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50th BMC talk

Arithmetic invariants and periodicity in stable homotopy theory

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§1 The homotopy problem

In this talk, spaces X are always based at points $x_0 \in X$ and maps $f\colon X \longrightarrow Y$ are always based maps for which $f(x_0) = y_0$.

Two maps $f, g\colon X \longrightarrow Y$ are *homotopic* (written $f \simeq g$) if there is a map $H\colon [0,1] \times X \longrightarrow Y$ for which

$$H(0,x) = f(x),\ H(1,x) = g(x),\ H(t,x_0) = y_0.$$

Homotopy is an equivalence relation on maps $X \longrightarrow Y$, the set of equivalence classes being denoted $[X,Y]$. For $n \geqslant 0$, $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the unit sphere based at $\mathbf{e}_1$, $\pi_n(X) = [\mathbb{S}^n, X]$. For $n \geqslant 1$, this is a group, and abelian if $n > 1$.

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The mapping set $[X, Y]$ is functorial in the variables X, Y; indeed, if $f_1 \simeq f_2 \colon X' \longrightarrow X$ and $g_1 \simeq g_2 \colon Y \longrightarrow Y'$, the induced maps satisfy

$$f_1{}^* = f_2{}^* \colon [X, Y] \longrightarrow [X', Y],$$
$$g_{1*} = g_{2*} \colon [X, Y] \longrightarrow [X, Y'].$$

The *smash product* $X \wedge Y$ is the space

$$X \wedge Y = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y},$$

based at the equivalence class class of (x_0, y_0). We have $\mathbb{S}^m \wedge \mathbb{S}^n \cong \mathbb{S}^{m+n}$.

We set $\Sigma^n X = \mathbb{S}^n \wedge X$ and $\Sigma X = \mathbb{S}^1 \wedge X$. $[\Sigma X, Y]$ is a group and $[\Sigma^n X, Y]$ is abelian for $n \geqslant 2$.

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For $n \geqslant 0$, there are suspension maps

$$\Sigma \colon [\Sigma^n X, \Sigma^n Y] \longrightarrow [\Sigma^{n+1} X, \Sigma^{n+1} Y]$$

which are group homomorphisms apart from the case where $n = 0$. The limit

$$\{X, Y\} = \varinjlim_n [\Sigma^n X, \Sigma^n Y]$$

is called the set of *stable homotopy classes* of maps $X \longrightarrow Y$ and is an abelian group. By the *Freudenthal Suspension Theorem,* this limit is actually attained. Notice that for $k \geqslant 0$,

$$\{\Sigma^k X, \Sigma^k Y\} \cong \{X, Y\}.$$

For $n \in \mathbb{Z}$, we set

$$\{X, Y\}_n = \{X, Y\}^{-n} = \begin{cases} \{\Sigma^n X, Y\} & \text{if } n \geqslant 0, \\ \{X, \Sigma^{-n} Y\} & \text{if } n < 0, \end{cases}$$

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and

$$\pi_n^S(X) = \{\mathbb{S}^n, X\}_0 = \{\mathbb{S}^0, X\}_n = \{\mathbb{S}^0, X\}_{-n}.$$

If X, Y are finite CW complexes, then $\{X, Y\}_*$ is isomorphic to $\pi_*^S(Z)$ for some Z, so it is important to determine or try to understand $\pi_*^S(X)$ for finite CW complexes X.

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§2 Homology theories and the Adams spectral sequence

A homology theory on finite CW complexes is a covariant functor $E_*(\ )$ for which $f \simeq g \colon X \longrightarrow Y$ induce $f_* = g_* \colon E_*(X) \longrightarrow E_*(Y)$ and there are Mayer–Vietoris sequences for cofibrations. A multiplicative homology theory $E_*(\ )$ is one where there are natural pairings $E_*(X) \otimes E_*(Y) \longrightarrow E_*(X \wedge Y)$ and $E_*(X)$ is naturally a module over the graded commutative ring $E_* = E_*(\mathbb{S}^0)$. From the Mayer–Vietoris sequence, for $k \geqslant 1$,

$$E_n(X) \cong E_{n+1}(\Sigma X) \cong E_{n+k}(\Sigma^k X),$$

so such homology theories are intrinsically ‘stable’.

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There is a spectral sequence (natural in X)

$$\mathrm{E}_2^{s,t}(X) = \mathrm{Ext}_{E_*E}^{s,t}(E_*, E_*(X)) \implies \pi_{t-s}^S(X).$$

For connective theory this spectral sequence converges to something like $\pi_*^S(X)$ or an arithmetic modification. In particular, when $X = \mathbb{S}^0$, the target is something like $\pi_*^S(\mathbb{S}^0)$, the stable homotopy groups of spheres. If E_* is periodic, the spectral sequence converges to something much less obvious.

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§3 Complex bordism

Complex bordism, $MU_*(\ )$, is an important homology theory in which $MU_n(X)$ is defined using maps of closed stably almost complex n-dimensional manifolds into X and imposing a bordism relation. The coefficient ring is $MU_* = \mathbb{Z}[u_n : n \geqslant 1]$ with $u_n \in MU_{2n}$. Moreover, there is a geometrically defined formal group law associated to this theory which is a universal FGL. This makes MU_* extremely interesting since there are many algebraic constructions that can be done using this universality. The Adams spectral sequence based on complex bordism is one of the most powerful tools in stable homotopy theory.

Theorem 1 (Devinatz–Hopkins–Smith) *For a finite CW complex X, $f \in \{X, X\}_n$ is nilpotent if and only if*

$$f_* = 0 \colon MU_*(X) \longrightarrow MU_*(X).$$

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§4 *K*-theory

K-theory provides a periodic homology theory $K_*(\ )$ where $K_* = \mathbb{Z}[t, t^{-1}]$ where $t \in K_2$ is the Bott periodicity generator. The Ext groups for spheres were calculated by Adams–Baird then Miller–Ravenel–Wilson and have

$$\mathrm{Ext}^{1,2n}_{K_*K}(K_*, K_*) = \begin{cases} \mathbb{Z}/\mathrm{m}(|n|) & \text{if } n \neq 0, \\ \mathbb{Q}/\mathbb{Z} & \text{if } n = 0. \end{cases}$$

Here

$$\mathrm{m}(|n|) = \text{denominator } \frac{B_n}{2n},$$

where B_n is the n th Bernoulli number. For an odd prime p,

$$\mathrm{m}(n)_p = \begin{cases} p^{r+1} & \text{if } n = (p-1)p^r n_0 \text{ with } p \nmid n_0, \\ 1 & \text{otherwise.} \end{cases}$$

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In fact,

$$\mathrm{Ext}^{1,2n} \subset \mathbb{Q}/\mathbb{Z} \otimes K_{2n} = \mathbb{Q}/\mathbb{Z} \otimes t^n$$

and a generator is provided by $(B_n/2n) \otimes t^n$.

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For $k > 0$, these generators detect maps

$$\alpha_{r,k} \in \pi^S_{2k(p-1)p^r-1}(\mathbb{S}^0)$$

whose construction we will now outline.

Let $M(d) = \mathbb{S}^0 \cup_d \mathbb{S}^1$ where $d \in \pi^S_0(\mathbb{S}^0)$ is a stable map of degree d. Then $K_*(M(d)) = K_*/(d)$.

There are stable maps

$$A^k_r \in \{M(p^{r+1}), M(p^{r+1})\}_{2k(p-1)p^r}$$

for which the induced map A^k_{r*} in K-homology is multiplication by the unit $t^{2k(p-1)p^r}$, hence A^k_r is not nilpotent.

We get $\alpha_{r,k}$ by using the following commutative diagram of stable

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maps.

$$\begin{array}{ccc}
\mathbb{S}^{2k(p-1)p^r} & \xrightarrow{\alpha_{r,k}} & \mathbb{S}^1 \\
{\scriptstyle \text{inclusion}}\downarrow & & \uparrow{\scriptstyle \text{projection}} \\
\Sigma^{2k(p-1)p^r} M(p^{r+1}) & \xrightarrow{A^k_r} & M(p^{r+1})
\end{array}$$

Then

$$\alpha_{r,k} \in \pi^S_{2k(p-1)p^r}(\mathbb{S}^1) = \pi^S_{2k(p-1)p^r-1}(\mathbb{S}^0).$$

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The K-theory Adams spectral sequence for $\mathbb{S}^0$ converges to something which contains terms of negative degree. For example, the above generators for $\mathrm{Ext}^{1,-2n}$ $(n > 0)$ survive to E_∞. Convergence here is to the homotopy of the K-localization of $\mathbb{S}^0$ which is a 'spectrum' not a space.

The K-theory Ext groups can be related to continuous group cohomology:

$$\mathrm{Ext}^{s,*}_{K_*K}(K_*, K_*/(p^k)) = H^s_c(\mathbb{Z}_p^\times; \mathbb{Z}/(p^k)[t, t^{-1}]).$$

The appearance of the Bernoulli numbers is 'explained' by this since they are known to arise in p-integration and this kind of continuous cohomology.

The action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}/(p^k)t^n$ is that of the nth power of the natural representation reduced modulo p^k. It is induced from the stable

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p-adic Adams operations

$$\psi^\alpha \quad (\alpha \in \mathbb{Z}_p^\times).$$

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§5 Elliptic homology

Let $p > 3$ be a prime. Elliptic homology $E\ell\ell_*(\ )$ is a multiplicative homology theory where in the coefficient ring $E\ell\ell_*$ (at least for the p-local version) $E\ell\ell_{2n}$ consists of the group of all modular forms of weight n for $\mathrm{SL}_2(\mathbb{Z})$, meromorphic at infinity and having q-expansion coefficients in $\mathbb{Z}_{(p)}$. Thus,

$$E\ell\ell_* = \mathbb{Z}_{(p)}[Q, R, \Delta^{-1}]$$

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where

$$Q = E_4 = 1 + 240 \sum_{1 \leqslant r} \sigma_3(r) q^r,$$

$$R = E_6 = 1 - 504 \sum_{1 \leqslant r} \sigma_5(r) q^r,$$

$$\Delta = \frac{(Q^3 - R^2)}{1728} = q \prod_{n \geqslant 1} (1 - q^n)^{24}.$$

We also set

$$A = E_{p-1} = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{1 \leqslant r} \sigma_{p-2}(r) q^r.$$

Notice that this q-expansion satisfies $A \equiv 1 \mod (p)$.

Elliptic homology is periodic with Δ providing a periodicity operator of degree 24.

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The Adams E_2-term for elliptic homology has

$$\mathrm{Ext}^{1,2n}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*) = \begin{cases} \mathbb{Z}/\mathrm{m}(|n|) & \text{if } n > 0, \\ 0 & \text{if } n \leqslant 0. \end{cases}$$

The proof of this involves reducing to

$$\varinjlim_k \mathrm{Ext}^{0,2n}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p^k)) \cong \mathrm{Ext}^{1,2n}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*),$$

and then using Hecke operators T_n $(p \nmid n)$ to show that only holomorphic modular forms modulo p^k can be in such groups then using Atkin's Hecke operator U_p and a result of Serre to show exactly which groups can occur.

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An alternative approach involves inverting A in the ring $E\ell\ell_*/(p^k)$ and reducing to calculations formally similar to those in K-theory, then using arguments like the above to rule out the negative degree part.

Both these approaches make use of the 'ordinary' theory of mod p and p-adic modular forms due to Swinnerton-Dyer–Serre–Katz.

The sequence p, A, Δ is regular in $E\ell\ell_*$. The periodicity can be interpreted in terms of yet another Eisenstein function (of weight $p+1$)

$$B = E_{p+1} = 1 - \frac{2(p+1)}{B_{p+1}} \sum_{1 \leqslant r} \sigma_p(r) q^r.$$

We have

$$B^{p-1} \equiv -\left(\frac{-1}{p}\right) \Delta^{(p^2-1)/12} \bmod (p, A).$$

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A calculation of the rest of

$$\mathrm{Ext}^{*,*}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*)$$

follows by reduction to

$$\mathrm{Ext}^{*,*}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p, A))$$

where $E\ell\ell_*/(p, A)$ is the *supersingular reduction* of $E\ell\ell_*$. This is calculated by interpreting stable operations in terms of isogenies of supersingular elliptic curves over $\bar{\mathbb{F}}_p$ and considering an appropriate kind of cohomology.

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This cohomology can be shown to agree with the continuous cohomology of the automorphism group of the formal group law associated to such an elliptic curve, agreeing with earlier cohomological approximations. There seems to be some work on p-adic interpolation in the supersingular context, but it seems to be less well developed than the ordinary theory.

As an example of what is obtained, we have

$$\mathrm{Ext}^{0,*}_{E\ell\ell_* E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p, A)) = \mathbb{F}_p[B^{p-1}, B^{-(p-1)}].$$

With the aid of similar techniques to those used for K-theory, elements of $\pi_*^S(\mathbb{S}^0)$ can be constructed to realise supersingular elements in the elliptic homology Ext groups. In particular, we can realise each $B^{k(p-1)}$ $(k > 0)$ with an element in homotopy.

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