures and these will be unique. This contrasts with the situation for Morava K-theory $K(n)$ dealt with in [12] where although A_∞ structures exist because the obstruction groups of (1.3) are zero, all of the elements in

$$\mathrm{HH}^{m,2-m}\left(K(n)_*(K(n)), K(n)_*\right) \neq 0$$

are required to parametrise the extensions from A_{m-2} to A_m structures– this leads to uncountably many distinct A_∞ structures.

§2 The spectrum $\widehat{E(n)}$.

Recall that for each prime p and $n > 0$, there is a multiplicative, complex oriented, cohomology theory $E(n)^*(\ )$ on the category of finite CW spectra $\mathbf{CW^f}$, for which the coefficient ring is

$$E(n)_* = v_n^{-1} BP_*/(v_k : k > n) \tag{2.1}$$

and by definition

$$E(n)^*(\ ) = E(n)_* \otimes_{BP_*} BP^*(\ ) \tag{2.2}$$

where the tensor product is taken with respect to the obvious module structures. In the above, we denote by $v_k \in BP_{2(p^k-1)}$ the kth *Araki generator*, uniquely specified by the requirement that

$$[p]_{BP}(X) = \sum_{0\leq k}^{BP} \left(v_k X^{p^k} \right) \tag{2.3}$$

and agreeing with the more commonly used *Hazewinkel generators* modulo p.

We can define a completed version of the theory $E(n)^*(\ )$ by

$$\widehat{E(n)}^*(\ ) = \varprojlim_r E(n)^*(\ )/I_n^r E(n)^*(\ ). \tag{2.4}$$

From [2] we have the following facts.

Proposition (2.5). *The functor $\widehat{E(n)}^*(\ )$ defines a multiplicative, complex oriented cohomology theory on $\mathbf{CW^f}$, taking values in the category of finitely generated, complete topological modules over $\widehat{E(n)}_*$. Moreover, this theory is uniquely (up to equivalence) representable by a commutative topological ring spectrum $\widehat{E(n)}$, and hence admits a unique (up to canonical natural equivalence) extension to the full stable category* **hSpectra**.

The coefficient ring $\widehat{E(n)}_*$ is the Noetherian completion of $E(n)_*$ with respect to the graded maximal ideal $I_n = (v_k : 0 \leq k < n)$, and is therefore *flat* over $E(n)_*$ by [6]– this is the essential idea in the proof. We also have

$$\begin{aligned}\widehat{E(n)}^*(\ ) &= \widehat{E(n)}_* \otimes_{BP_*} BP^*(\ ) \\ &= \widehat{E(n)}_* \otimes_{v_n^{-1}BP_*} v_n^{-1}BP^*(\ ).\end{aligned}$$

A fundamental property of this cohomology theory (on **hSpectra**) is that it is totally determined by its restriction to $\mathbf{CW^f}$. This is the import of the following crucial result.

Proposition (2.6). *Let $F^*(\ )$ be a cohomology theory on $\mathbf{CW^f}$, represented by the spectrum F. Let $\Phi\colon F^*(\ ) \longrightarrow \widehat{E(n)}^*(\ )$ be a natural transformation. Then there is a unique morphism of spectra $F \longrightarrow \widehat{E(n)}$ inducing Φ.*

Again the proof appears in [2] and depends upon the fact that (continuous) inverse limits of *linearly compact* modules are linearly compact and have vanishing higher derived functors of $\varprojlim$.

Notice that for any spectrum $Z = \operatorname*{colim}_{\overrightarrow{\alpha}} Z_\alpha$ where $Z_\alpha \in \mathbf{CW^f}$ we have

$$\begin{aligned}\widehat{E(n)}_*(Z) &\cong \operatorname*{colim}_{\overrightarrow{\alpha}} \widehat{E(n)}_*(Z_\alpha)\\ &\cong \operatorname*{colim}_{\overrightarrow{\alpha}} \widehat{E(n)}_* \otimes_{E(n)_*} E(n)_*(Z_\alpha)\,.\end{aligned}$$

From this we can deduce the important

Lemma (2.7). *There is an isomorphism of topological $\widehat{E(n)}_*$ bimodules*

$$\widehat{E(n)}_*\left(\widehat{E(n)}\right) \cong \widehat{E(n)}_* \otimes_{E(n)_*} E(n)_*(E(n)) \otimes_{E(n)_*} \widehat{E(n)}_*.$$

Now recall from [7] that

$$E(n)_*(E(n)) = E(n)_*(t_k : 1 \leq k) \tag{2.8}$$

where the generators t_k satisfy polynomial relations over $E(n)_*$ of form

$$t_k^{p^n} \equiv v_n^{p^k-1} t_k \mod I_n. \tag{2.9}$$

Here we abuse notation and use I_n to denote the ideal in $E(n)_*(E(n))$ generated by the image of $I_n \lhd E(n)_*$ under either of the left or right units $\eta_{\mathrm{L}}, \eta_{\mathrm{R}}$ which coincide as I_n is an *invariant* ideal in $E(n)_*$. Notice that $E(n)_*(E(n))$ is generated as a module over $E(n)_*$ by the elements

$$t_1^{r_1} t_2^{r_2} \dots t_d^{r_d}$$

with $0 \leq r_k \leq p^n - 1$ for all k. Similarly, $\widehat{E(n)}_*(E(n))$ is generated over $\widehat{E(n)}_*$ by the same elements, and $\widehat{E(n)}_*(\widehat{E(n)})$ is topologically generated by these elements. However, it is not clear if these are *free* modules over the stated rings. Instead they are *flat* modules over the rings $E(n)_*$ and $\widehat{E(n)}_*$ respectively. To see this we reproduce the following argument from [7,remark 3.7].

Let $E = E(n)$ or $\widehat{E(n)}$. Then by [4], the ring E_* is flat on the category of finitely presented $BP_*(BP)$ comodules. Hence we have for any module M_* over E_*, the following sequence of isomorphisms (of left E_* modules):

$$\begin{aligned}E_*(E) \otimes_{E_*} M_* &\cong (E_*(BP) \otimes_{BP_*} E_*) \otimes_{E_*} M_*\\ &\cong (E_* \otimes_{BP_*} BP_*(BP)) \otimes_{BP_*} M_*\\ &\cong E_* \otimes_{BP_*} (BP_*(BP) \otimes_{BP_*} M_*).\end{aligned}$$

Now as the BP_* module $BP_*(BP) \otimes_{BP_*} M_*$ is an *extended* $BP_*(BP)$ comodule (and hence a colimit of finitely presented $BP_*(BP)$ comodules), the flatness of E_* and the freeness of $BP_*(BP)$ over BP_* implies that for $s > 0$,

$$\mathrm{Tor}^{s,*}_{E_*}(E_*(E), M_*) \cong \mathrm{Tor}^{s,*}_{BP_*}(E_*, BP_*(BP) \otimes_{BP_*} M_*) = 0.$$

Thus we have established the flatness of $E_*(E)$. Of course this argument is equally valid for any algebra E_* over BP_* satisfying the conditions required for Landweber's Exact Functor Theorem to apply. We can easily modify this argument for the case of $\widehat{E(n)}_*(E(n))$.

Of course, $\widehat{E(n)}_*\left(\widehat{E(n)}\right)$ is a left topological $\widehat{E(n)}_*$ module with a second topology inherited from the right hand factor of $\widehat{E(n)}$, which happens to agree with the left hand topology! This common topology is not *Hausdorff*; indeed, the intersection

$$\bigcap_r I_n^r \widehat{E(n)}_*\left(\widehat{E(n)}\right)$$

contains infinitely I_n divisible elements and is a summand. An analogous construction worth considering is $\mathbb{Z}_p \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$, which has similar features. Despite this seeming pathology we will be able to make use of $\widehat{E(n)}_*\left(\widehat{E(n)}\right)$ in a universal coefficient type result.

Using the above result that $\widehat{E(n)}_*(\widehat{E(n)})$ is flat over $\widehat{E(n)}_*$, we see that the functor

$$\widehat{E(n)}_*(\widehat{E(n)}) \otimes_{\widehat{E(n)}_*} \widehat{E(n)}^*(\ ) \cong \widehat{E(n)}_*(\widehat{E(n)}) \otimes_{E(n)_*} E(n)^*(\ )$$

is a cohomology theory on $\mathbf{CW^f}$, represented by $\widehat{E(n)} \wedge \widehat{E(n)}$. Similarly, we have

$$\widehat{E(n)}_*(\widehat{E(n)}^{\wedge m}) \cong \widehat{E(n)}_*(\widehat{E(n)})^{\otimes m},$$

where the tensor power is defined over the ring $\widehat{E(n)}_*$.

We have

Theorem (2.10). *Let F be a topological module spectrum over $\widehat{E(n)}$ such that for each $Z \in \mathbf{CW^f}$, $F^*(Z)$ is a linearly compact $\widehat{E(n)}_*$ module. Then there is an isomorphism*

$$F^*_{\mathrm{c}}(\widehat{E(n)}) \cong \mathcal{H}om_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right), F_*\right)$$

where $\mathcal{H}om$ denotes topological homomorphism.

Proof. By methods of [2], a continuous natural transformation (on $\mathbf{CW^f}$)

$$\overline{\theta} : \widehat{E(n)}^*(\ ) \longrightarrow F^*(\ )$$

determines a unique (up to natural equivalence) morphism of spectra $\theta : \widehat{E(n)} \longrightarrow F$, and this must be continuous. Moreover, such a morphism determines an $\widehat{E(n)}_*$ module homomorphism which is the composite

$$\widehat{E(n)}_*\left(\widehat{E(n)}\right) \xrightarrow{\theta_*} \widehat{E(n)}_*(F) \xrightarrow{\cong} \pi_*\left(\widehat{E(n)} \wedge F\right) \to \pi_*(F) = F_*$$

defined using the module structure.

Conversely, a continuous homomorphism $h_* : \widehat{E(n)}_*\left(\widehat{E(n)}\right) \longrightarrow F_*$ determines a continuous natural transformation $\overline{h} : \widehat{E(n)}^*(\ ) \longrightarrow F^*(\ )$ as the composite

$$\begin{aligned}
\widehat{E(n)}^*(\ ) \xrightarrow{\cong} \left(S^0 \wedge \widehat{E(n)}\right)^*(\ ) &\xrightarrow{\bar{e}} \left(\widehat{E(n)} \wedge \widehat{E(n)}\right)^*(\ ) \\
&\xrightarrow{\cong} \widehat{E(n)}_*\left(\widehat{E(n)}\right) \otimes_{\widehat{E(n)}_*} \widehat{E(n)}^*(\ ) \\
&\xrightarrow{1\otimes h} \widehat{E(n)}^*(\ ) \otimes_{\widehat{E(n)}_*} F_* \\
&\xrightarrow{\cong} \widehat{E(n)}^*(\ ) \otimes_{\widehat{E(n)}_*} F^*(S^0) \\
&\longrightarrow F^*\left(\ \wedge S^0\right) \xrightarrow{\cong} F^*(\ ).
\end{aligned}$$

Here the last map uses the external multiplication coming from the $\widehat{E(n)}$ module structure on F. This proves the theorem. $\square$

More generally we have

Theorem (2.11). *There is an isomorphism*

$$\widehat{E(n)}^*_{\mathrm{c}}(\widehat{E(n)}^{\wedge m}) \cong \mathcal{H}om_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right)^{\otimes m}, \widehat{E(n)}_*\right).$$

The proof is similar to that of (2.10) and uses our earlier remark that $\widehat{E(n)}_*(\widehat{E(n)})$ is flat over $E(n)_*$ on the category of $E(n)_*(E(n))$ comodules. Of course, this verifies (1.2) for $\widehat{E(n)}$.

§3 Topological A_∞ structures on $\widehat{E(n)}$.

The principal goal of the present section is to prove the following result.

Theorem (3.1). *The ring spectrum $\widehat{E(n)}$ has a unique equivalance class of A_∞ structures compatible with its canonical ring spectrum structure. Moreover, the natural morphism of ring spectra $\widehat{E(n)} \longrightarrow K(n)$ is an A_∞ morphism for* **any** *of the A_∞ structures on $K(n)$ compatible with the canonical ring spectrum structure on the latter.*

The proof of this will follow from (1.3) and (1.4) together with the fact that certain continuous Hochschild cohomology groups vanish.

Theorem (3.2). *The following continuous Hochschild cohomology groups are as stated:*

$$\mathcal{HH}^{r,*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right), \widehat{E(n)}_*\right) = \begin{cases} \widehat{E(n)}_* & \text{if } r = 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathcal{HH}^{r,*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right), K(n)_*\right) = \begin{cases} K(n)_* & \text{if } r = 0, \\ 0 & \text{otherwise,} \end{cases}$$

where in the second equation Hochschild cohomology is taken with respect to the discrete topology on $K(n)_*$ and hence agrees with the ordinary version.

To compute such groups, we must first recall that the continuous Hochschild cochains in this case are given by

$$
\begin{aligned}
\mathcal{C}^{k,*}\left(\widehat{E(n)}_*\right) &= \mathcal{H}om^*_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right)^{\otimes k}, \widehat{E(n)}_*\right) \\
&= \varprojlim_r \mathcal{H}om^*_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right)^{\otimes k}, \widehat{E(n)}_*/I_n^r\right) \\
&= \varprojlim_r \mathrm{Hom}^*_{E(n)_*/I_n^r}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right)^{\otimes k}/I_n^r, \widehat{E(n)}_*/I_n^r\right) \\
&= \varprojlim_r C^{k,*}\left(E(n)_*/I_n^r\right).
\end{aligned} \tag{3.3}
$$

Observe that each projection

$$C^{k,*}\left(E(n)_*/I_n^{r+1}\right) \longrightarrow C^{k,*}\left(E(n)_*/I_n^r\right)$$

is surjective and hence

$$\varprojlim_r{}^1\, C^{k,*}\left(E(n)_*/I_n^r\right) = 0.$$

By results of [5, Part I Chap 3] we have an exact sequence for each k of the form

$$
\begin{aligned}
0 \to \varprojlim_r{}^1\, \mathrm{HH}^{k-1,*}\left(E(n)_*\big(E(n)\big)/I_n^r, E(n)_*/I_n^r\right) &\longrightarrow \mathcal{HH}^{k,*}\left(\widehat{E(n)}_*\left(\widehat{E(n)}\right), \widehat{E(n)}_*\right) \\
&\longrightarrow \varprojlim_r \mathrm{HH}^{k,*}\left(E(n)_*\big(E(n)\big)/I_n^r, E(n)_*/I_n^r\right) \to 0
\end{aligned}
$$

We will now prove

Lemma (3.4). *For each $r \geq 1$, we have the following Hochschild cohomology groups:*

$$\mathrm{HH}^{k,*}\left(E(n)_*\big(E(n)\big)/I_n^r, E(n)_*/I_n^r\right) = \begin{cases} E(n)_*/I_n^r & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will require a version of the *infinite dimensional Hensel's Lemma* of [2]. In fact, we only need a very simple case, namely that where the equations each involve a single variable; we make do with a modification of the argument used in that earlier work.

Recall from (2.8) that

$$E(n)_*\left(E(n)\right) = E(n)_*(t_k : k \geq 1)$$

where the t_k satisfy relations

$$f_k(\mathbf{t}) = 0$$

with $f_k(\mathbf{X}) \in E(n)_*[X_k : k \geq 1]$ a polynomial satisfying

$$f_k(\mathbf{X}) \equiv v_n X_k^{p^n} - v_n^{p^k} X_k \mod I_n. \tag{3.5}$$

Thus, the sequence $\mathbf{t} = (t_i)_{i\geq 1}$ in $E(n)_*(E(n))$ is a zero modulo I_n of the sequence of polynomials $\mathbf{g} = (g_j)_{j\geq 1}$ where

$$g_j(\mathbf{X}) = v_n X_j^{p^n} - v_n^{p^j} X_j. \tag{3.6}$$

Now notice that the derivative matrix of $\mathbf{g}$ has the form

$$\mathrm{d}\mathbf{g} \equiv \begin{pmatrix} -v_n^p & 0 & 0 & \cdots \\ 0 & -v_n^{p^2} & 0 & \cdots \\ 0 & 0 & -v_n^{p^3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mod I_n. \tag{3.7}$$

Now define a sequence (of sequences) $(\mathbf{s}_m)_{m\geq 1}$ by

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{t} \\ \mathbf{s}_{m+1} &\equiv \mathbf{s}_m - \mathrm{d}\mathbf{g}(\mathbf{t})^{-1}\mathbf{g}(\mathbf{s}_m) \mod I_n^{m+1} \end{aligned} \tag{3.8}$$

where

$$\mathrm{d}\mathbf{g}(\mathbf{t})^{-1} = \begin{pmatrix} -v_n^{-p} & 0 & 0 & \cdots \\ 0 & -v_n^{-p^2} & 0 & \cdots \\ 0 & 0 & -v_n^{-p^3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to verify that for each $m \geq 1$,

$$\mathbf{g}(\mathbf{s}_m) \equiv \mathbf{0} \mod I_n^m$$

and that we have thus define a *Cauchy sequence* of sequences in $E(n)_*(E(n))$ with respect to the I_n-adic topology. Of course, this topology is neither complete nor Hausdorff so the sequence does not have a limit!

Now for each $k \geq 1$ we can consider $E(n)_*(E(n))/I_n^k$ which *is* complete and Hausdorff with respect to the I_n-adic topology (in fact it is discrete) and hence the above procedure gives a sequence with limit term $\mathbf{s}_k$ and this is a simultaneous zero of the polynomials g_j. Notice that we have $\mathbf{s}_k = (s_{k,i})_{i\geq 1}$ with

$$s_{k,i} \equiv t_i \mod I_n$$

and hence the elements $s_i = s_{k,i}$ can be taken as algebra generators in place of the t_i.

We now see that there is an isomorphism of algebras over $E(n)_*/I_n^k$,

$$E(n)_*(E(n))/I_n^k \cong \bigotimes_{j\geq 1} \left[(E(n)_*/I_n^k) \times (E(n)_*/I_n^k)[S_j]/(S_j^{p^n-1} - v_n^{p^j-1})\right]. \tag{3.9}$$

Here, in each tensor product factor the augmentation onto $E(n)_*/I_n^k$ sends the (ring) summand of the form $E(n)_*(S_j)$ to 0 and is the identity on the summand of the form $E(n)_*/I_n^k$.

Just as in [12] we have

$$\mathrm{HH}^{*,*}\left(E(n)_*(E(n))/I_n^k, E(n)_*/I_n^k\right) = E(n)_*/I_n^k \tag{3.10}$$

with all terms in bidegrees not of form $(0,*)$ being 0.

This completes the proof of Lemma (3.3), and Theorem (3.1) follows from this. $\square$

§4 Some discrete module spectra over $\widehat{E(n)}$.

We now turn our attention to the construction of certain module spectra over $\widehat{E(n)}$, both as a demonstration of the usefulness of the techniques and because we will use these spectra in [3]. The simplest example is provided by $K(n)$ which we already know is an A_∞ algebra spectrum over $\widehat{E(n)}$ (see (3.2)), hence is a module spectrum. Let $\varphi_1\colon \widehat{E(n)} \wedge K(n) \longrightarrow K(n)$ denote the A_∞ product map. We will need some information on the spectrum $\mathrm{REnd}_{\widehat{E(n)}}(K(n)) = \mathrm{RHom}_{\widehat{E(n)}}(K(n), K(n))$ of [10]. We will determine the homotopy of this spectrum using a spectral sequence of [10].

Recall the *Koszul complex* of $K(n)_*$ as a module over $\widehat{E(n)}_*$:

$$\mathbf{K}\left\langle K(n)_*\right\rangle_{*,*} = \widehat{E(n)}_*\left(e_0, e_1, \dots, e_{n-1}\right) \xrightarrow{\varepsilon} K(n)_*$$

where $e_k \in \mathbf{K}\left\langle K(n)_*\right\rangle_{1,2(p^k-1)}$ is an exterior generator, the differential ∂ is given by

$$\partial(e_k) = v_k$$

and ε is the reduction of the augmentation map given by

$$\begin{aligned} &\varepsilon(e_k) = 0 \\ &\varepsilon(u) = \overline{u} \in \widehat{E(n)}_*/I_n = K(n)_* \quad \text{if } u \in \widehat{E(n)}_*. \end{aligned}$$

Thus, we can calculate

$$\mathrm{Ext}^{*,*}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right)$$

as the cohomology of the complex whose rth term is

$$\mathrm{Hom}^*_{\widehat{E(n)}_*}\left(\mathbf{K}\left\langle K(n)_*\right\rangle_{r,*}, K(n)_*\right)$$

with trivial differential $\delta = \partial^*$. So we have

$$\mathrm{Ext}^{*,*}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right) = K(n)_*\{\varepsilon_{i_1 i_2 \dots i_r} : 0 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq n-1\} \tag{4.1}$$

where for each sequence $0 \leq j_1 \leq j_2 \leq \dots \leq j_r$ we have

$$\varepsilon_{i_1 i_2 \dots i_r}(e_{j_1} e_{j_2} \cdots e_{j_r}) = \begin{cases} 1 & \text{if } j_k = i_k \text{ for each } k, \\ 0 & \text{otherwise,} \end{cases}$$

thus defining a $K(n)_*$ basis.

Now recall from [13] that there is spectral sequence

$$\mathrm{E}_2^{s,t}\left(K(n), K(n)\right) = \mathrm{Ext}^{-s,-t}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right) \Longrightarrow \pi_{s+t}\left(\mathrm{REnd}_{\widehat{E(n)}}\left(K(n)\right)\right) \tag{4.2}$$

which does converge here because modules over $\widehat{E(n)}_*$ have finite homological dimension (always less than or equal to n). We will show that this has trivial differentials.

Theorem (4.3). *The natural morphism of spectra*

$$\mathrm{REnd}_{\widehat{E(n)}}(K(n)) \longrightarrow \mathbf{Spectra}(K(n), K(n))$$

induces a monomorphism

$$\pi_*\left(\mathrm{REnd}_{\widehat{E(n)}}(K(n))\right) \longrightarrow K(n)^*(K(n)).$$

and the spectral sequence $\mathrm{E}_r^{*,*}(K(n), K(n))$ *collapses from* E_2 *onwards.*

Proof. The spectrum $\widehat{E(n)} \wedge K(n)$ is a left A_∞ module spectrum over $\widehat{E(n)}$ and the multiplication map $\varphi_1 \colon \widehat{E(n)} \wedge K(n) \longrightarrow K(n)$ is a morphism of A_∞ module spectra over $\widehat{E(n)}$, thus it induces a morphism of spectral sequences

$$\text{(4.4)} \qquad \begin{array}{ccc} \mathrm{E}_2^{s,t}(K(n), K(n)) & \Longrightarrow & \pi_{s+t}\left(\mathrm{REnd}_{\widehat{E(n)}}(K(n))\right) \\ {\scriptstyle \varphi_1^*}\downarrow & & \downarrow \\ \mathrm{E}_2^{s,t}\left(\widehat{E(n)} \wedge K(n), K(n)\right) & \Longrightarrow & K(n)^{-s-t}(K(n)) \end{array}$$

– here the second spectral sequence has

$$\mathrm{E}_2^{s,t}\left(\widehat{E(n)} \wedge K(n), K(n)\right) = \mathrm{Ext}^{-s,-t}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(K(n)), K(n)_*\right)$$

and converges to

$$\text{(4.5)} \qquad \begin{aligned} \pi_{s+t}\left(\mathrm{RHom}_{\widehat{E(n)}}\left(\widehat{E(n)} \wedge K(n), K(n)\right)\right) &\cong \pi_{s+t}\left(\mathbf{Spectra}(K(n), K(n))\right) \\ &\cong K(n)^{-(s+t)}(K(n)) \end{aligned}$$

as described in [10] and [11] (this is just a universal coefficient type spectral sequence).

Now consider the free $\widehat{E(n)}_*$ algebra

$$\Gamma_* = \widehat{E(n)}_*(t'_k : k \geq 1)$$

subject to relations of the form

$$t_k'^{\,p^n} = v_n^{p^k-1} t'_k.$$

Observing that

$$K(n)_* \otimes_{\widehat{E(n)}_*} \Gamma_* \cong K(n)_*(E(n))$$

we see that the Koszul resolution $\mathbf{K}\langle K(n)_*\rangle_{*,*} \longrightarrow K(n)_*$ gives rise to a free $\widehat{E(n)}_*$ resolution

$$\Gamma_* \otimes_{\widehat{E(n)}_*} \mathbf{K}\langle K(n)_*\rangle_{*,*} \longrightarrow \Gamma_* \otimes_{\widehat{E(n)}_*} K(n)_*.$$

We also have that

$$\begin{aligned} \widehat{E(n)}_*(K(n)) &\cong K(n)_*(\widehat{E(n)}) \\ &\cong K(n)_*(E(n)) \\ &= K(n)_*(t_k : k \geq 1) \\ &\cong \Gamma_* \otimes_{\widehat{E(n)}_*} K(n)_*. \end{aligned} \tag{4.6}$$

It is now easy to see that

$$\mathrm{Ext}^{r,*}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*\left(K(n)\right), K(n)_*\right) \cong \mathrm{Hom}_{\widehat{E(n)}_*}\left(\Gamma_*, \mathrm{Ext}^{r,*}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right)\right).$$

Now recall from [14] that we have

$$K(n)_*\left(K(n)\right) = K(n)_*\left(t_k : k \geq 1\right) \otimes \mathrm{E}\left(a_0, a_1, \ldots, a_{n-1}\right) \tag{4.7}$$

where $t_k^{p^n} = v_n^{p^k-1} t_k$ and a_k is exterior of degree $|a_k| = 2p^k - 1$. Clearly φ_1^* induces a monomorphism of E_2 terms, since it visibly maps an element

$$h \in \mathrm{Ext}^{*,*}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right)$$

to the element

$$\widetilde{h}(x) = \varepsilon(x)h \in \mathrm{Hom}_{\widehat{E(n)}_*}\left(\Gamma_*, \mathrm{Ext}^{*,*}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right)\right)$$

where $\varepsilon\colon \Gamma_* \longrightarrow K(n)_*$ is the $\widehat{E(n)}_*$ algebra augmentation determined by

$$\varepsilon(t'_k) = 0 \quad k \geq 1.$$

Hence, all that remains to do is to show that the second spectral sequence has trivial differentials from E_2 on.

From (4.7) we see that

$$K(n)^*\left(K(n)\right) \cong \mathrm{Hom}_{K(n)_*}\left(K(n)_*\left(K(n)\right), K(n)_*\right)$$

and it is straightforward to show that the associated graded object of $K(n)^*\left(K(n)\right)$ is isomorphic to the E_2 term of the second spectral sequence, which must therefore collapse. □

We will use the last result to construct a spectrum $E(n)/I_n^2$ with good properties. In the case $n = 1$, we are simply constructing $E(1)/(p^2)$ but in the general case we obtain much more information than is be available from alternative methods.

Theorem (4.8). *For $0 \leq j \leq n-1$, let $\delta_j\colon K(n) \longrightarrow \Sigma^{2p^j-1}K(n)$ be any map of spectra which is a morphism of A_∞ modules over $\widehat{E(n)}$ and is detected by the element*

$$\varepsilon_j \in \mathrm{Ext}^{1,2(p^j-1)}_{\widehat{E(n)}_*}\left(K(n)_*, K(n)_*\right).$$

Then there is a cofibre sequence of A_∞ module spectra over $\widehat{E(n)}$,

$$\bigvee_{0\leq j\leq n-1}\Sigma^{2p^j-1}K(n)\longrightarrow M(\delta_0,\delta_1,\dots,\delta_{n-1})\longrightarrow K(n)$$

with cofibre map

$$K(n)\xrightarrow{\bigvee_{0\leq j\leq n-1}\delta_j}\Sigma^{2p^j-1}K(n)$$

and

$$\pi_*\left(M(\delta_0,\delta_1,\dots,\delta_{n-1})\right)\cong\widehat{E(n)}_*/I_n^2$$

as an $\widehat{E(n)}_*$ module.

We will now assume we have such a sequence $\delta_0,\delta_1,\dots,\delta_{n-1}$ and denote the spectrum $M(\delta_0,\delta_1,\dots,\delta_{n-1})$ by $E(n)/I_n^2$.

More generally, we can iteratively define a sequence of spectra $E(n)/I_n^k$ for which

$$\pi_*\left(E(n)/I_n^k\right)\cong E(n)/I_n^k$$

and having good properties. To see this we choose a minimal resolution of the $\widehat{E(n)}_*$ module $E(n)/I_n^k$,

$$\mathrm{P}^k_{**}\longrightarrow E(n)/I_n^k\longrightarrow 0$$

for which

$$\mathrm{P}^k_{0\,*}=\widehat{E(n)}_*$$

and

$$\mathrm{P}^k_{1\,*}=\widehat{E(n)}_*\left\{e_{(r_0,r_1,\dots,r_{n-1})}:0\leq r_i\text{ and }\sum_{i=0}^{n-1}r_i=k\right\}$$

with differential

$$\partial\, e_{(r_0,r_1,\dots,r_{n-1})}=v_0^{r_0}v_1^{r_1}\cdots v_{n-1}^{r_{n-1}}.$$

We can also assume that Ker $\partial\colon\mathrm{P}^k_{1\,*}\longrightarrow\mathrm{P}^k_{0\,*}$ lies in $I_n\mathrm{P}^k_{1\,*}$. Now forming

$$\mathrm{Hom}_{\widehat{E(n)}}\left(\mathrm{P}^k_{*\,*},K(n)_*\right)$$

and taking the cohomology with respect to ∂^* gives

$$\mathrm{Ext}^{*,*}_{\widehat{E(n)}_*}\left(E(n)/I_n^k,K(n)_*\right).$$

The dual classes $\varepsilon_{(r_0,r_1,\dots,r_{n-1})}$ where

$$\varepsilon_{(r_0,r_1,\dots,r_{n-1})}\left(e_{(r_0,r_1,\dots,r_{n-1})}\right)=\begin{cases}1 & \text{if } s_i=r_i\ \forall i,\\ 0 & \text{otherwise}\end{cases}$$

are then cocycles representing a $K(n)_*$ basis for Ext^1. A careful inspection of the spectral sequences for $\pi_*\left(\mathrm{RHom}_{\widehat{E(n)}}\left(E(n)/I_n^k, K(n)\right)\right)$ and $K(n)^*\left(E(n)/I_n^k\right)$ shows that these elements are non-zero infinite cycles in both. Hence we can form a cofibre sequence of A_∞ module spectra over $\widehat{E(n)}$,

$$\bigvee_{\sum_i r_i=k} \Sigma^{2d(r_0,r_1,\dots,r_{n-1})}K(n) \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$$

where $d(r_0,r_1,\dots,r_{n-1})=\sum_i r_i(p^i-1)$. Moreover in homotopy this realises the extension of $\widehat{E(n)}_*$ modules

$$I_n^k/I_n^{k+1} \longrightarrow E(n)_*/I_n^{k+1} \longrightarrow E(n)_*/I_n^k.$$

We incorporate state this into the next theorem.

Theorem (4.9). *For each $k \geq 1$, there is a cofibre sequence of A_∞ module spectra over $\widehat{E(n)}$,*

$$\bigvee_{\sum_i r_i=k} \Sigma^{2d(r_0,r_1,\dots,r_{n-1})}K(n) \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$$

which in homotopy realises the following extension of modules over $\widehat{E(n)}_*$:

$$I_n^k/I_n^{k+1} \longrightarrow E(n)_*/I_n^{k+1} \longrightarrow E(n)_*/I_n^k.$$

Notice that we have

$$\pi_*\left(\mathrm{RHom}_{\widehat{E(n)}}\left(\widehat{E(n)}, E(n)/I_n^k\right)\right) \cong E(n)_*/I_n^k$$

since $\widehat{E(n)}_*$ is free over itself; furthermore, the generators can be taken to be compatible under the natural maps $E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$. We can now deduce

Theorem (4.10). *On the category $\mathbf{CW^f}$, the functor*

$$\varprojlim_k \left(E(n)/I_n^k\right)^*(\ )$$

is a cohomology theory which is a continuous module theory over $\widehat{E(n)}_*$, uniquely (up to equivalence) representable by the topological spectrum

$$\operatorname*{holim}_{\overleftarrow{k}} E(n)/I_n^k.$$

The canonical continuous natural transformation

$$\widehat{E(n)}^*(\ ) \longrightarrow \varprojlim_k \left(E(n)/I_n^k\right)^*(\ )$$

is an equivalence of functors, induced by an equivalence of topological module spectra over $\widehat{E(n)}$.

The proof uses the above together with results from [2].

We end by remarking that the above sequence of spectra and cofibrations gives rise to a spectral sequence for each spectrum X whose E_1 term has the form

$$\mathrm{E}_1^{s\,*} = \bigoplus_{\alpha_s} K(n)^*(X),$$

with E_2 term obtained as the cohomology with respect to certain sums of "Bockstein operations"

$$\bigoplus_{\alpha_s} K(n)^*(X) \longrightarrow \bigoplus_{\alpha_{s+1}} K(n)^*(X),$$

and which converges to $\widehat{E(n)}^*(X)$. The existence of such a spectral sequence was pointed out to the author by Urs Würgler, and we return to it in joint work [3], where we show that the boundary morphisms can be explicitly constructed so as to yield derivations in cohomology.

Concluding remarks.

In the above we have restricted attention to the ring spectrum $\widehat{E(n)}$; we can also use our results to give an A_∞ structure on $\widehat{v_n^{-1}BP}$, the *Artinian completion* of $v_n^{-1}BP$, constructed in [2]– this can be made consistent with the two morphisms of ring spectra constructed in [2],

$$\widehat{E(n)} \longrightarrow \widehat{v_n^{-1}BP} \longrightarrow \widehat{E(n)}$$

being morphisms of A_∞ ring spectra. We do not prove this here since we feel that there should be a more generally applicable result encompassing other examples such as $BP \longrightarrow MU_{(p)}$ whereas our present proof is rather *ad hoc*. We hope to return to this in future work.

The results of §4 are used in [3], and we have also applied them, together with results from [11], to construct various interesting A_∞ ring spectra. For example, the subalgebra

$$K(n)_*\left(Q^0, Q^1, \dots, Q^{n-1}\right) \subset K(n)^*(K(n))$$

can be realised as the homotopy ring of an A_∞ ring spectrum, as also can

$$K(n)_*\left(a_0, a_1, \dots, a_{n-1}\right) \subset K(n)_*(K(n))$$

(see [14]).

Finally, we remark that A. Robinson's general theory of A_∞ module spectra assures us that there are Künneth and Universal Coefficient spectral sequences associated to module spectra over $\widehat{E(n)}$ which promise to be of great use in future calculations.

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