Complex cobordism of Hilbert manifolds with some applications to flag varieties of loop groups

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Abstract. We develop a version of Quillen's geometric cobordism theory for infinite dimensional separable Hilbert manifolds. This cobordism theory has a graded group structure under the topological union operation and has push-forward maps for Fredholm maps. We discuss transverse approximations and products, and the contravariant property of this cobordism theory. We define Euler classes for finite dimensional complex vector bundles and describe some applications to the complex cobordism of flag varieties of loop groups.

Introduction

In [**17**], Quillen gave a geometric interpretation of complex cobordism groups which suggests a way of defining the cobordism of separable Hilbert manifolds. In order that such an extension be reasonable, it ought to reduce to his construction for finite dimensional manifolds and also be capable of supporting calculations for important types of infinite dimensional manifolds such as homogeneous spaces of free loop groups of finite dimensional Lie groups and Grassmannians.

In this paper, we outline an extension of Quillen's work to separable Hilbert manifolds and discuss its main properties. Although we are able to verify some expected features, there appears to be a serious gap in the literature on infinite dimensional transversality and without appropriate transverse approximations of Fredholm and smooth maps we are unable to obtain contravariance or product structure. However, covariance along Fredholm maps does hold as does contravariance along submersions. If the relevant infinite dimensional transversality results are indeed true then our version of Quillen's theory may be of wider interest. A major motivation for the present work lay in the desire to generalize to loop groups the finite dimensional results of Bressler & Evens [**2, 3**], and as a sample of applications, we describe some cobordism classes for flag varieties of loop groups and related spaces which appeared in the second author's PhD thesis [**15**].

We would like to thank Jack Morava for bringing the volume 'Global Analysis' [**9**] to our attention at a crucial moment.

2000 *Mathematics Subject Classification.* Primary 55N22 57R77 57R19 58B05; Secondary 58B15.

Key words and phrases. cobordism, Fredholm map, Hilbert manifold, flag variety, loop group.

Contemporary Mathematics **258** (2000), 1–19. [Version 9: 24/12/2003].

1. Fredholm maps and cobordism of separable Hilbert manifolds

By a manifold, we mean a smooth manifold locally modelled on a separable Hilbert space; see Lang [**12**] for details on infinite dimensional manifolds. We begin by reviewing some facts about Fredholm maps which can be and found in Conway [**4**]. Let X and Y be manifolds.

Definition 1.1. A smooth map $f\colon X \longrightarrow Y$ is *proper* if the preimages of compact sets are compact.

Definition 1.2. A linear operator $A\colon U \longrightarrow V$ between the normed vector spaces U and V is *Fredholm* if both $\dim\ker A$ and $\dim\operatorname{coker} A$ are finite and then *index* of A is defined by

$$\operatorname{index} A = \dim\ker A - \dim\operatorname{coker} A.$$

The set of Fredholm operators $U \longrightarrow V$ will be denoted $\mathrm{Fred}(U,V)$ viewed as a subspace of space of all bounded operators $\mathrm{L}(U,V)$ in the norm topology.

Proposition 1.3. $\mathrm{Fred}(U,V)$ *is open in* $\mathrm{L}(U,V)$ *and the index map*

$$\operatorname{index}\colon \mathrm{Fred}(U,V) \longrightarrow \mathbb{Z}$$

is locally constant, hence continuous.

Definition 1.4. A smooth map $f\colon X \longrightarrow Y$ is *Fredholm* if for each $x \in X$, $df_x\colon T_xX \longrightarrow T_{f(x)}Y$ is a Fredholm operator. For such a map, the *index* of f at $x \in X$ is defined by

$$\operatorname{index} f_x = \dim\ker df_x - \dim\operatorname{coker} df_x.$$

Proposition 1.5. *The function* $X \longrightarrow \mathbb{Z}$ *given by* $x \longmapsto \operatorname{index} df_x$ *is locally constant, hence continuous.*

The following results are well known and can be found in Zeidler [**21**].

Proposition 1.6. *Let* U, V, W *be finite dimensional vector spaces and let* X, Y *be finite dimensional smooth manifolds.*

a) *Then every linear operator* $A\colon U \longrightarrow V$ *is Fredholm and*
$$\operatorname{index} A = \dim U - \dim V.$$
b) *Let* $U \xrightarrow{A} V \xrightarrow{B} W$ *be a sequence of Fredholm operators. Then the composite linear operator* $U \xrightarrow{BA} W$ *is also Fredholm and*
$$\operatorname{index} BA = \operatorname{index} B + \operatorname{index} A.$$
c) *Let* $f\colon X \longrightarrow Y$ *be a smooth map. Then* f *is a Fredholm map and for* $x \in X$,
$$\operatorname{index} f_x = \dim X_x - \dim Y_{f(x)}.$$

Proposition 1.6 implies

Proposition 1.7. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a sequence of Fredholm maps, where* X, Y, Z *are smooth manifolds. Then the composite map* $X \xrightarrow{gf} Z$ *is also Fredholm and for* $x \in X$,

$$\operatorname{index}(gf)_x = \operatorname{index} g_{f(x)} + \operatorname{index} f_x.$$

DEFINITION 1.8. Suppose that $f\colon X \longrightarrow Y$ is a proper Fredholm map with even index at each point. Then f is an *admissible complex orientable map* if there is a smooth factorization

$$f\colon X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y,$$

where $q\colon \xi \longrightarrow Y$ is a finite dimensional smooth complex vector bundle and $\tilde{f}$ is a smooth embedding endowed with a complex structure on its normal bundle $\nu(\tilde{f})$.

A complex orientation for a Fredholm map f of odd index is defined to be one for the map $(f, \varepsilon)\colon X \longrightarrow Y \times \mathbb{R}$ given by $(f, \varepsilon)(x) = (f(x), 0)$ for every $x \in X$. Then for $x \in X$, $\operatorname{index}(f, \varepsilon)_x = (\operatorname{index} f_x) - 1$ and the finite dimensional complex vector bundle ξ in the smooth factorization will be replaced by $\xi \times \mathbb{R} \longrightarrow Y \times \mathbb{R}$.

Suppose that f is an admissible complex orientable map with a factorisation $\tilde{f}$ as in Definition 1.8. Since f is a Fredholm map and ξ is a finite dimensional vector bundle, $\tilde{f}$ is also a Fredholm map. By Proposition 1.7 and the surjectivity of q,

$$\operatorname{index} \tilde{f} = \operatorname{index} f - \dim \xi.$$

Before giving a notion of equivalence of such factorizations $\tilde{f}$ of f, we require yet more definitions.

DEFINITION 1.9. Let $F\colon X \times \mathbb{R} \longrightarrow Y$ a smooth map where X and Y are separable Hilbert manifolds. Then F is an *isotopy* if it satisfies the following conditions.

a) For every $t \in \mathbb{R}$, the map F_t given by $F_t(x) = F(x, t)$ is an embedding.
b) There exist numbers $t_0 < t_1$ such that $F_t = F_{t_0}$ for $t \leqslant t_0$ and $F_t = F_{t_1}$ for $t \geqslant t_1$.

The closed interval $[t_0, t_1]$ is called a *proper domain* for isotopy. Two embeddings $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ are *isotopic* if there is an isotopy $F_t\colon X \times \mathbb{R} \longrightarrow Y$ with proper domain $[t_0, t_1]$ such that $f = F_{t_0}$ and $g = F_{t_1}$.

From Lang [**12**] we have

PROPOSITION 1.10. *The relation of isotopy between smooth embeddings is an equivalence relation.*

DEFINITION 1.11. Two factorizations $f\colon X \xrightarrow{\tilde{f}} \xi \xrightarrow{q} Y$ and $f\colon X \xrightarrow{\tilde{f}'} \xi' \xrightarrow{q'} Y$ are *equivalent* if ξ and ξ' can be embedded as subvector bundles of a vector bundle $\xi'' \longrightarrow Y$ such that $\tilde{f}$ and $\tilde{f}'$ are isotopic in ξ'' and this isotopy is compatible with the complex structure on the normal bundle. That is, there is an isotopy F such that for all $t \in [t_0, t_1]$, $F_t\colon X \longrightarrow \xi''$ is endowed with a complex structure on its normal bundle matching that of $\tilde{f}$ and $\tilde{f}'$ in ξ'' at t_0 and t_1 respectively.

Proposition 1.10 gives

PROPOSITION 1.12. *The relation of equivalence of admissible complex orientability of proper Fredholm maps between separable Hilbert manifolds is an equivalence relation.*

This generalizes Quillen's notion of complex orientability for maps of finite dimensional manifolds. We can also define a notion of cobordism of admissible complex orientable maps between separable Hilbert manifolds. First we recall some ideas on the transversality.

Definition 1.13. Let $f_1\colon M_1 \longrightarrow N$, $f_2\colon M_2 \longrightarrow N$ be smooth maps between Hilbert manifolds. Then f_1, f_2 are *transverse at* $y \in N$ if

$$df_1(T_{x_1}M_1) + df_2(T_{x_2}M_2) = T_yN$$

whenever $f_1(x_1) = f_2(x_2) = y$. The maps f_1, f_2 are said to be *transverse* if they are transverse at every point of N.

Lemma 1.14. *Smooth maps $f_i\colon M_i \longrightarrow N$ $(i = 1, 2)$ are transverse if and only if $f_1 \times f_2\colon M_1 \times M_2 \longrightarrow N \times N$ is transverse to the diagonal map $\Delta\colon N \longrightarrow N \times N$.*

Definition 1.15. Let $f_1\colon M_1 \longrightarrow N$, $f_2\colon M_2 \longrightarrow N$ be transverse smooth maps between Hilbert manifolds. The *topological pullback*

$$M_1 \underset{N}{\sqcap} M_2 = \{(x_1, x_2) \in M_1 \times M_2 \colon f_1(x_1) = f_2(x_2)\}$$

is a submanifold of $M_1 \times M_2$ and the diagram

$$\begin{array}{ccc} M_1 \underset{N}{\sqcap} M_2 & \xrightarrow{{f_2}^* f_1} & M_2 \\ \Big\downarrow {f_1}^* f_2 & & \Big\downarrow f_2 \\ M_1 & \xrightarrow{f_1} & N \end{array}$$

is commutative, where the map ${f_i}^* f_j$ is the pull-back of f_j by f_i.

Definition 1.16. Let $f_i\colon X_i \longrightarrow Y$ $(i = 0, 1)$ be admissible complex oriented maps. Then f_0 is *cobordant* to f_1 if there is an admissible complex orientable map $h\colon W \longrightarrow Y \times \mathbb{R}$ such that the maps $\varepsilon_i\colon Y \longrightarrow Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i = 0, 1$, are transverse to h and the pull-back map ${\varepsilon_i}^* h$ is equivalent to f_i. The cobordism class of $f\colon X \longrightarrow Y$ will be denoted $[X, f]$.

Proposition 1.17. *If $f\colon X \longrightarrow Y$ is an admissible complex orientable map and $g\colon Z \longrightarrow Y$ a smooth map transverse to f, then the pull-back map*

$$g^* f : Z \underset{Y}{\sqcap} X \longrightarrow Z$$

is an admissible complex orientable map with finite dimensional pull-back vector bundle

$$g^*\xi = Z \underset{Y}{\sqcap} \xi = \{(z, v) \in Z \times \xi : g(z) = q(v)\}$$

in the factorization of $g^* f$, where $q\colon \xi \longrightarrow Y$ is the finite-dimensional complex vector bundle in a factorization of f as in Definition 1.8.

The next result was proved in [**15**] by essentially the same argument as in the finite dimensional situation using the Implicit Function Theorem [**12**].

Theorem 1.18. *Cobordism is an equivalence relation.*

Definition 1.19. For a separable Hilbert manifold Y, $\mathcal{U}^d(Y)$ is the set of cobordism classes of the admissible complex orientable proper Fredholm maps of index $-d$.

In the above definition, instead of proper maps, closed maps could be used for infinite dimensional Hilbert manifolds, because of the following result of Smale [**19**].

Theorem 1.20. *When X and Y are infinite dimensional, every closed Fredholm map $X \longrightarrow Y$ is proper.*

By Proposition 1.7 together with the fact that composites of proper maps are proper, we have the following theorem.

THEOREM 1.21. *If $f\colon X \longrightarrow Y$ is an admissible complex orientable Fredholm map of index d_1 and $g\colon Y \longrightarrow Z$ is an admissible complex orientable Fredholm map with index d_2, then $g \circ f\colon X \longrightarrow Z$ is an admissible complex orientable map with index $d_1 + d_2$.*

PROOF. For the complex orientation of the composition map $g \circ f$, see Dyer [**6**]. $\square$

Let $g\colon Y \longrightarrow Z$ be an admissible complex orientable Fredholm map of index r. By Theorem 1.21, we have the *push-forward* (or *Gysin*) map

$$g_*\colon \mathcal{U}^d(Y) \longrightarrow \mathcal{U}^{d-r}(Z)$$
$$g_*[X, f] = [X, g \circ f].$$

It is straightforward to verify that this is well-defined. In fact, if $g'\colon Y \longrightarrow Z$ is a second such map cobordant to g then $g'_* = g_*$; in particular, if g and g' are homotopic through proper Fredholm maps they induce the same Gysin maps.

The graded cobordism set $\mathcal{U}^*(Y)$ of the separable Hilbert manifold Y has a group structure given as follows. Let $[X_1, f_1]$ and $[X_2, f_2]$ be cobordism classes. Then $[X_1, f_1] + [X_2, f_2]$ is the class of the map $f_1 \amalg f_2\colon X_1 \amalg X_2 \longrightarrow Y$, where $X_1 \amalg X_2$ is the disjoint union of X_1 and X_2. As in the finite dimensional theory, the class of the empty set is the zero element and the negative of $[X, f]$ is itself with the opposite orientation on the normal bundle of the embedding $\tilde{f}$.

THEOREM 1.22. *The graded cobordism set $\mathcal{U}^*(Y)$ of the admissible complex orientable maps of Y is a graded abelian group.*

If our cobordism functor $\mathcal{U}^*(\ )$ of admissible complex orientable Fredholm maps is restricted to finite dimensional Hilbert manifolds, it agrees Quillen's complex cobordism functor $\mathrm{MU}^*(\ )$.

THEOREM 1.23. *If Y is a finite dimensional separable Hilbert manifold, there is a natural isomorphism*

$$\mathcal{U}^*(Y) \cong \mathrm{MU}^*(Y).$$

2. Transversality, contravariance and cup products

We would like to define a product structure on the graded cobordism group $\mathcal{U}^*(Y)$. Given cobordism classes $[X_1, f_1] \in \mathcal{U}^{d_1}(Y_1)$ and $[X_2, f_2] \in \mathcal{U}^{d_2}(Y_2)$, their external product is

$$[X_1, f_1] \times [X_2, f_2] = [X_1 \times X_2, f_1 \times f_2] \in \mathcal{U}^{d_1+d_2}(Y_1 \times Y_2).$$

We cannot necessarily define an internal product on $\mathcal{U}^*(Y)$ unless Y is a finite dimensional manifold. However, if admissible complex orientable Fredholm maps f_1 and f_2 are transverse, then they do have an internal (cup) product

$$[X_1, f_1] \cup [X_2, f_2] = \Delta^*[X_1 \times X_2, f_1 \times f_2],$$

where Δ is the diagonal embedding. If Y is finite dimensional, then by Thom's Transversality Theorem [**20**], every complex orientable Fredholm map to Y has a transverse approximation, hence the cup product $\cup$ induces a graded ring structure on $\mathcal{U}^*(Y)$. The unit element 1 is represented by the identity map $Y \longrightarrow Y$ with index 0; note that this also element exists when Y is infinite dimensional.

The following result was proved by F. Quinn [**18**].

THEOREM 2.1. *Let $f\colon M \longrightarrow N$ be a Fredholm map and $g\colon W \longrightarrow N$ an inclusion of a finite-dimensional submanifold. Then there exists an approximation g' of g in $C^\infty(W,N)$ with the fine topology such that g' is transverse to f.*

Details on the fine topology and the space of smooth maps $C^\infty(W,N)$ can be found in Michor [**13**]. In this topology the derivatives of the difference function between the function g and its approximation g' are bounded. We would like to interpret this approximation in the fine topology. We need some notation to describe this situation.

DEFINITION 2.2. Let X and Y be smooth manifolds. A *k-jet from X to Y* is an equivalence class $[f,x]_k$ of pairs (f,x) where $f\colon X \longrightarrow Y$ is a smooth mapping and $x \in X$. The pairs (f,x) and (f',x') are *equivalent* if $x = x'$, f and f' have same Taylor expansion of order k at x in some pair of coordinate charts centered at x and $f(x)$ respectively. We will write $J^k f(x) = [f,x]_k$ and call this the *k-jet* of f at x.

There is an equivalent definition of this equivalence relation:
$[f,x]_k = [f',x']_k$ if $x = x'$ and $T^k_x f = T^k_x f'$ where T^k is the kth tangent mapping.

DEFINITION 2.3. For a topological space X, a covering of X is *locally finite* if every point has a neighbourhood which intersects only finitely many elements of the covering.

Approximation g' of g in the smooth fine topology means the following. Let $\{L_i\}_{i\in I}$ be a locally finite cover of W. For every open set L_i, there is a bounded continuous map $\varepsilon_i\colon L_i \longrightarrow [0,\infty)$ such that for every $x \in L_i$ and $k > 0$,

$$\|J^k g(x) - J^k g'(x)\| < \varepsilon_i(x).$$

By Theorem 2.1, for a finite dimensional manifold Z, a smooth map $g\colon Z \longrightarrow Y$, can be deformed by a smooth homotopy until it is transverse to an admissible complex orientable map $f\colon X \longrightarrow Y$. Thus the cobordism functor is contravariant for any map from a finite dimensional manifold to a Hilbert manifold.

THEOREM 2.4. *Let $f\colon X \longrightarrow Y$ be an admissible complex oriented map and let $g\colon Z \longrightarrow Y$ be a smooth map from a finite dimensional manifold Z. Then the cobordism class of the pull-back $Z \underset{Y}{\sqcap} X \longrightarrow Z$ depends only on the cobordism class of f, hence there is a map $g^*\colon \mathcal{U}^d(Y) \longrightarrow \mathcal{U}^d(Z)$ given by*

$$g^*[X,f] = [Z \underset{Y}{\sqcap} X, {g'}^*(f)],$$

where g' is an approximation of g which is transverse to f.

PROOF. Suppose that $f\colon X \longrightarrow Y$ be an admissible complex orientable map. By Theorem 2.1, there exists an approximation $g_0\colon Z \longrightarrow Y$ of g which is transverse to f. We will show that $g_0^*[X,f]$ depends only on the cobordism class $[X,f]$.

Assume that $f\colon X \longrightarrow Y$ and $f'\colon X' \longrightarrow Y$ are cobordant and that $g_1\colon Z \longrightarrow Y$ is another approximation of g is transverse to f'. Then there is an admissible complex orientable map $h\colon W \longrightarrow Y \times \mathbb{R}$ such that $\varepsilon_i\colon Y \longrightarrow Y \times \mathbb{R}$ given by

$\varepsilon_i(y) = (y, i)$ for $i = 0, 1$, is transverse to h and the pull-back map $\varepsilon_0^* h$ is equivalent to f and $\varepsilon_1^* h$ is equivalent to f'. So,

$$W \underset{Y\times\mathbb{R}}{\sqcap} Y = \{(w, y) : h(w) = (y, 0)\} \cong X,$$

and

$$W \underset{Y\times\mathbb{R}}{\sqcap} Y = \{(w, y) : h(w) = (y, 1)\} \cong X'.$$

There is a smooth map

$$(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})\colon Z \times \mathbb{R} \longrightarrow Y \times \mathbb{R}$$

transverse to h. By Proposition 1.17, the map

$$(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})^* h\colon W \underset{Y\times\mathbb{R}}{\sqcap} Z \times \mathbb{R} \longrightarrow Z \times \mathbb{R}$$

is an admissible complex orientable map transverse to $\varepsilon_i\colon Z \longrightarrow Z \times \mathbb{R}$ for $i = 0, 1$. By Proposition 1.17, there is an induced map

$$\varepsilon_0^*(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})^* h\colon \left(W \underset{Y\times\mathbb{R}}{\sqcap} Z \times \mathbb{R}\right) \underset{Z\times\mathbb{R}}{\sqcap} Z \longrightarrow Z.$$

We have the product manifold

$$\begin{aligned}
&\left(W \underset{Y\times\mathbb{R}}{\sqcap} Z \times \mathbb{R}\right) \underset{Z\times\mathbb{R}}{\sqcap} Z \\
&= \{(w, (z_1, t), z_2) : h(w) = (g_0(z_1), t) \text{ or } h(w) = (g_1(z_1), t), (z_1, t) = (z_2, 0)\} \\
&= \{(w, (z_1, 0) : h(w) = (g_0(z_1), 0) \text{ or } h(w) = (g_1(z_1), 0)\} \\
&\cong Z \underset{Y}{\sqcap} X.
\end{aligned}$$

Similarly, we have the induced map

$$\varepsilon_1^*(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})^* h\colon \left(W \underset{Y\times\mathbb{R}}{\sqcap} Z \times \mathbb{R}\right) \underset{Z\times\mathbb{R}}{\sqcap} Z \longrightarrow Z.$$

The product manifold $\left(W \underset{Y\times\mathbb{R}}{\sqcap} Z \times \mathbb{R}\right) \underset{Z\times\mathbb{R}}{\sqcap} Z$ is diffeomorphic to $Z \underset{Y}{\sqcap} X'$. The two induced functions $\varepsilon_0^*(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})^* h$ and $\varepsilon_1^*(g_0 \amalg g_1, \mathrm{Id}_{\mathbb{R}})^* h$ are equivalent to $g_0^* f$ and ${g_1}^* f'$ respectively. $\square$

Using the concept of Sard functions of Definition 3.2, Quinn [**18**] studied this situation when g is a smooth map between infinite dimensional separable Hilbert manifolds.

Theorem 2.5. *Let H be a separable infinite dimensional Hilbert space with $U \subseteq H$ an open subset and $f\colon M \longrightarrow N$ be a proper Fredholm map between separable infinite dimensional Hilbert manifolds M and N. Then the set of maps transverse to f is dense in the closure of Sard function space $\overline{\mathcal{S}(U, N)}$ in the C^∞ fine topology.*

We will require the Open Embedding Theorem of Eells & Elworthy [**7**].

Theorem 2.6. *Let X be a smooth manifold modelled on the separable infinite dimensional Hilbert space H. Then X is diffeomorphic to an open subset of H.*

Using this result, we have the transverse smooth approximation of Sard functions in the C^∞ fine topology. From [**7**], we have

Theorem 2.7. *Let X and Y be two smooth manifold modelled on the separable infinite dimensional Hilbert space H. If there is a homotopy equivalence $\varphi\colon X \longrightarrow Y$, then φ is homotopic to a diffeomorphism.*

Theorems 2.5, 2.6 and 2.7 together imply the following result.

Theorem 2.8. *Suppose X, Y, Z are infinite dimensional smooth separable Hilbert manifolds, $f\colon X \longrightarrow Y$ is an admissible complex orientable map and $g\colon Z \longrightarrow Y$ a Sard function. Then the cobordism class of the pull-back $Z \underset{Y}{\sqcap} X \longrightarrow Z$ only depends on the cobordism class of f. Hence there is a well defined group homomorphism $g^*\colon \mathcal{U}^d(Y) \longrightarrow \mathcal{U}^d(Z)$ given by*

$$g^*[X, f] = [Z \underset{Y}{\sqcap} X, g^*(f)].$$

Then, $\mathcal{U}^*(\ )$ is a contravariant functor for Sard functions on the infinite dimensional separable Hilbert manifolds. The question of whether it agrees with other cobordism functors such as representable cobordism seems not so easily answered and there is also no obvious dual bordism functor.

3. Euler classes of finite dimensional bundles

In this section, we will show how to define Euler classes in complex cobordism for finite dimensional complex vector bundles over separable Hilbert manifolds. We begin with some definitions.

Definition 3.1. Let E be a Banach space. We say that a collection $\mathcal{S}$ of smooth functions $\alpha\colon E \longrightarrow \mathbb{R}$ is a *Sard class* if it satisfies the following conditions.

a) For $r \in \mathbb{R}$, $y \in E$ and $\alpha \in \mathcal{S}$, the function $x \longmapsto \alpha(rx + y)$ is also in the class $\mathcal{S}$.
b) If $\alpha_1, \ldots, \alpha_n \in \mathcal{S}$, then the rank of the differential $D_x(\alpha_1, \ldots, \alpha_n)$ is constant for all x not in some closed finite dimensional submanifold of E.

Definition 3.2. Let $\mathcal{S}$ be a Sard class on E, $U \subseteq E$ open, and M a smooth Banach manifold. A *Sard function* $f\colon U \longrightarrow M$ is one for which for each $x \in U$ there is a neighbourhood $V \subseteq U$ of x, functions $\alpha_1, \ldots, \alpha_n \in \mathcal{S}$, and a smooth map $g\colon W \longrightarrow M$, with $W \subseteq \mathbb{R}^n$ open, $(\alpha_1, \ldots, \alpha_n)(V) \subseteq W$ and $f_{|V} = g \circ (\alpha_1, \ldots, \alpha_n)_{|V}$. The collection of all Sard functions $f\colon U \longrightarrow M$ will be denoted $\mathcal{S}(U, M)$; in particular we will consider $\mathcal{S}(E, \mathbb{R})$.

Recall that the *support* of a function $f\colon X \longrightarrow \mathbb{R}$ is the closure of the set of points $x \in X$ such that $f(x) \neq 0$. From [**18**], we have

Theorem 3.3. *If $\mathcal{S}(E, \mathbb{R})$ contains a function with nonempty bounded support, then E admits a Sard class $\mathcal{S}$. In particular, every separable Hilbert space admits Sard classes.*

Let X be a topological space. Recall that a *refinement* of a covering of X is a second covering, each element of which is contained in an element of the first covering. Also, X is *paracompact* if it is Hausdorff and every open covering has a locally finite open refinement.

Definition 3.4. A smooth *partition of unity* on a manifold X consists of a covering $\{U_i\}_{i\in I}$ of X and a collection of smooth functions $\{\psi_i\colon X \longrightarrow \mathbb{R}\}_{i\in I}$ satisfying the following conditions.

a) For $x \in X$ and $i \in I$, $\psi_i(x) \geqslant 0$.
b) For $i \in I$, the support of ψ_i is contained in U_i.
c) The covering is locally finite.
d) For $x \in X$, we have

$$\sum_{i \in I} \psi_i(x) = 1.$$

DEFINITION 3.5. A paracompact manifold X *admits partitions of unity* if, given a locally finite open covering $\{U_i\}_{i \in I}$, there exists a partition of unity $\{\psi_i\}_{i \in I}$ such that the support of each ψ_i is contained in some U_i.

From Lang [**12**], we have

THEOREM 3.6. *Every paracompact smooth manifold X modelled on a separable Hilbert space H admits smooth partitions of unity.*

From Eells & McAlpin [**8**], we have

THEOREM 3.7. *For a separable Hilbert manifold X, the functions on X constructed using smooth partitions of unity form a Sard class.*

Global sections of a vector bundle on a smooth separable Hilbert manifold can be constructed using partitions of unity, and all sections are Sard. Given a smooth vector bundle $\pi \colon E \longrightarrow B$ over a separable Hilbert manifold B, we know from Theorem 2.6, that B can be embedded as a open subset of a separable Hilbert space H. By Theorem 2.5, we have

COROLLARY 3.8. *Let $\pi \colon E \longrightarrow B$ be a finite dimensional complex vector bundle over a separable Hilbert manifold B and let $i \colon B \longrightarrow E$ be the zero-section. Then there is an approximation $\tilde{i}$ of i with $\tilde{i}$ transverse to i.*

By Theorems 2.7 and 2.8, we can define the Euler class of a finite dimensional complex vector bundle on a separable Hilbert manifold. Note that Theorem 2.8 implies that this Euler class is a well-defined invariant of the bundle π.

DEFINITION 3.9. Let $\pi \colon \xi \longrightarrow B$ be a finite dimensional complex vector bundle of dimension d on a separable Hilbert manifold B with zero-section $i \colon B \longrightarrow \xi$. The *$\mathcal{U}$-theory Euler class* of ξ is the element

$$\chi(\pi) = i^* i_* 1 \in \mathcal{U}^{2d}(B).$$

We have the following projection formula for the Gysin map of a submersion.

THEOREM 3.10. *Let $f \colon X \longrightarrow Y$ be an admissible complex orientable Fredholm submersion and let $\pi \colon \xi \longrightarrow Y$ be a finite dimensional smooth complex vector bundle of dimension d. Then*

$$\chi(\xi) \cup [X, f] = f_* \chi(f^* \xi).$$

PROOF. Let s be a smooth section of π transverse to the zero section $i \colon Y \longrightarrow \xi$. Then $Y' = \{y \in Y : s(y) = i(y)\}$ is a submanifold of complex codimension d and $\chi(\xi) = [Y', j]$, where $j \colon Y' \longrightarrow Y$ is the inclusion. Setting

$$X' = f^{-1} Y' = \{x \in X : s(f(x)) = i(f(x))\},$$

which is also a submanifold of X of complex codimension d, we have

$$\begin{aligned} \chi(\xi) \cup [X, f] &= [Y', j] \cup [X, f] \\ &= [X', f_{|_{X'}}]. \end{aligned}$$

Now we determine $f_*\chi(f^*\xi)$. Since f is a submersion, the composite section $s \circ f\colon X \longrightarrow f^*\xi$ is transverse to the zero section and they agree on X', hence by definition we have $\chi(f^*\xi) = [X', j]$ where $j\colon X' \longrightarrow X$ is the inclusion. Hence, $f_*\chi(f^*\xi) = [X', f_{|_{X'}}]$ by definition of the Gysin map f_*. □

4. The relationship between $\mathcal{U}$-theory and MU-theory

In this section we consider the relationship between $\mathcal{U}$-theory and MU-theory. Later we will discuss the particular cases of Grassmannians and LG/T.

First we will discuss the general relationship between $\mathcal{U}^*(\ )$ and $\mathrm{MU}^*(\ )$. Let X be a separable Hilbert manifold and recall Theorem 2.1. Then for each proper smooth map $f\colon M \longrightarrow X$ where M is a *finite* dimensional manifold, there is a pullback homomorphism

$$f^*\colon \mathcal{U}^*(X) \longrightarrow \mathcal{U}^*(M) = \mathrm{MU}^*(M).$$

If we consider all such maps into X, then there is a unique homomorphism

$$\rho\colon \mathcal{U}^*(X) \longrightarrow \varprojlim_{M\downarrow X} \mathrm{MU}^*(M),$$

where the limit is taken over all proper smooth maps $M \longrightarrow X$ from finite dimensional manifolds, which form a directed system along commuting diagrams of the form

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{=} & X \end{array}$$

and hence give rise to an inverse system along induced maps $f^*\colon \mathrm{MU}^*(M_2) \longrightarrow \mathrm{MU}^*(M_1)$ in cobordism.

Let X be a separable Hilbert manifold. Each of the following conjectures appears reasonable and is consistent with examples we will discuss later. We might also hope that surjectivity could be replaced by isomorphism, but we do not have any examples supporting this.

Conjecture 1. *ρ is always a surjection.*

Conjecture 2. *If $\mathcal{U}^{\mathrm{ev}}(X) = 0$ or $\mathcal{U}^{\mathrm{odd}}(X) = 0$, then ρ is a surjection.*

Conjecture 3. *If $\mathrm{MU}^{\mathrm{ev}}(X) = 0$ or $\mathrm{MU}^{\mathrm{odd}}(X) = 0$, then ρ is a surjection.*

5. Some examples of cobordism classes for infinite dimensional manifolds

Now we discuss some important special cases. Let H be a separable complex Hilbert space, with H^n $(n \geqslant 1)$ an increasing sequence of finite dimensional subspaces with $\dim H^n = n$ with $H^\infty = \bigcup_{n\geqslant 1} H^n$ dense in H. We will use a theorem of Kuiper [**11**].

Theorem 5.1. *The unitary group $\mathrm{U}(H)$ of a separable Hilbert space H is contractible.*

Let $\mathrm{Gr}_n(H)$ be the space of all n-dimensional subspaces of H, which is a separable Hilbert manifold. Then

$$\mathrm{Gr}_n(H^\infty) = \bigcup_{k\geqslant n} \mathrm{Gr}_n(H^k)$$

is a dense subspace of $\mathrm{Gr}_n(H)$ which we will take it to be a model for $B\mathrm{U}(n)$.

Theorem 5.2. *The natural embedding* $\mathrm{Gr}_n(H^\infty) \longrightarrow \mathrm{Gr}_n(H)$ *is a homotopy equivalence, and the natural n-plane bundle* $\zeta_n \longrightarrow \mathrm{Gr}_n(H)$ *is universal.*

Proof. By a theorem of Pressley & Segal [**16**], the unitary group $\mathrm{U}(H)$ acts on $\mathrm{Gr}(H)$ transitively and hence $\mathrm{U}(H)$ acts on $\mathrm{Gr}_n(H)$ transitively. Let H' be the orthogonal complement of H^n in H. The stabilizer group of H^n is $\mathrm{U}(H^n)\times\mathrm{U}(H')$ which acts freely on the contractible space $\mathrm{U}(H)$. Hence

$$\begin{aligned}\mathrm{Gr}_n(H) &= \mathrm{U}(H)/(\mathrm{U}(H^n)\times\mathrm{U}(H'))\\ &= \mathrm{B}(\mathrm{U}(H^n)\times\mathrm{U}(H'))\\ &= \mathrm{BU}(H^n)\times\mathrm{BU}(H').\end{aligned}$$

By Kuiper's Theorem 5.1, $\mathrm{U}(H')$ is contractible, hence so is $\mathrm{BU}(H')$. Hence

$$\mathrm{Gr}_n(H) \simeq \mathrm{BU}(H^n) = \mathrm{BU}(n).$$

On the other hand,

$$\mathrm{Gr}_n(H^\infty) = \bigcup_{k\geqslant n} \mathrm{U}(H^k)/(\mathrm{U}(H^n)\times\mathrm{U}(H'')) \subseteq \mathrm{Gr}_n(H),$$

where H'' is the orthogonal complement of H^n in H^k.

By construction, the natural n-plane bundle $\zeta_n \longrightarrow \mathrm{Gr}_n(H)$ is universal. Also, the natural bundle $\zeta_n^\infty \longrightarrow \mathrm{Gr}_n(H^\infty)$ is classified by the inclusion $\mathrm{Gr}_n(H^\infty) \longrightarrow \mathrm{Gr}_n(H)$ and since the latter is universal, this inclusion is a homotopy equivalence. $\square$

In particular, the inclusion of the projective space

$$\mathrm{P}(H^\infty) = \bigcup_{n\geqslant 1} \mathrm{P}(H^n) \subseteq \mathrm{P}(H)$$

is a homotopy equivalence.

Theorem 5.3. *The natural homomorphism*

$$\rho\colon \mathcal{U}^*(\mathrm{P}(H)) \longrightarrow \varprojlim_n \mathrm{MU}^*(\mathrm{P}(H^n)) = \mathrm{MU}^*(\mathrm{P}(H^\infty))$$

is surjective.

Proof. We will show by induction that

$$\mathcal{U}^*(\mathrm{P}(H)) \xrightarrow{i_n^*} \mathrm{MU}^*(\mathrm{P}(H^{n+1})),$$

is surjective for each n. It will suffice to show that $x^i \in \operatorname{im} i_n^*$ for $i = 0,\dots,n$. For $n = 0$, this is trivial.

Now we verify it for $n = 1$. By Theorem 5.2, since the natural line bundle $\lambda \longrightarrow \mathrm{P}(H) \simeq \mathrm{P}(H^\infty)$ is universal, the following diagram commutes for each $n \geqslant 1$

$$\begin{tikzcd}
\eta_n = i_n^*(\lambda) \arrow[r, "i_n^*"] \arrow[d, "i_n^*(\lambda)"] & \lambda \arrow[d] \\
\mathbb{C}\mathrm{P}^n = \mathrm{P}(H^{n+1}) \arrow[r, "i_n"] & \mathrm{P}(H),
\end{tikzcd}$$

where $i_n \colon \mathbb{C}\mathrm{P}^n = \mathrm{P}(H^{n+1}) \longrightarrow \mathrm{P}(H)$ is the inclusion map. By the compatibility of induced bundles, for $n \geqslant 1$ and the generator $x = \chi(\eta_n) \in \mathrm{MU}^*(\mathrm{P}(H^{n+1}))$, there exists an Euler class $\tilde{x} = \chi(\lambda) \in \mathcal{U}^2(\mathrm{P}(H))$ satisfying $i_n^*(\tilde{x}) = x$, where $i_n \colon \mathrm{P}(H^{n+1}) \longrightarrow \mathrm{P}(H)$ is the inclusion map.

Assume that i_n^* is surjective. Then there are elements

$$y_i \in \mathcal{U}^{2i}(\mathrm{P}(H)), \quad i = 0, \ldots, n,$$

such that

$$i_n^* y_i = x^i \in \mathrm{MU}^{2i}(\mathrm{P}(H^{n+1})).$$

Also,

$$i_{n+1}^* y_i = x^i + z_i x^{n+1} \in \mathrm{MU}^{2i}(\mathrm{P}(H^{n+2})),$$

where $z_i \in \mathrm{MU}_{2(n+1-i)}$.

In particular, let $y_n = [W, f] \in \mathcal{U}^{2(n)}(\mathrm{P}(H))$. Then the following diagram commutes

$$\begin{tikzcd}
f^*\lambda \arrow[r, "f^*"] \arrow[d] & \lambda \arrow[d] \\
W \arrow[r, "f"] & \mathrm{P}(H)
\end{tikzcd}$$

and there is an Euler class $\chi(f^*\lambda) = [W', g] \in \mathcal{U}^2(W)$. Now by Theorem 3.10,

$$y_{n+1} = f_* \chi(f^*\lambda) \in \mathcal{U}^{2n+2}(\mathrm{P}(H))$$

satisfies

$$i_{n+1}^* y_{n+1} = x^n \chi(\eta_n) = x^{n+1}.$$

Hence, $\operatorname{im} i_{n+1}^*$ contains the MU^*-submodule generated by x^i $(i = 0, \ldots, n)$ and so i_{n+1}^* is surjective. This completes the induction.

This shows the induced homomorphism

$$\rho \colon \mathcal{U}^*(\mathrm{P}(H)) \longrightarrow \varprojlim_n \mathrm{MU}^*(\mathrm{P}(H^n)) = \mathrm{MU}^*(\mathrm{P}(H^\infty)).$$

is surjective. $\square$

This proof shows that Theorem 3.10 can be used to work with elements which have the appearance of products even though these may not always exist. It is also possible (and probably more natural) to prove this result by using the projective spaces $\mathrm{P}(H^{n\perp}) \subseteq \mathrm{P}(H)$ to realise cobordism classes restricting to the classes x^n in $\mathrm{MU}^*(\mathrm{P}(H^\infty))$.

Next we will discuss some geometry of Grassmannians from Pressley & Segal [**16**], whose ideas and notation we assume. We take for our separable Hilbert space

$H = \mathrm{L}^2(S^1;\mathbb{C})$ and let H_+ to be the closure of the subspace of Hcontaining the functions $z^n\colon z \longmapsto z^n$ $(n \geqslant 0)$. Then

$$\mathrm{Gr}_0(H) = \varinjlim_{k\geqslant 1} \mathrm{Gr}(H_{-k,k}),$$

where $\mathrm{Gr}(H_{-k,k})$ is the Grassmannian of the finite dimensional vector space

$$H_{-k,k} = z^{-k}H_+/z^k H_+.$$

$\mathrm{Gr}_0(H)$ is dense in $\mathrm{Gr}(H)$ and is also known to be homotopic to the classifying space of K-theory, $\mathrm{BU}\times\mathbb{Z}$.

Theorem 5.4. *For $n \geqslant 1$, the natural homomorphism*

$$\rho\colon \mathcal{U}^*(\mathrm{Gr}_n(H)) \longrightarrow \mathrm{MU}^*(\mathrm{Gr}_n(H))$$

is surjective.

Proof. For $k \geqslant n$, the inclusion $i\colon\ \mathrm{Gr}_n(H_{-k,k}) \longrightarrow \mathrm{Gr}_n(H)$ induces a contravariant map

$$\mathcal{U}^*(Gr_n(H)) \longrightarrow \mathcal{U}^*(\mathrm{Gr}_n(H_{-k,k}) = \mathrm{MU}^*(\mathrm{Gr}_n(H_{-k,k})).$$

For $k \geqslant n$, since $C_S \subseteq \mathrm{Gr}_n(H_{-k,k})$ is transverse to Σ_S, there exists a stratum $\Sigma_{S'}$ such that

$$\sigma_{S',k} = [\mathrm{Gr}_n(H_{-k,k}) \cap \Sigma_{S'} \longrightarrow \mathrm{Gr}_n(H_{-k,k})] \in \mathrm{MU}^*(\mathrm{Gr}_n(H_{-k,k}))$$

are the classical Schubert cells. By an argument using the Atiyah–Hirzebruch spectral sequence and results on Schubert cells in cohomology [**14**], the cobordism classes $\sigma_{S',k}$ provide generators for the MU^*-module $\mathrm{MU}^*(\mathrm{Gr}_n(H_{-k,k}))$. Thus i^* is surjective. For each k,

$$\mathrm{MU}^{\mathrm{odd}}(\mathrm{Gr}_n(H_{-k,k})) = 0,$$

hence

$$\begin{aligned}\mathcal{U}^*(\mathrm{Gr}_n(H)) \longrightarrow &\varprojlim_{k} \mathrm{MU}^*(\mathrm{Gr}_n(H_{-k,k}))\\ &= \mathrm{MU}^*(\mathrm{Gr}_n(H^\infty))\\ &\cong \mathrm{MU}^*(\mathrm{Gr}_n(H))\end{aligned}$$

is surjective. □

Theorem 5.5. *For a compact connected semi-simple Lie group G,*

$$\rho\colon \mathcal{U}^*(LG/T) \longrightarrow \mathrm{MU}^*(LG/T)$$

is surjective.

Proof. As LG/T has no odd dimensional cells, the Atiyah–Hirzebruch spectral sequence for $\mathrm{MU}^*(LG/T)$ collapses. Hence it suffices to show that the composition

$$\mathcal{U}^*(LG/T) \longrightarrow \mathrm{MU}^*(LG/T) \longrightarrow H^*(LG/T,\mathbb{Z})$$

is surjective. Since $H^*(LG/T,\mathbb{Z})$ is generated by the Schubert classes ε^w $(w \in W)$ dual to the Schubert cells C_w, and Σ_w is dual to C_w, the image of the stratum Σ_w under the composition map gives ε^w, establishing the desired surjectivity. □

Similarly, we have

THEOREM 5.6. *For a compact connected semi-simple Lie group G,*

$$\rho\colon \mathcal{U}^*(\Omega G) \longrightarrow \mathrm{MU}^*(\Omega G)$$

is surjective.

6. Bott–Samelson resolutions and operators of Bernstein–Gelfand–Gelfand type

In this section, we will construct some families of elements in the complex cobordism of the smooth Hilbert manifold LG/T, where G is a compact connected semi-simple Lie group G and T with maximal torus T. We will assume familiarity with notation and ideas of Pressley & Segal [**16**].

The homogeneous space LG/T is a complex manifold by the diffeomorphism $LG/T \cong LG_{\mathbb{C}}/\widetilde{B}$, where $LG_{\mathbb{C}}$ is the loop group of the complexification of G and $\widetilde{B}$ is a (positive) Borel subgroup containing T.

Associated to each simple affine root α_i is a parabolic subgroup P_i of $LG_{\mathbb{C}}$ containing $\widetilde{B}$. The projection map $p_i\colon LG_{\mathbb{C}}/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/P_i$ is then a smooth fibre bundle with fibre $P_i/\widetilde{B} \cong \mathbb{C}\mathrm{P}^1$.

As usual, the identity map on $LG_{\mathbb{C}}/\widetilde{B}$ represents the cobordism class $1 \in \mathcal{U}^0(LG/T)$ and

$$p_{i*}1 = [LG_{\mathbb{C}}/\widetilde{B}, p_i \circ \mathrm{Id}] \in \mathcal{U}^{-2}(LG_{\mathbb{C}}/P_i).$$

Since the bundle map p_i is a submersion, it is transverse to any smooth map, so

$$p_i{}^*(p_{i*}1) = [LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_i}{\sqcap} LG_{\mathbb{C}}/\widetilde{B}, p_i{}^*p_i] \in \mathcal{U}^{-2}(LG/T).$$

By definition of the transverse intersection $\sqcap$,

$$\begin{aligned} LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_i}{\sqcap} LG_{\mathbb{C}}/\widetilde{B} &\cong \{(\xi_1\widetilde{B}, \xi_2\widetilde{B}) : \xi_1 P_i = \xi_2 P_i\} \\ &= \{(\xi_1\widetilde{B}, \xi_2\widetilde{B}) : \xi_2{}^{-1}\xi_1 \in P_i\}. \end{aligned}$$

The last space is diffeomorphic to

$$LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B}$$

under the smooth correspondence

$$(\xi_1\widetilde{B}, \xi_2\widetilde{B}) \longleftrightarrow [\xi_2, \xi_2{}^{-1}\xi_1\widetilde{B}],$$

where $\widetilde{B}$ acts on $LG_{\mathbb{C}} \times P_i/\widetilde{B}$ by

$$b \cdot (\xi, x\widetilde{B}) = (\xi b, b^{-1}x\widetilde{B}).$$

Hence the pull-back map

$$p_i{}^*p_i\colon LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/\widetilde{B}$$

is given by

$$[\xi, x\widetilde{B}] \longrightarrow \xi \cdot x\widetilde{B}.$$

For $i \neq j$, let

$$p_j\colon LG_{\mathbb{C}}/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/P_j$$

be the $\mathbb{C}\mathrm{P}^1$-bundle associated with a different parabolic subgroup $P_j \neq P_i$. Then ${p_j}^* {p_j}_* {p_i}^* {p_i}_* 1$ is represented by the smooth map

$$s\colon LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_j}{\sqcap} LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/\widetilde{B}.$$

By the definition of product, the smooth manifold

$$LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_j}{\sqcap} LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B}$$

agrees with

$$\{(\xi_1\widetilde{B}, [\xi_2, x\widetilde{B}]) : \xi_1 P_j = \xi_2 \cdot xP_j\} = \{(\xi_1\widetilde{B}, [\xi_2, x\widetilde{B}]) : \xi_2 x^{-1}\xi_1 \in P_j\}.$$

The space

$$LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_j}{\sqcap} LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B}$$

is diffeomorphic to

$$LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i \underset{\widetilde{B}}{\times} P_j/\widetilde{B}$$

under the correspondence

$$(\xi_1\widetilde{B}, [\xi_2, x\widetilde{B}]) \longleftrightarrow [\xi_2, x, \xi_2 x^{-1}\xi_1\widetilde{B}].$$

The smooth map

$$LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i \underset{\widetilde{B}}{\times} P_j/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/\widetilde{B} \underset{LG_{\mathbb{C}}/P_j}{\sqcap} LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i/\widetilde{B}$$

is given by

$$[\xi, x, x'\widetilde{B}] \longmapsto (\xi x x'\widetilde{B}, [\xi, x'\widetilde{B}]).$$

The cobordism class of

$$s\colon LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_i \underset{\widetilde{B}}{\times} P_j/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/\widetilde{B}.$$

is given by

$$[\xi, x, x'\widetilde{B}] \longmapsto \xi x x'\widetilde{B}.$$

Continuing in a similar way by induction, for $I = (i_1, i_2, \ldots, i_n)$ satisfying $i_k \neq i_{k+1}$, we construct a space

$$Z_I = LG_{\mathbb{C}} \underset{\widetilde{B}}{\times} P_{i_1} \underset{\widetilde{B}}{\times} \cdots \underset{\widetilde{B}}{\times} P_{i_n}/\widetilde{B}$$

together with a smooth map

$$z_I\colon Z_I \longrightarrow LG_{\mathbb{C}}/\widetilde{B}$$

given by

$$[\xi, x_{i_1}, \ldots, x_{i_n}\widetilde{B}] \longmapsto \xi \cdot x_{i_1} \cdots x_{i_n}\widetilde{B}.$$

Here $\widetilde{B}$ acts by inverse multiplication on the right hand side of the each term in the sequence and by multiplication on the left hand side of each term for any $i \in I$.

Proposition 6.1. *For any sequence $I = (i_1, i_2, \ldots, i_n)$ such that $i_k \neq i_{k+1}$, Z_I is a smooth complex manifold and*

$$z_I\colon Z_I \longrightarrow LG_{\mathbb{C}}/\widetilde{B}$$

is a proper holomorphic map.

Definition 6.2. Let $r_{i_1} r_{i_2} \cdots r_{i_n}$ be the reduced decomposition of the element w of the affine Weyl group W. Then

$$\widetilde{Z}_w = P_{i_1} \underset{\widetilde{B}}{\times} P_{i_2} \underset{\widetilde{B}}{\times} \cdots \underset{\widetilde{B}}{\times} P_{i_n} / \widetilde{B} \subseteq Z_I$$

is called the *Bott–Samelson variety* associated to w.

Definition 6.3. A map $f \colon M \longrightarrow \mathbb{C}\mathrm{P}^n$ whose domain M is a complex manifold is *rational* if it has the form

$$f(x) = [1, f_1(z), \ldots, f_n(z)]$$

for meromorphic functions $f_1, \ldots, f_n$ on $M \longrightarrow \mathbb{C}$. A rational map $g \colon M \longrightarrow N$ to an algebraic variety $N \subseteq \mathbb{C}\mathrm{P}^n$ is one for which the composition with the inclusion $j \colon N \longrightarrow \mathbb{C}\mathrm{P}^n$ is a rational map $j \circ g \colon M \longrightarrow \mathbb{C}\mathrm{P}^n$.

A rational map $f \colon M \longrightarrow N$ is *birational* if there exists a rational map $g \colon N \longrightarrow M$ such that $f \circ g$ and $g \circ f$ are the identity maps. Two algebraic varieties are said to be *birationally isomorphic*, or simply *birational*, if there exists a birational map between them.

Definition 6.4. Let Y be a complex manifold with singularities and $\Theta \colon X \longrightarrow Y$ a holomorphic map. Then (X, Θ) is a *resolution of singularities* of Y if X is smooth and the map Θ is proper and birational.

The next result is from Demazure [**5**].

Theorem 6.5. *The map $z_w \colon \widetilde{Z}_w \longrightarrow \overline{C_w}$ is a resolution of singularities of the closure of the cell C_w of $LG_{\mathbb{C}}/\widetilde{B}$ in the usual complex topology.*

Since the resolution Z_I is a complex manifold and the map z_I is a holomorphic map, z_I has a natural complex orientation and so $[Z_I, z_I]$ is a element of $\mathcal{U}^*(LG/T)$. For any sequence $I = (i_1, \ldots, i_n)$, this class will be denoted x_I.

Let the $\mathbb{C}\mathrm{P}^1$-bundle associated with the parabolic subgroup P_i be

$$p_i \colon LG_{\mathbb{C}}/\widetilde{B} \longrightarrow LG_{\mathbb{C}}/P_i.$$

We will denote by A_i the operator

$$p_i{}^* p_{i*} \colon \mathcal{U}^*(LG/T) \longrightarrow \mathcal{U}^{*-2}(LG/T).$$

This operator is analogous to one introduced by Bernstein, Gelfand & Gelfand [**1**] and used by Kač and others in their work on the ordinary cohomology of flag spaces. Bressler & Evens [**2, 3**] defined similar operators in complex cobordism for finite dimensional flag spaces.

Proposition 6.6. *For each sequence $I = (i_1, \ldots, i_n)$, there is a cobordism class $x_I = A_I(1)$.*

Proposition 6.7. *For two sequences of the form $I = (i_1, \ldots, i_n)$ and $J = (i_1, \ldots, i_{n+1})$,*

$$A_{n+1}(x_I) = x_J.$$

Now we will describe a method for computing products of the cobordism classes x_I with characteristic classes of line bundles on LG/T. Let $L_\lambda \longrightarrow LG/T$ be the line bundle associated with a weight λ. Then $i^* i_* 1$ is the Euler class in the $\mathcal{U}^2(LG/T)$, where i is the zero-section of the line bundle L_λ.

THEOREM 6.8. *We have*

$$\chi(L_\lambda) \cup x_I = z_{I*}\chi(z_I^* L_\lambda).$$

PROOF. Since z_I is a complex orientable submersion, by Theorem 3.10 we have the equality. □

Given a index $I = (i_1, \ldots, i_n)$, we define new indices $I^k, I_{>k}$ by

$$I^k = (i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n), \quad I_{>k} = (i_{k+1}, \ldots, i_n).$$

A subindex J of I of length k is determined by a one-to-one order preserving map

$$\sigma\colon \{1, \ldots, k\} \longrightarrow \{1, \ldots, n\}$$

by the rule $j_m = i_{\sigma(m)}$. For the subindex J of I of length k there is a natural embedding $i_{J,I}\colon Z_J \longrightarrow Z_I$ defined by converting a $(k+1)$-tuple $(\xi, x_{j_1}, \ldots, x_{j_k})$ to the $n+1$-tuple in the $i_{\sigma(m)}$ th slot for $1 \leqslant m \leqslant k$ and the identity element elsewhere. A pair $x_J = [Z_J, i_{J,I}]$ represents an element of $\mathcal{U}^*(Z_I)$. The classes of the form x_J are used to obtain an expression for $\chi(L_\lambda)$.

A complex line bundle is determined up to isomorphism by its first Chern class $c_1(L)$ in integral cohomology. The Picard group $\mathrm{Pic}(Z_I)$ of line bundles on Z_I is isomorphic to $H^2(Z_I, \mathbb{Z})$, which is free with a basis consisting of elements with liftings to $\mathcal{U}^*(Z_I)$ that can be chosen to be the x_{I^k}. The first Chern class is given as

$$c_1\colon \mathrm{Pic}(Z_I) \longrightarrow H^2(Z_I, \mathbb{Z}).$$

$H^2(Z_I, \mathbb{Z})$ is free with basis consisting of classes x_{I^k} ($1 \leqslant k \leqslant n$). Therefore we can choose a basis for $\mathrm{Pic}(Z_I)$ consisting of line bundles L_k, where $1 \leqslant k \leqslant n$, satisfying

$$c_1(L_k) = x_{I^k}.$$

We take L_k to be the line bundle associated with the divisor Z_{I^k}. This means L_k has a section which intersects the zero section transversely on Z_{I^k} so that $\chi(L_k) = x_{I^k}$ in complex cobordism. This basis relates line bundles to a basis for $\mathcal{U}^*(Z_I)$.

In ordinary cohomology, the Euler class of a line bundle on Z_I is a linear combination of the elements x_{I^k}. The proof of the following is essentially to be found in Bressler & Evens [**3**].

THEOREM 6.9. *Let λ be a weight. Let $I = (i_1, \ldots, i_n)$ and r_I be the corresponding product of reflections. Then the line bundle L_λ on Z_I decomposes as*

$$L_\lambda = \bigotimes_{k=1}^{n} L_k^{-\langle r_{I_{>k}}\lambda, \mathbf{a}_{i_k}\rangle}.$$

7. Cobordism classes related to Pressley–Segal stratifications

In this section, we will show that stratifications introduced by Pressley & Segal [**16**] give rise some further interesting cobordism classes in $\mathcal{U}^*(LG/T)$, where LG is the smooth loop group of the finite dimensional compact connected semi-simple Lie group G with maximal torus T. Again we will assume familiarity with notation and ideas of Pressley & Segal [**16**].

Let H be a fixed separable Hilbert space. The Grassmannian $\mathrm{Gr}\, H$ of [**16**] is a separable Hilbert manifold, and the stratum $\Sigma_S \subseteq \mathrm{Gr}(H)$ is a locally closed contractible complex submanifold of codimension $\ell(S)$ and the inclusion map $\Sigma_S \longrightarrow \mathrm{Gr}(H)$ is a proper Fredholm map. Therefore, we have

Theorem 7.1. *The stratum* $\Sigma_S \longrightarrow \mathrm{Gr}(H)$ *represents a class in* $\mathcal{U}^{2\ell(S)}(\mathrm{Gr}(H))$.

These strata Σ_S are dual to the Schubert cells C_S in the following sense:

- the dimension of C_S is the codimension of Σ_S;
- C_S meets Σ_S transversely in a single point, and meets no other stratum of the same codimension.

The loop group LG acts via the adjoint representation on the Hilbert space

$$H_{\mathfrak{g}} = \mathrm{L}^2(S^1; \mathfrak{g}_{\mathbb{C}}),$$

where $\mathbf{g}_{\mathbb{C}}$ is the complexified Lie algebra of G. If $\dim G = n$, we can identify $H_{\mathfrak{g}}$ with H^n and since the adjoint representation is unitary for a suitable Hermitian inner product, this identifies LG with a subgroup of $L\mathrm{U}(n)$. Then [**16**] shows how to identify the based loop group ΩG with a submanifold of $\Omega\mathrm{U}(n)$, which can itself be identified with a submanifold $\mathrm{Gr}(H_{\mathfrak{g}})$.

Then ΩG inherits a stratification with strata Σ_λ indexed by homomorphisms $\lambda\colon \mathbb{T} \longrightarrow T$. Each stratum $\Sigma_\lambda \subseteq \Omega G$ is a locally closed contractible complex submanifold of codimension d_λ, and the inclusion map $\Sigma_\lambda \longrightarrow \Omega G$ is an admissible Fredholm map. Then

Theorem 7.2. *For each* λ*, the inclusion* $\Sigma_\lambda \longrightarrow \Omega G$ *represents a class in* $\mathcal{U}^{2d_\lambda}(\Omega G)$.

Such stratifications also exist for the homogeneous space LG/T for every compact connected semi-simple Lie group G with $T \leqslant G$ a maximal torus.

Theorem 7.3. *For* $w \in \widetilde{W}$*, the inclusion* $\Sigma_w \longrightarrow LG/T$ *represents a class in* $\mathcal{U}^{2\ell(w)}(LG/T)$.

Concluding remarks

The evidence of this paper suggests that it is possible to extend Quillen's definition of cobordism to some class of infinite dimensional manifolds perhaps with a slightly weaker equivalence relation to ensure that the conjectural surjections or isomorphisms of Section 4 hold. However, it may be that the problems with transversality encountered can be overcome and provided that reasonable generalizations of the standard finite dimensional results are indeed true then our approach may lead to more precise understanding of the theory we describe. We hope that our efforts may at least lead to further consideration of these matters.

References

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