BOCKSTEIN OPERATIONS IN MORAVA K-THEORIES

Andrew Baker & Urs Würgler

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§0 Introduction.

The Morava K-theories $K(n)^*(\ )$ (for p a prime and $0 < n < \infty$) form a collection of multiplicative cohomology theories whose central rôle in homotopy theory is now well established. However, even though they have been intensively studied there are still many aspects of their structure which remain undeveloped. In this paper we give a construction for families of operations reminiscent of the classical Bockstein operations in ordinary mod p cohomology which is related both to the formal group theoretic techniques exploited by Jack Morava [8] and also the more recently discovered A_∞ structures on $K(n)$ and the related spectra $\widehat{E(n)}$. Given the underlying algebra involved in both of these areas we are tempted to speculate that there are as yet undiscovered connections with crystalline cohomology and the de Rham complex–however, in the present work we merely hint at this.

In §1 we give an exposition of the theory of liftings of Lubin-Tate formal group laws from algebras over $K(n)_*$ to local Artinian rings A_* for which the maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^2 = 0$. In particular, such deformations give rise to automorphisms of the group scheme of p th roots of 0 for the lifted formal group law and in turn this produces a sequence of n derivations which are essentially our Bocksteins in $K(n)$-theory.

In §2 we define our family of operations $Q_k\colon K(n)^*(\ ) \longrightarrow K(n)^{*+2p^k-1}(\ )$ for $0 \leqslant k \leqslant n-1$, and verify that they are $K(n)_*$ derivations, determined upon the spaces $\mathbb{C}\mathrm{P}^\infty$ and the skeleton $L^{(1)} = (B\mathbb{Z}/p)^{[2p^n-1]}$. This shows also that they agree with earlier constructions.

In §3 we construct higher order Bocksteins which correspond to morphisms of A_∞ module spectra over $\widehat{E(n)}$ of the form

$$Q_v^k\colon E(n)/I_n^k \longrightarrow \Sigma^{|v|+1}K(n)$$

where $v = v_0^{r_0}\cdots v_{n-1}^{r_{n-1}} \in I_n \triangleleft \widehat{E(n)}_*$. These are determined by their effect on both $\mathbb{C}\mathrm{P}^\infty$ and certain skeleta of the classifying spaces $B\mathbb{Z}/p^k$. Closely related to these operations

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is a tower of A_∞ module spectra over $\widehat{E(n)}$

$$* \longleftarrow K(n) = E(n)/I_n \longleftarrow E(n)/I_n^2 \longleftarrow \cdots \longleftarrow E(n)/I_n^k \longleftarrow \cdots$$

and for each spectrum X an associated spectral sequence with $\mathrm{E}_1(X)$ a direct sum of copies of $K(n)^*(X)$, d_1 a direct sum of our generalised Bocksteins and converging to $\widehat{E(n)}^*(X)$; thus, the functors $K(n)^*(\ )$ and $\widehat{E(n)}^*(\ )$ determine each other.

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§1 Liftings of height n Lubin-Tate formal group laws.

Let p be an odd prime and $0 < n < \infty$. Recall that the nth Morava K-theory at the prime p is a complex oriented multiplicative cohomology theory $K(n)^*(\ )$ whose coefficient ring is the graded field

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$$

where $v_n \in K(n)_{2(p^n-1)}$. The spectrum $K(n)$ representing this theory is a commutative ring spectrum. Moreover, the canonical orientation class $x^{K(n)} \in K(n)^2(\mathbb{C}\mathrm{P}^\infty)$ has an associated formal group law $F^{K(n)}$ over $K(n)_*$ whose p-series is given by

$$[p]_{K(n)}X = v_n X^{p^n}$$

and is therefore a *height n Lubin-Tate* group law [6]. In general, a height n Lubin-Tate group law over a (graded) $\mathbb{F}_p$ algebra A_* is a formal group law F over A_* which has p-series of the from

$$[p]_F X = uX^{p^n},$$

with u a unit in A_*; the canonical p-typification of F then has the same p-series as F and is induced by a homomorphism (of $\mathbb{F}_p$ algebras) $K(n)_* \longrightarrow A_*$ sending v_n to u. (For those unfamiliar with Lubin-Tate theory this will suffice as a definition.)

The full algebra of stable operations in $K(n)$-theory was determined in [17], [15] and [16]; its most convenient description is as the $K(n)_*$ dual of the cooperation algebra $K(n)_*(K(n))$ whose structure we now recall.

Theorem (1.1). *There is an isomorphism of $K(n)_*$ algebras*

$$K(n)_*(K(n)) \cong \Sigma(n)_* \otimes_{K(n)_*} \Lambda_{K(n)_*}(a_i : 0 \leqslant i \leqslant n-1)$$

where

$$\Sigma(n)_* = K(n)_*(t_i : 1 \leqslant i)$$

subject to relations

$$t_i^{p^n} = v_n^{p^i-1} t_i$$

and with $|t_i| = 2(p^i-1),\ |a_i| = 2p^i - 1$.

In fact, the subalgebra $\Sigma(n)_*$ can be identified with the image of the $K(n)$ homology of a spectrum $E(n)$; in the literature, it is also frequently denoted $K(n)_*K(n)$, however this is likely to be confused with $K(n)_*(K(n))$ and so we avoid such notation. Recall from [10] the ring spectrum $E(n)$ for which there are canonical morphisms of ring spectra

$$BP \longrightarrow E(n) \xrightarrow{\rho} K(n).$$

Proposition (1.2). *The map $\rho_*\colon K(n)_*(E(n)) \longrightarrow K(n)_*(K(n))$ is a monomorphism of $K(n)_*$ algebras with image equal to $\Sigma(n)_*$.*

Corollary (1.3). *There is an isomorphism of $K(n)_*$ modules*

$$K(n)^*(E(n)) \cong \operatorname{Hom}_{K(n)_*}(\Sigma(n)_*, K(n)_*)\,.$$

Now $K(n)_*(K(n))$ actually has the further structure of a Hopf algebra over $K(n)_*$ (see [1] and [9]). The coproduct

$$\psi\colon K(n)_*(K(n)) \longrightarrow K(n)_*(K(n)) \otimes_{K(n)_*} K(n)_*(K(n))$$

is determined by our next result. We use the symbol $\sum^F$ to denote summation with respect to the formal group law F.

Proposition (1.4). *For a suitable choice of the generators t_i, a_i we have*

$$\sum_{0\leqslant k}^{F^{K(n)}} \left(\psi(t_k)X^{p^k}\right) = \sum_{\substack{0\leqslant i\\ 0\leqslant j}}^{F^{K(n)}} \left(t_i \otimes t_j^{p^i} X^{p^{i+j}}\right)$$

and

$$\psi(a_k) = \sum_{0\leqslant i\leqslant k} a_i \otimes t_{k-i}^{p^i} + 1 \otimes a_k.$$

The sub-Hopf algebra $\Sigma(n)_*$ was studied closely in [7], [8], and [9]. A crucial aspect of such work is its relationship with Lubin-Tate theory.

Now the generators a_k are dual to Bockstein type operations in $K(n)^*(\ )$, constructed by Johnson and Wilson [5], Würgler [16], and Yagita [18]. Their constructions all proceed by first constructing the BP module spectra $P(m)$ fitting into cofibre sequences of module spectra

$$P(m) \xrightarrow{\cdot v_m} P(m) \longrightarrow P(m+1) \xrightarrow{\partial_m} \Sigma^{2p^m-1}P(m)$$

where the coboundaries ∂_m induce operations in $K(n)^*(\ )$ having properties analogous to those of the classical Bocksteins in ordinary mod p cohomology. One of the aims of the present work is to give a construction for these Bocksteins in Morava K-theory using the existence of A_∞ structures on certain spectra $\widehat{E(n)}$, as defined in [2]; indeed, our results show that such operations are intimately related to these A_∞ structures.

We now describe the lifting theory of Lubin-Tate group laws which underlies our construction of Bockstein operations. We remark that a similar description also applies to the case of ordinary mod p cohomology.

Let $f\colon K(n)_* \longrightarrow A_*$ be homomorphism of commutative graded $\mathbb{F}_p$ algebras. Then f induces a formal group law $F = f_*F^{K(n)}$ upon A_*, also a height n Lubin-Tate law. Now an extension to a homomorphism $\widetilde{f}\colon \Sigma(n)_* \longrightarrow A_*$ is equivalent to a *strict automorphism* $\varphi_f\colon F \longrightarrow F$, i.e., a power series $\varphi_f(X) \in A_*[X]$ with leading term X and satisfying

$$\varphi_f(F(X,Y)) = F(\varphi_f(X), \varphi_f(Y));$$

the precise relationship is given by the formula

$$\varphi_f{}^{-1}(X) = \sum_{0\leqslant k}^{F} \left(\widetilde{f}(t_k)X^{p^k}\right)$$

or equivalently,

$$\varphi_f(X) = \sum_{0\leqslant k}^{F} \left(\widetilde{f}(\chi(t_k))X^{p^k}\right)$$

where $\chi\colon \Sigma(n)_* \longrightarrow \Sigma(n)_*$ is the restriction to $\Sigma(n)_*$ of the conjugation (or antipode) on the Hopf algebra $K(n)_*(K(n))$.

Following the notation of [4], let $\mathbf{L}\text{-}\mathbf{T}_n(A_*)$ denote the category of all such height n Lubin-Tate group laws over A_* and their strict isomorphisms. Then $\mathbf{L}\text{-}\mathbf{T}_n(\ )$ is a groupoid valued functor on the category of graded $\mathbb{F}_p$ algebras, which is in fact a coproduct of disjoint automorphism groups (this is equivalent to the fact that $\Sigma(n)_*$ is a Hopf *algebra* over $K(n)_*$ rather than just a Hopf *algebroid*–see [9]). Thus there is an isomorphism of categories

$$\operatorname{Mor}\mathbf{L}\text{-}\mathbf{T}_n(A_*) \cong \coprod_{F\in\operatorname{Obj}\mathbf{L}\text{-}\mathbf{T}_n(A_*)} \operatorname{Aut}(F).$$

We have

Proposition (1.5). *There is an isomorphism of groupoids (natural in A_*)*

$$\mathbf{L}\text{-}\mathbf{T}_n(A_*) \cong \mathbf{Alg}_{\mathbb{F}_p}\left(\Sigma(n)_*, A_*\right).$$

In this statement, the second functor inherits its multiplication from the coproduct on the Hopf algebra $\Sigma(n)_*$.

In [2], we use (1.5) to obtain all multiplicative operations

$$E(n)^*(\ ) \longrightarrow \overline{K(n)}^{\,*}(\ )$$

where the latter theory is defined by

$$\overline{K(n)}^{\,*}(\ ) = \overline{K(n)}_* \otimes_{K(n)_*} K(n)^*(\ )$$

with

$$\overline{K(n)}_* = K(n)_*[u]/(u^{p^n-1} - v_n)$$

for $u \in \overline{K(n)}_2$. As the representing spectrum for this new theory we can take

$$\overline{K(n)} = \bigvee_{\substack{\overline{a}\in\mathbb{Z}/r_n\\ 0\leqslant a\leqslant r_n-1}} \Sigma^{2a}K(n)$$

where $r_n = (p^n-1)/(p-1) = 1+p+\cdots p^{n-1}$; this is given a ring structure by extending that of $K(n)$ using the cyclic group structure of $\mathbb{Z}/r_n$. From [2] we recall the following result, essentially due to Jack Morava:

Theorem (1.6). *There is a bijection of sets*

$$\left\{\begin{matrix} \textit{Morphisms of ring spectra} \\ E(n) \longrightarrow \overline{K(n)} \end{matrix}\right\} \cong \left\{\begin{matrix} \textit{Strict automorphisms of} \\ F^{K(n)} \quad \textit{over} \quad \overline{K(n)} \end{matrix}\right\}.$$

Now let $\mathbf{Art}_p^2$ denote the category of graded Artinian local rings A_* with residue field $A_*/\mathfrak{m}$ of characteristic p and for which $\mathfrak{m}^2 = 0$. Clearly this is a full subcategory of the category $\mathbf{Art}_p$ of graded Artinian local rings with residue characteristic p used in [4]. Now fix an $A_* \in \operatorname{Obj} \mathbf{Art}_p^2$ and consider the category $\mathbf{lift}_n(A_*)$ of p-typical lifts of elements of $\mathbf{L\text{-}T}_n(A_*/\mathfrak{m})$ with strict isomorphisms as the morphisms. Within this groupoid we have the full subcategory $\mathbf{lift}_n^{(n)}(A_*)$ of *coheight* n lifts, i.e., those whose p-series have the form

$$[p]_F X = \sum_{0\leqslant k\leqslant n}^{F} \left(u_k X^{p^k}\right)$$

where $u_k \in \mathfrak{m}$ for $0 \leqslant k \leqslant n-1$ and u_n a unit in A_*. A central result of [4] is

Theorem (1.7). *There is an idempotent natural equivalence of groupoids (natural in A_*)*

$$\mathbf{e}\colon \mathbf{lift}_n(A_*) \longrightarrow \mathbf{lift}_n(A_*)$$

with image equal to $\mathbf{lift}_n^{(n)}(A_*)$.

For $F \in \operatorname{Obj} \mathbf{lift}_n(A_*)$, $\mathbf{e}(F)$ is canonically $*$-*isomorphic* to F, i.e. the isomorphism reduces modulo $\mathfrak{m}$ to the identity. Such an $\mathbf{e}(F)$ is then unique within its $*$-isomorphism class.

Now let $A_* \in \mathbf{Art}_p^2$ and $F \in \operatorname{Obj} \mathbf{lift}_n^{(n)}(A_*)$. Then for any $F' \in \operatorname{Obj} \mathbf{lift}_n^{(n)}(A_*)$, a $*$-isomorphism (not necessarily strict) $\varphi\colon F \longrightarrow F'$ has the form

$$\varphi(X) = X +_{F'} \sum_{0\leqslant k}^{F'} \left(a_k X^{p^k}\right)$$

with $a_k \in \mathfrak{m}$. But from [4] we learn that F' can only have coheight n if $F' = F$!

Now consider the A_* algebra

$$A_*(y) = A_*[[Y]]/([p]_F Y)$$

where $y = \overline{Y}$. Then as

$$[p]_F X \equiv u_n X^{p^n} \mod \mathfrak{m}$$

and A_* is $\mathfrak{m}$-adically complete, the *Weierstrass Preparation Theorem* applies and assures us of the existence of a polynomial $P_F(X) \in A_*[X]$ satisfying

$$\deg_{A_*} P_F = p^n,$$
$$P_F(X) \equiv u_n X^{p^n} \mod \mathfrak{m}$$

and

$$[p]_F X = P_F(X)(1 + Xg(X)) \tag{1.8}$$

where $g(X) \in A_*[[X]]$. Hence, $A_*(y)$ is a finite rank, free A_* algebra (the rank is p^n). Moreover, the formal group law F gives rise to a coproduct

$$y \longmapsto F(y \otimes 1, 1 \otimes y)$$

since the ideal of relations is clearly a Hopf ideal.

Lemma (1.9). *A $*$-isomorphism $\varphi\colon F \longrightarrow F'$ induces a Hopf algebra automorphism over A_* such that*

$$y \longmapsto \varphi(y).$$

Proof. We have from (1.8) that

$$y^{p^n} \in \mathfrak{m}A_*(y)$$

and an easy calculation now shows that

$$\varphi(y) = y + \sum_{0 \leqslant k \leqslant n-1} a_k y^{p^k}.$$

Similarly, we find that

$$\begin{aligned} \varphi\left(F(y \otimes 1, 1 \otimes y)\right) &= F'\left(\varphi(y) \otimes 1, 1 \otimes \varphi(y)\right) \\ &\in A_*(y) \otimes A_*(y). \end{aligned}$$

The result now follows. $\square$

Now observe that given such a $*$-isomorphism, the assignment

$$y \longmapsto \sum_{0 \leqslant k \leqslant n-1} a_k \otimes y^{p^k - 1}$$

extends to an A_* derivation

$$\partial_\varphi\colon A_*(y) \longrightarrow \mathfrak{m} \otimes_{A_*} A_*(y).$$

Upon reducing modulo $\mathfrak{m}$, we obtain an $A_*/\mathfrak{m}$ derivation

$$\overline{\partial}_\varphi\colon A_*/\mathfrak{m}(\overline{y}) \longrightarrow \mathfrak{m} \otimes_{A_*/\mathfrak{m}} A_*/\mathfrak{m}(\overline{y}).$$

This is equivalent to an $A_*/\mathfrak{m}$ module homomorphism

$$\mathrm{d}_\varphi\colon \Omega^1_{A_*/\mathfrak{m}}\left(A_*/\mathfrak{m}(\overline{y})\right) \longrightarrow \mathfrak{m} \otimes_{A_*/\mathfrak{m}} A_*/\mathfrak{m}(\overline{y})$$

where $\Omega^1_{A_*/\mathfrak{m}}$ denotes the module of 1-*forms* for $A_*/\mathfrak{m}(\overline{y})$, well known to be of rank 1 on the generator $\mathrm{d}\overline{y}$ over $A_*/\mathfrak{m}(\overline{y})$.

Now let $\theta\colon F'' \longrightarrow F$ be any strict isomorphism over A_*, where both F and F'' have coheight n. Upon reducing modulo $\mathfrak{m}$ we have an automorphism $\overline{\theta}\colon \overline{F} \longrightarrow \overline{F}$. Write

$$\theta(X) = \sum_{0\leqslant k}^{F} b_k X^{p^k}$$

for some $b_k \in A_*$.

Let $c_k \in \mathfrak{m}$ for $0 \leqslant k$; then there is an isomorphism of formal groups

$$\widetilde{\theta}\colon F''' \longrightarrow F$$
$$\widetilde{\theta}(X) = \sum_{0\leqslant k}^{F} (b_k + c_k) X^{p^k}$$

where F''' is defined by the formula

$$F'''(X,Y) = \widetilde{\theta}^{-1} F\left(\widetilde{\theta}(X), \widetilde{\theta}(Y)\right).$$

Calculation shows that

$$\begin{aligned}\varphi\widetilde{\theta}(y) &= \theta(y) + \sum_{0\leqslant k\leqslant n-1} a_k \theta(y)^{p^k} + \sum_{0\leqslant k\leqslant n-1} c_k y^{p^k} \\ &= \sum_{0\leqslant k\leqslant n-1} \left(b_k + \sum_{0\leqslant j\leqslant k} a_j b_{k-j}^{p^j} + c_k \right) y^{p^k}\end{aligned}$$

and hence

$$\overline{\partial}_{\varphi\widetilde{\theta}}(\overline{y}) = \partial_{\varphi}(\overline{y}) + \sum_{0\leqslant k\leqslant n-1} \left(a_k \otimes \theta(\overline{y})^{p^k} + c_k y^{p^k} \right).$$

Hence, writing $\overline{\partial}_{\varphi}$ in the form

$$\overline{\partial}_{\varphi} = \sum_{0\leqslant k\leqslant n-1} \delta_{\varphi}^{k} \otimes y^{p^k}$$

we have

$$\delta_{\varphi\widetilde{\theta}} = \sum_{0\leqslant j\leqslant i} b_{i-j}^{p^j} \delta_{\varphi}^{j} + c_i.$$

As a concrete example of this, we can take $A_* = E(n)_*/I_n^2$ and $\mathfrak{m} = I_n$, together with the canonical group law $F^{E(n)}$ induced by the natural homomorphism $E(n)_* \longrightarrow E(n)_*/I_n^2$. Associated to this is the Hopf algebra $E(n)_*/I_n^2(y)$ whose reduction modulo I_n is $K(n)_*(\overline{y})$. We can also consider the Hopf algebra $K(n)_*(\overline{y}) \otimes \Lambda(z)$ where we set $z = \mathrm{d}\overline{y}$ in degree -1 (y has degree 2) and declare the coproduct upon z to be trivial (i.e., $z \mapsto z \otimes 1 + 1 \otimes z$). There is a multiplicative coaction

$$\Psi_0\colon K(n)_*(\overline{y}) \otimes \Lambda(z) \longrightarrow K(n)_*(K(n)) \otimes_{K(n)_*} K(n)_*(\overline{y}) \otimes \Lambda(z),$$

defined by

$$\Psi_0(\overline{y}\,) = \sum_{0\leqslant k\leqslant n-1}^{F^{K(n)}} \left(t_k \otimes \overline{y}^{\,p^k}\right)$$
$$\Psi_0(z) = 1\otimes z + \sum_{0\leqslant k\leqslant n-1} a_k \otimes \overline{y}^{\,p^k}.$$

We have now described the basic algebraic setting underlying our discussion of Bockstein operations in $K(n)^*(\ )$ of §2.

§2 Bockstein operations in $K(n)^*(\ )$.

In this section we relate the ideas of §1 to a family of operations in $K(n)^*(\ )$. We will make use of results from [3] and indeed improve upon them.

Recall from [3] that the ring spectrum $\widehat{E(n)}$ admits an essentially unique A_∞ structure and that $K(n)$ is an A_∞ algebra spectrum whichever of the uncountably many A_∞ ring structures from [14] we impose. We also have

Theorem (2.1). *There is a cofibre sequence of A_∞ module spectra over $\widehat{E(n)}$,*

$$\bigvee_{0\leqslant k\leqslant n-1} \Sigma^{2p^k-2}K(n) \longrightarrow E(n)/I_n^2 \longrightarrow K(n)$$

which realises the exact sequence of $\widehat{E(n)}_*$ modules

$$\bigoplus_{0\leqslant k\leqslant n-1} \Sigma^{2p^k-2}K(n)_* \cong I_n/I_n^2 \longrightarrow E(n)_*/I_n^2 \longrightarrow K(n)_*.$$

In this statement, we use the n homomorphisms of $\widehat{E(n)}_*$ modules

$$K(n)_* \longrightarrow I_n/I_n^2$$
$$1 \longmapsto \overline{v_k} \in E(n)_*/I_n^2$$

with $0\leqslant k\leqslant n-1$, and denote the cofibre map by

$$Q = \bigvee_k Q_k\colon K(n) \to \bigvee_{0\leqslant k\leqslant n-1} \Sigma^{2p^k-2}K(n)$$

We remark that it is an A_∞ module spectrum morphism. As it stands, the morphisms $Q_k\colon K(n) \longrightarrow K(n)$ are somewhat mysterious, but we will show that they enjoy good properties.

Let $L^{(1)} = (B\mathbb{Z}/p)^{[2p^n-1]}$ be the $2p^n-1$ skeleton of $B\mathbb{Z}/p$. Then

$$K(n)^*({L^{(1)}}^+) \cong K(n)_*[Y]/(Y^{p^n}) \otimes \Lambda(z_1)$$

and we will denote the class of Y by Y_1. Here $|Y_1| = 2$ and $|z_1| = 1$.

Theorem (2.2). *The cohomology operation*

$$\overline{Q}\colon K(n)^*(\ ) \longrightarrow \bigoplus_k K(n)^*(\ )$$

induced by Q is a derivation over $K(n)_*$, satisfying

(1) *as an operation defined upon $K(n)^*(\mathbb{C}P^\infty)$, we have $\overline{Q} = 0$;*
(2) *as an operation defined upon $K(n)^*(L^{(1)})$, we have $\overline{Q}\,(Y_1) = 0$ and $\overline{Q}\,(z_1) = \sum_{0\leqslant k\leqslant n-1} {Y_1}^{p^k}$.*

We begin by observing that a $K(n)_*$ derivation $\mathcal{D}\colon K(n)_*(K(n)) \longrightarrow K(n)_*$ is determined by its effect upon the elements t_k, a_k of (1.1). Now in fact these are in the image of the homomorphism

$$f_*\colon K(n)_*\left(\mathbb{C}\mathrm{P}^\infty \times L^{(1)}\right) \longrightarrow K(n)_*(K(n))$$

where $f\colon \mathbb{C}\mathrm{P}^\infty \times L^{(1)} \longrightarrow \Sigma^3 K(n)$ is the external product of the two canonical maps $x^{K(n)}\colon \mathbb{C}\mathrm{P}^\infty \longrightarrow \Sigma^2 K(n)$ and $z_1\colon L^{(1)} \longrightarrow \Sigma K(n)$. This follows from the fact that $z_{1*}\colon K(n)_*(L^{(1)}) \longrightarrow K(n)_*(K(n))$ has the elements a_k modulo decomposables in its image (see [17]). We can now deduce that a stable operation

$$\overline{D}\colon K(n)^*(\ ) \longrightarrow K(n)^*(\ )$$

which is a derivation on each space X is determined by its effect upon $\mathbb{C}\mathrm{P}^\infty$ and $L^{(1)}$, since the derivation

$$\varepsilon \circ D_*\colon K(n)_*(K(n)) \xrightarrow{D_*} K(n)_*(K(n)) \xrightarrow{\varepsilon} K(n)_*$$

(where ε is the augmentation $K(n)_*(K(n)) \longrightarrow K(n)_*$) is forced from the $K(n)$-theory of the above spaces, since the image of f_* generates $K(n)_*(K(n))$ as a $K(n)_*$ algebra. The existence and uniqueness of a morphism of spectra $D\colon K(n) \longrightarrow \Sigma K(n)$ inducing $\overline{D}$ follows from the fact that $K(n)_*$ is a graded field, hence $K(n)^*(\ )$ theory is determined upon the category of finite CW complexes $\mathbf{CW^f}$.

Conversely, given such a derivation D we can define a stable operation

$$\overline{D}\colon K(n)^*(\ ) \longrightarrow K(n)^*(\ )$$

to be the composite

$$\begin{aligned} K(n)^*(\ ) \xrightarrow{\cong} \left(S^0 \wedge K(n)\right)^*(\ ) \xrightarrow{\overline{\eta_{K(n)}}} (K(n) \wedge K(n))^*(\ ) \xrightarrow{\cong} \\ K(n)_*(K(n)) \otimes_{K(n)_*} K(n)^*(\ ) \xrightarrow{\varepsilon} K(n)_* \otimes K(n)^*(\ ) \xrightarrow{\cong} K(n)^*(\ ) \end{aligned}$$

where $\overline{\eta_{K(n)}}$ is induced by the unit $S^0 \longrightarrow K(n)$. In fact, this operation is a $K(n)_*$ derivation on each $K(n)^*(X)$. Hence, we have

Proposition (2.3). *Let D_k, with $0 \leqslant k \leqslant n-1$, be a $K(n)_*$ derivation*

$$D_k\colon K(n)^*(L^{(1)}) \longrightarrow K(n)^*(L^{(1)})$$

for which

$$D_k(z_1) = {Y_1}^{p^k}$$

and

$$D_k(Y_1) = 0.$$

Then there is a unique stable cohomology operation

$$\widetilde{D_k}\colon K(n)^*(\ ) \longrightarrow K(n)^*(\ )$$

which is a derivation for each space X, and satisfying

(1) *as an operation defined upon $K(n)^*(\mathbb{C}P^{\infty+})$, we have $\widetilde{D_k} = 0$;*
(2) *as an operation defined upon $K(n)^*({L^{(1)}}^+)$, we have $\widetilde{D_k}(Y_1) = 0$ and $\widetilde{D_k}(z_1) = {Y_1}^{p^k}$.*

At this stage we have certainly established the existence of a family of operations which behave as we might expect; we shall now relate these to the cofibre maps Q_k. To do this we must recall the definition of the extension of (2.1),

$$\bigvee_{0\leqslant k\leqslant n-1} \Sigma^{2p^k-2}K(n) \longrightarrow E(n)/I_n^2 \longrightarrow K(n).$$

We will give modified account of the result in [3].

In the spectral sequence

$$\mathrm{E}_2^{*\,*} = \mathrm{Ext}^{*\,*}_{\widehat{E(n)}_*}\left(K(n)_*, \bigoplus K(n)_*\right) \Longrightarrow \pi_*\left(\mathrm{RHom}(K(n), \bigvee K(n))\right)$$

of [12], the cofibre Q is detected by the class of the extension

$$\bigoplus_{0\leqslant k\leqslant n-1} \Sigma^{2p^k-2}K(n)_* \cong I_n/I_n^2 \longrightarrow E(n)^*/I_n^2 \longrightarrow K(n)_*.$$

We must show that the effect of $\overline{Q}$ upon $K(n)^*(L^{(1)})$ is as stated in (2.3). Now from [11] and [12] we obtain a Universal Coefficient spectral sequence

$$\mathrm{E}_2^{*\,*}(L^{(1)}) = \mathrm{Ext}^{*\,*}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(L^{(1)}), M_*\right) \Longrightarrow M^*(L^{(1)})$$

which is natural in the A_∞ module spectrum M over $\widehat{E(n)}$, in the sense that an A_∞ morphism $f\colon M_1 \longrightarrow M_2$ induces a morphism of spectral sequences agreeing at E_2 with the homomorphism induced by $f_*\colon \pi_*(M_1) \longrightarrow \pi_*(M_2)$. Moreover, the effect of Q is determined modulo filtration by the effect of the connecting homorphism of the above extension on E_2.

The space $L^{(1)}$ can be identified with the sphere bundle of the pth tensor power of the Hopf bundle $\eta \longrightarrow \mathbb{C}\mathrm{P}^{p^n-1} = \mathbb{C}\mathrm{P}^{(1)}$, and so the Gysin sequence gives rise to a free resolution

$$\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)}) \xrightarrow{(j_1)_*} \widehat{E(n)}_*(M\lambda_1) \to \widehat{E(n)}_*(\Sigma L^{(1)})$$

of $\widehat{E(n)}_*$ modules, where we are throughout using *reduced* homology and $M\lambda_1$ denotes the Thom complex of $\lambda_1 = \eta^p$. The $K(n)_*$ basis of $K(n)^*(L^{(1)})$ consisting of the classes z_1, ${Y_1}^t$, $z_1{Y_1}^t$ with $1 \leqslant t \leqslant p^n - 1$, are easily represented in this spectral sequence. We recall from [1] that as a $\widehat{E(n)}_*$ module,

$$\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^N) = \widehat{E(n)}_*\{\beta_k : N \geq k \geq 1\}$$

is the free module on the dual classes β_k to ${x^{\widehat{E(n)}}}^k \in \widehat{E(n)}^{2k}(\mathbb{C}\mathrm{P}^N)$ (here $1 \leqslant N \leqslant \infty$). Then the generators ${Y_1}^k$ are represented by the elements

$$\beta_k^* \in \mathrm{Hom}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)}), K(n)_*\right)$$
$$\beta_k^*(\beta_j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}.$$

which are all infinite cycles in E_2^1. The elements $z_1{Y_1}^k$ and z_1 are similarly given by the homomorphisms

$$\widehat{\beta}_k^* \in \mathrm{Hom}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(M\lambda_1), K(n)_*\right)$$
$$\widehat{\beta}_k^*(\widehat{\beta}_j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise} \end{cases}$$

where we use the Thom isomorphism

$$\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)^+}) \cong \widehat{E(n)}_*(M\lambda_1)$$
$$\beta_k \longleftrightarrow \widehat{\beta}_{k+1}$$

(with $\beta_0 = 1$). Thus, z_1 is represented by the natural $\widehat{E(n)}$ theory Thom class $u \in \widehat{E(n)}^*(M\lambda_1)$.

There is a commutative diagram

$$\begin{array}{ccc} \widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)^+}) & \xrightarrow{[p]_*} & \widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{\infty+}) \\ \downarrow & & \downarrow \\ \widehat{E(n)}_*(M\lambda_1) & \xrightarrow{\widehat{[p]}_*} & \widehat{E(n)}_*(MU(1)) \end{array}$$

where $[p]$ denotes the map classifying λ_1 and the vertical maps are Thom isomorphisms. Now an easy calculation shows that

$$[p]_*\left(\sum_{0\leqslant k\leqslant p^n-1}\beta_k T^k\right)\equiv\sum_{0\leqslant k}\beta_k([p]_{\widehat{E(n)}}(T))^k \mod T^{p^n}$$

and hence

$$\widehat{[p]}_*\left(\sum_{1\leqslant k\leqslant p^n}\widehat{\beta}_k T^{k-1}\right)\equiv\sum_{0\leqslant k}\beta_k([p]_{\widehat{E(n)}}(T))^k \mod T^{p^n}.$$

The zero section $j_1\colon \mathbb{C}\mathrm{P}^{(1)}\longrightarrow M\lambda_1$ induces

$$(j_1)_*\left(\sum\beta_k T^k\right)\equiv[p]_{\widehat{E(n)}}(T)\sum\widehat{\beta}_k T^k,$$

and this in turn induces the cohomological boundary for computing Ext.

To compute the effect of the coboundary, we lift $\widehat{\beta}_1$ to the element

$$B_1^*\in\mathrm{Hom}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(M\lambda_1),E(n)/I_n^2\right)$$
$$B_1^*(\widehat{\beta}_k)=\begin{cases}1 & \text{if } j=1,\\ 0 & \text{otherwise}\end{cases}$$

and then take the coboundary of this by precomposing with $(j_1)_*$, yielding

$$\beta_j\longmapsto\begin{cases}v_r & \text{if } j=p^r-1,\\ 0 & \text{if } j\neq p^r-1,\end{cases}$$

as an element of

$$\mathrm{Hom}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)}),E(n)/I_n^2\right)$$

which comes from

$$\mathrm{Hom}_{\widehat{E(n)}_*}\left(\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(1)}),I_n/I_n^2\right).$$

Hence, remembering our identification of I_n/I_n^2 with $\sum_{0\leqslant k\leqslant n-1}K(n)_*$ we easily determine that modulo filtration in the Universal Coefficient spectral sequence, the coboundary behaves as desired on $L^{(1)}$. However,in this case we see that the filtration is trivial, so Theorem (2.2) follows.

Notice that we have proved rather more. In [3] we only assumed that the cofibre Q was detected in filtration 1 of the spectral sequence for $\mathrm{RHom}(K(n),\bigvee K(n))$, however (2.2) actually tells us that the cofibre is determined totally by this condition and is always a derivation with the stated properties–a somewhat stronger result than might have been expected.

§3 Higher order Bocksteins.

In this section we describe Bockstein operations

$$(E(n)/I_n^k)^*(\ ) \longrightarrow K(n)^*(\ ),$$

where $E(n)/I_n^k$ is the A_∞ module spectrum over $\widehat{E(n)}$ defined in [3]. In fact we will describe these operations in such a way that they allow us to define the spectra $E(n)/I_n^k$ from cofibre sequences of module spectra whose coboundaries are the Bocksteins, just as we did in §2. In fact, these fit together to form a tower of module spectra

$$* \longleftarrow K(n) = E(n)/I_n \longleftarrow E(n)/I_n^2 \longleftarrow \cdots \longleftarrow E(n)/I_n^k \longleftarrow E(n)/I_n^{k+1} \longleftarrow \cdots$$

whose homotopy inverse limit is $\widehat{E(n)}$; applying the functor $[X,\ ]$ for any spectrum X then yields a spectral sequence (see (3.8)) converging to $\widehat{E(n)}^*(X)$ which is a generalisation of the Bockstein spectral sequence.

Our first task is to compute the set of morphisms of spectra $E(n)/I_n^k \longrightarrow K(n)$, namely $K(n)^*(E(n)/I_n^k)$. As Morava K cohomology is dual to the associated homology theory over the ring $K(n)_*$, it suffices to determine the $K(n)_*$ module $K(n)_*(E(n)/I_n^k)$ for each $k \geq 1$. From [3] we know that

$$K(n)_*\left(\widehat{E(n)}\right) \cong K(n)_*\left(E(n)\right) \otimes_{E(n)_*} \widehat{E(n)}_*$$

and moreover there is the the Koszul resolution

$$\mathbf{K}\left\langle K(n)_*\right\rangle_{*,*} \longrightarrow K(n)_*$$

which is a free $\widehat{E(n)}_*$ resolution. Tensoring with the flat $\widehat{E(n)}_*$ module $\widehat{E(n)}_*\left(\widehat{E(n)}\right)$ we obtain a Koszul resolution

$$\mathbf{K}\left\langle K(n)_*\left(\widehat{E(n)}\right)\right\rangle_{*,*} \longrightarrow K(n)_*\left(\widehat{E(n)}\right).$$

Hence the homology of the complex

$$\mathbf{K}\left\langle K(n)_*\left(\widehat{E(n)}\right)\right\rangle_{*,*} \otimes_{\widehat{E(n)}_*} E(n)/I_n^k$$

is the Tor algebra

$$\mathrm{Tor}_{*\,*}^{\widehat{E(n)}_*}\left(K(n)_*\left(\widehat{E(n)}_*\right), E(n)/I_n^k\right)$$

which is the E_2 term of a Künneth spectral sequence

$$\mathrm{E}^2_{*,*} \Longrightarrow K(n)_*\left(\widehat{E(n)} \wedge_{\widehat{E(n)}} E(n)/I_n^k\right) \cong K(n)_*\left(E(n)/I_n^k\right).$$

described in [11].

To describe this E^2 term explicitly we recall that

$$\mathbf{K}\left\langle K(n)_*\right\rangle_{*,*} = \Lambda_{\widehat{E(n)}_*}(e_k : 0 \leqslant k \leqslant n-1)$$

where the differential is given by

$$\partial(e_k) = v_k.$$

This extends to the differential ∂' on the complex for $K(n)_*\left(\widehat{E(n)}\right)$. The chain complex for our Tor algebra has differential d' which is easily seen to have a kernel given by

$$\mathrm{Z}^{r,*} = \begin{cases} \mathbf{K}\left\langle K(n)_*\left(\widehat{E(n)}\right)\right\rangle_{0,*} \otimes_{\widehat{E(n)}_*} E(n)/I_n^k & \text{if } r = 0, \\ I_n^{k-1}\mathbf{K}\left\langle K(n)_*\left(\widehat{E(n)}\right)\right\rangle_{r,*} \otimes_{\widehat{E(n)}_*} E(n)/I_n^k & \text{if } r > 0. \end{cases}$$

From this we deduce that

$$\mathrm{E}^2_{r,*} = \begin{cases} K(n)_*K(n) & \text{if } r = 0, \\ \Lambda_{r,*} \otimes_{K(n)_*} I_n^{k-1}/I_n^k & \text{if } r > 0. \end{cases} \tag{3.1}$$

where

$$\Lambda_{*,*} = \Lambda_{\Sigma(n)_*}(e_k : 0 \leqslant k \leqslant n)$$

is the exterior algebra upon the e_k, graded as in §2. Here we use the evident fact that I_n^{k-1}/I_n^k is a $K(n)_*$ module. Of course, for $k > 1$ we are not asserting that $E(n)/I_n^k$ is a ring spectrum so these isomorphisms are as $\widehat{E(n)}_*$ modules.

Upon dualising over $K(n)_*$ we obtain

$$K(n)^*(E(n)/I_n^k) \cong \mathrm{Hom}_{K(n)_*}\left(K(n)_*(E(n)/I_n^k), K(n)_*\right).$$

We will realise the duals of the classes e_k as stable operations

$$(E(n)/I_n^k)^*(\ ) \longrightarrow K(n)^*(\ ),$$

which are $\widehat{E(n)}_*$ derivations–this is clearly sufficient to define morphisms of $\widehat{E(n)}$ module spectra $E(n)/I_n^k$.

Notice that as $(E(n)/I_n^k)^*(\ )$ is a module theory over $\widehat{E(n)}^*(\ )$, the canonical natural transformation

$$\widehat{E(n)}^*(\ ) \longrightarrow (E(n)/I_n^k)^*(\ )$$

sends the orientation class $x^{\widehat{E(n)}} \in \widehat{E(n)}^2(\mathbb{C}\mathrm{P}^\infty)$ to a class $x_k \in (E(n)/I_n^k)^2(\mathbb{C}\mathrm{P}^\infty)$. Then as a module over $\widehat{E(n)}^*(\mathbb{C}\mathrm{P}^\infty)$, we have that $(E(n)/I_n^k)^*(\mathbb{C}\mathrm{P}^\infty)$ is generated by x_k (we are using reduced cohomology here). Thus we can write elements as $\widehat{E(n)}_*$ linear combinations of the products $(x^{\widehat{E(n)}})^r x_k$.

Now let $L^{(k)} = (B\mathbb{Z}/p^k)^{[2p^{kn}-1]}$ be the $2p^{nk}-1$ skeleton of $B\mathbb{Z}/p^k$. The case of $k = 1$ was dealt with in §2 and we will sketch the details for $k > 1$. We have the bundle

$\lambda_k = \eta^{p^k} \longrightarrow \mathbb{C}\mathrm{P}^{(k)}$, where $\mathbb{C}\mathrm{P}^{(k)} = \mathbb{C}\mathrm{P}^{p^{kn}-1}$. Then $L^{(k)}$ is equivalent to the sphere bundle of λ_k. There is a cofibre sequence

$$\mathbb{C}\mathrm{P}^{(k)} \xrightarrow{j_k} M\lambda_k \longrightarrow \Sigma L^{(k)}$$

where $M\lambda_k$ is the Thom space of λ_k. Applying the homology theory $\widehat{E(n)}_*(\ )$ we obtain a short exact sequence which is a free $\widehat{E(n)}_*$ resolution,

$$0 \longrightarrow \widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(k)}) \xrightarrow{j_{k*}} \widehat{E(n)}_*(M\lambda_k) \longrightarrow \widehat{E(n)}_*(\Sigma L^{(k)}) \longrightarrow 0.$$

To determine $(E(n)/I_n^k)^*(\Sigma L^{(k)})$, we now apply the Universal Coefficient spectral sequence of [12] and find

$$\mathrm{E}_2^{r,*}(\Sigma L^{(k)}) = \begin{cases} \ker\,(j_{k*})^* & \text{if } r=0, \\ \operatorname{coker}\,(j_{k*})^* & \text{if } r=1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

where

$$(j_{k*})^*\colon \operatorname{Hom}_{\widehat{E(n)}_*}(\widehat{E(n)}_*(M\lambda_k), E(n)/I_n^k) \longrightarrow \operatorname{Hom}_{\widehat{E(n)}_*}(\widehat{E(n)}_*(\mathbb{C}\mathrm{P}^{(k)}), E(n)/I_n^k)$$

is induced from $(j_k)_*$. This spectral sequence is clearly trivial and so we can describe all elements in the target in terms of their representatives.

To do this, we first make explicit the effect of the homomorphism j_{k*}. If we denote the Thom image of the generator $\beta_r \in \widehat{E(n)}_{2r}(\mathbb{C}\mathrm{P}^{(k)})$ by $\widehat{\beta}_{r+1} \in \widehat{E(n)}_{2r+2}(M\lambda_k)$, then we have

Lemma (3.3). *We have*

$$\sum_{r=1}^{p^{kn}-1} (j_k)_*\beta_r T^r \equiv ([p^k]_{\widehat{E(n)}_*} T) \sum_{r=1}^{p^{kn}} \widehat{\beta}_{r+1} T^r$$

where $\equiv$ means modulo $T^{p^{kn}}$. Moreover, we also have

$$[p^k]_{\widehat{E(n)}_*} T \equiv \sum_{r=0}^{n-1} v_r (v_n^{t(k)} T^{p^{(k-1)n}})^{p^r} \mod (I_n^2, T^{p^{kn}})$$

where $t(k) = (p^{(k-1)n}-1)/(p^n-1)$.

The proof is an easy exercise which we leave to the reader–but see §2 for the case of $k=1$.

Now we see from (3.2) that the reduced Thom class

$$u_k = \widehat{\beta}_1^* \in \operatorname{Hom}_{\widehat{E(n)}_*}(\widehat{E(n)}_*(M\lambda_k), E(n)/I_n^k)$$

gives rise to a family of elements $z_v \in (E(n)/I_n^k)^1(L^{(k)})$ represented by vu_k, where we use the monomials $v = v_0^{r_0} v_1^{r_1} \cdots v_{n-1}^{r_{n-1}}$ with $r_0 + r_1 + \cdots + r_{n-1} = k-1$. We will denote by M_k the set of all such monomials. Also, there is the element Y_k represented by the reduction of $x_k \in \widehat{E(n)}^2(\mathbb{C}\mathrm{P}^{(k)})$. Then we have

Theorem (3.4). *As a module over* $\widehat{E(n)}^*(\mathbb{C}P^{(k)+})$ *via the projection* $L^{(k)} \longrightarrow \mathbb{C}P^{(k)}$*, we have that* $(E(n)/I_n^k)^*(L^{(k)})$ *is free on the generators* Y_k *and* z_v *for* $v \in M_k$*.*

We will write module products in the form $Y_k^{r+1} = x_k^r Y_k$ and $Y_k^r z_v = x_k^r z_v$. We also have the following result.

Theorem (3.5). *For each* $k \geq 1$*, there is a unique family of stable operations*

$$\overline{Q_v^k} : (E(n)/I_n^k)^*(\ ) \longrightarrow K(n)^*(\ )$$

for $v = v_0^{r_0} v_1^{r_1} \cdots v_{n-1}^{r_{n-1}}$ *with* $r_0 + r_1 + \cdots r_{n-1} = k - 1$*, and possessing the following properties:*

a) $\overline{Q_v^k}$ *has degree* $|v| + 1$*;*
b) $\overline{Q_v^k}$ *is a derivation over* $\widehat{E(n)}^*(\ )$*;*
c) *on the kernel of* $(E(n)/I_n^k)^*(\ ) \longrightarrow (E(n)/I_n^{k-1})^*(\ )$*, the reduction homomorphism* $\overline{Q_v^k}$ *is a* $K(n)^*(\ )$ *derivation;*
d) *as an operation on* $(E(n)/I_n^k)^*(\mathbb{C}P^\infty)$ *we have* $\overline{Q_v^k} = 0$*;*
e) *as an operation on* $(E(n)/I_n^k)^*(L^{(k)})$ *we have*

$$\overline{Q_v^k}\,(Y_k^r) = 0$$

$$\overline{Q_v^k}\,(Y_k^r z_u) = \begin{cases} \sum_{t=0}^{n-1} Y_k^{p^{(k-1)n}+r} & \text{if } v = u, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, these are induced by morphisms of A_∞ *module spectra over* $\widehat{E(n)}$ *of the form*

$$Q_v^k \colon E(n)/I_n^k \longrightarrow \Sigma^{|v|} K(n),$$

and whose wedge $Q^k = \bigvee_v Q_v^k$ *is the coboundary for a cofibre sequence*

$$\bigvee_v \Sigma^{|v|} K(n) \longrightarrow E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k.$$

We leave the proof of this to the reader as all the techniques required are exhibited in §2 in the case of $k = 1$.

The tower of module spectra over $\widehat{E(n)}$,

(3.6) $$* \longleftarrow K(n) \longleftarrow E(n)/I_n^2 \longleftarrow \cdots \longleftarrow E(n)/I_n^k \longleftarrow E(n)/I_n^{k+1} \longleftarrow \cdots$$

has associated to it the cofibre sequences

(3.7) $$E(n)/I_n^k \longleftarrow E(n)/I_n^{k+1} \longleftarrow \bigvee_v \Sigma^{|v|} K(n)$$

where the boundary is $Q^k \colon E(n)/I_n^k \longrightarrow \bigvee_v \Sigma^{|v|+1} K(n)$. For any spectrum X we can now apply the functor $[X,\ ]$ to this tower and thus obtain a spectral sequence with

$$\mathrm{E}_1^{s\,*}(X) = I_n^s/I_n^{s+1} \otimes_{K(n)_*} K(n)^*(X)$$

and differential

$$\mathrm{d}_1 = \widehat{Q}^{s+1} \colon \mathrm{E}_1^{s\,*}(X) \longrightarrow \mathrm{E}_1^{s+1\,*}(X),$$

which factors through $\widehat{Q^{s+1}}$. The proof of the next result is now routine.

Theorem (3.8). *The spectral sequence* $\mathrm{E}_r(X)$ *converges to* $\widehat{E(n)}^*(X)$*, and the associated filtration on* $\widehat{E(n)}^*(X)$ *generates the natural topology.*

Of course, this spectral sequence is a form of Bockstein spectral sequence. We record one immediate consequence of its existence which has an application in recent work of J. Hunton on the Morava K-theory of classifying spaces of finite groups (to appear).

Proposition (3.9). *Let* X *be a spectrum for which* $K(n)^{\mathrm{odd}}(X) = 0$*. Then the natural homomorphism*

$$\widehat{E(n)}^*(X) \longrightarrow K(n)^*(X)$$

is onto.

We also record the following result which was prompted by a question of N. Kuhn.

Proposition (3.10). *Let* $x \in \widehat{E(n)}^r(X)$ *be an element annihilated by some* v_k *with* $0 \leqslant k \leqslant n-1$*. Then we have*

$$K(n)^r(X) \neq 0 \quad \text{and} \quad K(n)^{r-2p^k+1}(X) \neq 0.$$

The proof makes use of the $\widehat{E(n)}$ module structure of the spectral sequence and we will return to such considerations in future work.

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Glasgow University, Glasgow G12 8QW, Scotland.
E-mail address: `andy@maths.gla.ac.uk`

Mathematische Institüt, Universität Bern, Bern, CH3012, Switzerland.
E-mail address: `wuergler@ch.unibe.mai`