EXOTIC MULTIPLICATIONS ON MORAVA K-THEORIES AND THEIR LIFTINGS

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Abstract. For each prime p and integer n satisfying $0 < n < \infty$, there is a ring spectrum $K(n)$ called the n th Morava K-theory at p. We discuss exotic multiplications upon $K(n)$ and their liftings to certain characteristic zero spectra $\widehat{E(n)}$.

Introduction.

The purpose of this paper is to describe exotic multiplications on Morava's spectrum $K(n)$ and certain "liftings" to spectra whose coefficient rings are of characteristic 0. Many of the results we describe are probably familiar to other topologists and indeed it seems likely that they date back to foundational work of Jack Morava in unpublished preprints, not now easily available. A published source for some of this is the paper of Urs Würgler [12]. We only give sketches of the proofs, most of which are straightforward modifications of existing arguments or to be found in [12]. For all background information and much notation that we take for granted, the reader is referred to [1] and [7].

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Convention: Throughout this paper we assume that p is an *odd* prime.

§1 Exotic Morava K-theories.

Morava K-theory is usually defined to be a multiplicative complex oriented cohomology theory $K(n)^*(\ )$ which has for its coefficient ring

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$$

where $v_n \in K(n)_{2p^n-2}$, and is canonically complex oriented by a morphism of ring spectra

$$\sigma^{K(n)} \colon BP \longrightarrow K(n)$$

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which on coefficients induces the ring homomorphism

$$\sigma_*^{K(n)} \colon BP_* \longrightarrow K(n)_*$$
$$\sigma_*^{K(n)}(v_k) = \begin{cases} v_n & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have $BP_* = \mathbb{Z}_{(p)}[v_k : k \geqslant 1]$ with $v_k \in BP_{2p^k-2}$ being the k th *Araki generator*, defined using the formal group sum

$$[p]_{BP} X = \sum_{0 \leqslant k}^{BP} \left(v_k X^{p^k} \right).$$

As a homomorphism of graded rings, we can regard $\sigma_*^{K(n)}$ as a quotient homomorphism

$$\sigma_*^{K(n)} \colon v_n^{-1} BP_* \longrightarrow v_n^{-1} BP_* / \mathcal{M}_n \cong K(n)_*$$

where $\mathcal{M}_n = (v_k : 0 \leqslant k \neq n) \lhd v_n^{-1} BP_*$ is a maximal graded ideal of the ring $v_n^{-1} BP_*$. Thus we can interpret $K(n)_*$ as a (graded) residue field for this maximal ideal.

Clearly this ideal $\mathcal{M}_n$ is not the only such maximal ideal and we can reasonably look at other examples and ask if the associated quotient (graded) fields occur as coefficient rings for cohomology theories is an analogous fashion. Notice that $\mathcal{M}_n$ contains the invariant prime ideal $I_n = (v_k : 0 \leqslant k \leqslant n-1)$ and the formal group law $F^{K(n)}$ therefore has height n. One way to construct $K(n)$-theory is by using Landweber's Exact Functor Theorem (LEFT) [6] in its modulo I_n version [14]; this allows us to make the definition

$$K(n)^*(\ ) = K(n)_* \otimes_{P(n)_*} P(n)^*(\ )$$

on the category of finite CW spectra $\mathbf{CW^f}$, where $P(n)$ is the spectrum for which

$$P(n)_* = BP_*/I_n.$$

We thus concentrate on maximal ideals $\mathcal{M}' \lhd v_n^{-1} BP_*$ containing the ideal I_n. We then have

Theorem (1.1). *Let $\mathcal{M}' \lhd v_n^{-1} BP_*$ be a maximal (graded) ideal containing I_n. Then there is a unique multiplicative cohomology theory $K(\mathcal{M}')^*(\ )$, defined on $\mathbf{CW^f}$, for which there is a multiplicative natural isomorphism*

$$K(\mathcal{M}')^*(\ ) \cong K(\mathcal{M}')_* \otimes_{P(n)_*} P(n)^*(\ )$$

and where $K(\mathcal{M}')_* = v_n^{-1} BP_* / \mathcal{M}'$ is the coefficient ring.

The proof is immediate using LEFT.

Of course, if there is an isomorphism of graded rings, $K(\mathcal{M}')_* \cong K(\mathcal{M}'')_*$, then we need to decide if the two theories arising from $\mathcal{M}'$ and $\mathcal{M}''$ can be naturally equivalent.

Theorem (1.2). *Let $\mathcal{M}', \mathcal{M}'' \lhd v_n^{-1} BP_*$ be graded maximal ideals containing I_n and let $f \colon K(\mathcal{M}')_* \longrightarrow K(\mathcal{M}'')_*$ be an isomorphism of graded rings. Then there is a natural isomorphism of multiplicative cohomology theories on $\mathbf{CW^f}$,*

$$\widetilde{f} \colon K(\mathcal{M}')^*(\ ) \longrightarrow K(\mathcal{M}'')^*(\ )$$

extending f if and only if the the formal group laws $f_* F^{v_n^{-1} BP_*/\mathcal{M}'}$ and $F^{v_n^{-1} BP_*/\mathcal{M}''}$ are strictly isomorphic over the ring $K(\mathcal{M}'')_*$.

The main observation required to prove this result is that these two formal group laws are associated to two complex orientations induced by the composite of the morphisms of ring spectra $BP \longrightarrow v_n^{-1} BP \longrightarrow K(\mathcal{M}'')$.

Corollary (1.3). *The theories $K(\mathcal{M}')^*(\ )$ and $K(\mathcal{M}'')^*(\ )$ are representable by ring spectra $K(\mathcal{M}')$ and $K(\mathcal{M}'')$, unique up to canonical equivalence in the stable category. Moreover, $K(\mathcal{M}'')$ and $K(\mathcal{M}'')$ are equivalent as ring spectra if and only if the formal group laws $f_*F^{v_n^{-1}BP_*/\mathcal{M}'}$ and $F^{v_n^{-1}BP_*/\mathcal{M}''}$ are strictly isomorphic over the ring $K(\mathcal{M}'')_*$.*

Let us now consider such ring spectra $K(\mathcal{M}')$ where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. By a result from [12] (see also [5]) these are precisely the ring spectra having the homotopy type of $K(n)$ (not necessarily multiplicatively). Thus, such ring spectra are classified to within equivalence as ring spectra by the set of maximal ideals $\mathcal{M}'$ modulo strict isomorphism of the associated formal group laws over $K(\mathcal{M}')_*$. We call the multiplicative cohomology theory associated to such a ring spectrum an *exotic Morava K-theory*.

Let us consider such a spectrum $K(\mathcal{M}')$, where $K(\mathcal{M}')_* \cong K(n)_*$ as graded rings. Then we have the following modification of a result of [13],

Theorem (1.4). *As an algebra over $K(n)_*$, we have*

$$K(\mathcal{M}')_*(K(\mathcal{M}')) \cong K(\mathcal{M}')_*(t'_k : k \geqslant 1) \otimes \Lambda_{K(n)_*}(a'_0, \dots, a'_{n-1})$$

where $|t'_k| = 2p^k - 2$, $|a'_k| = 2p^k - 1$, and there are polynomial relations of the form

$$t'^{\,p^n}_k - v_n^{(p^k-1)/(p-1)} t'_k = h_k(t'_1, \dots, t'_{k-1})$$

over $K(n)_*$.

The symbol $\Lambda_{K(n)_*}$ denotes an exterior algebra over $K(n)_*$ on the indicated generators.

To prove this result, we rework the proof for the case of $K(n)$ (see [13], [7]) and define the generators t'_k by using the identity

$$\sum_{\substack{r\geqslant 0\\ s\geqslant n}}^{K(\mathcal{M}')} (v'_s t'^{\,p^s}_r X^{p^{r+s}}) = \sum_{\substack{r\geqslant 0\\ s\geqslant n}}^{K(\mathcal{M}')} (t'_r v'^{\,p^r}_s X^{p^{r+s}})$$

where

$$[p]_{F^{K(\mathcal{M}')}} X = \sum_{s\geqslant n}^{K(\mathcal{M}')} (v'_s X^{p^s}).$$

The exterior generators are similarly derived.

We can interpret the algebra

$$K(\mathcal{M}')_*(t'_k : k \geqslant 1)$$

as representing the strict automorphisms of the group law $F^{K(\mathcal{M}')}$, in a way analogous to the case of $K(n)$ (see [7]).

§2 Liftings of exotic Morava K-theories.

Recall that there is a ring spectrum $E(n)$ for which

$$E(n)^*(\ ) \cong E(n)_* \otimes_{BP_*} BP^*(\ )$$

on $\mathbf{CW^f}$. Here we have

$$E(n)_* = v_n^{-1} BP_*/(v_{n+k} : k \geqslant 1).$$

We showed in joint work with Urs Würgler (see [4]) that the *Noetherian completion* $\widehat{E(n)}$ of $E(n)$, characterised by the formula

$$\widehat{E(n)}^{*}(\ ) = \varprojlim_{k} \left(E(n)^*(\ )/I_n^k E(n)^*(\ )\right)$$

on $\mathbf{CW^f}$, is a summand of the *Artinian completion* $\widehat{v_n^{-1}BP}$ of $v_n^{-1}BP$ and indeed there is a product splitting

$$\widehat{v_n^{-1}BP} \simeq \prod_{v} \Sigma^{2d(v)} \widehat{E(n)}$$

of (topological) ring spectra for v ranging over a suitable indexing set and d a suitable numerical function. The algebra underlying the proof is intimately related to liftings of *Lubin–Tate* group laws, i.e. group laws over $\mathbb{F}_p$ algebras classified by homomorphisms from $K(n)_*$. In this section we describe the analogous situation for liftings of exotic Morava K-theories of the form $K(\mathcal{M}')$ as in §1.

Now if $K(\mathcal{M}')_* \cong K(n)_*$ as a ring, then the natural homomorphism

$$\theta_{\mathcal{M}'} \colon v_n^{-1}BP \longrightarrow K(\mathcal{M}')_*$$

given by

$$\theta_{\mathcal{M}'}(v_k) = \begin{cases} c_k v_n^{r(k/n)} & \text{if } n \mid k \\ 0 & \text{otherwise} \end{cases}$$

for $k \geqslant n$ and integers c_k. Here, the numerical function r is given by

$$r(m) = \frac{(p^{mn}-1)}{(p^n-1)}.$$

and we set $c_k = 0 = r(k/n)$ whenever k is not divisible by n or $k = 0$.

Now consider an ideal of the form

$$J = (v_k - c_k v_n^{r(k/n)} + g_k : k > n) \subset \mathcal{M}' \triangleleft v_n^{-1}BP_*$$

and satisfying

$$J + I_n = \mathcal{M}'.$$

Here $g_k \in \mathcal{M}'$ are certain elements chosen so that the last condition holds. Set $E(J)_* = v_n^{-1}BP_*/J$.

We now define a cohomology theory

$$E(J)^*(\ ) = E(J)_* \otimes_{BP_*} BP^*(\ )$$

on $\mathbf{CW^f}$. This *is* a cohomology theory by Landweber's Exact Functor Theorem, and is moreover multiplicative and canonically complex oriented by the obvious natural transformation

$$BP^*(\ ) \longrightarrow E(J)^*(\ ).$$

Furthermore there is a canonical multiplicative natural transformation

$$E(J)^*(\ ) \longrightarrow K(\mathcal{M}')^*(\ ).$$

We can form the Noetherian completion

$$\widehat{E(J)}^{*}(\ ) = \varprojlim_{k} \left(E(J)^*(\ )/I_n^k E(J)^*(\ )\right)$$

and also the Artinian completion of $v_n^{-1}BP$ with respect to the maximal ideal $\mathcal{M}'$, $v_n^{-1}\widehat{BP(\mathcal{M}')}$ (see [4]). Then we have

Theorem (2.1). *There is a splitting of topological ring spectra*

$$v_n^{-1}\widehat{BP(\mathcal{M}')} \simeq \prod_w \Sigma^{2e(w)}\widehat{E(J)}$$

where w ranges over an appropriate indexing set and e is a numerical function.

The proof is a modification of that in [4] which rests on the fact that in the ring $v_n^{-1}BP_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} v_n^{-1}BP_*$, the generators t_k satisfy relations modulo $\mathcal{M}'$ of the form given in the statement of (1.4). Of course, in the case where $\mathcal{M}' = \mathcal{M} = (v_k : 0 \leqslant k \neq n)$, this shows that $\widehat{E(n)}$ is just one amongst many ring spectra splitting off of $\widehat{v_n^{-1}BP}$ in this way.

In [12] and [5] it was proved that any ring spectrum whose homotopy ring is isomorphic to $K(n)_*$ agrees with $K(n)$ up to equivalence as a spectrum. In fact we can lift such results to show

Theorem (2.2). *Let F be a complex oriented topological ring spectrum such that as graded topological groups*

$$\pi_*(F) \cong \widehat{E(n)}_*,$$

and there is a maximal ideal $\mathcal{M}' \lhd v_n^{-1}BP_*$ for which there is a morphism of ring spectra $F \longrightarrow K(\mathcal{M}')$ which is surjective in homotopy. Then there is an ideal $J \subset \mathcal{M}'$ such that there is an equivalance of topological ring spectra $F \simeq \widehat{E(J)}$.

The proof of this makes use of a tower

$$\cdots \longrightarrow E(J)/I_n^{k+1} \longrightarrow E(J)/I_n^k \longrightarrow \cdots \longrightarrow E(J)/I_n = K(\mathcal{M}')$$

of A_∞ module spectra over $\widehat{E(J)}$ generalising that constructed in [3], together with the existence of an A_∞ structure on $\widehat{E(J)}$ (see §3).

§3 $\boldsymbol{A_\infty}$ structures on exotic Morava $\boldsymbol{K}$-theories.

In [8] it was shown that for any odd prime p, the standard ring spectrum structure on $K(n)$ admits uncountably many distinct A_∞ structures in the sense of [9], [10] and [11]. One can similarly ask if this is true for any of the exotic structures discussed in our earlier sections. In fact, by (1.4), the arguments of [8] can be used in the more general context. Indeed, this is also true for the results of [3] and the liftings $\widehat{E(J)}$ which have unique topological A_∞ structures, and the natural morphisms of ring spectra $\widehat{E(J)} \longrightarrow K(\mathcal{M}')$ admit A_∞ structures whichever of the A_∞ structures is put on $K(\mathcal{M}')$.

One consequence of the existence of A_∞ structures is that there are Künneth and Universal Coefficient spectral sequences for A_∞ module theories over these ring spectra. For example, if M is a (topological) A_∞ module spectrum over $\widehat{E(J)}$, then for any spectrum X, there is a spectral sequence

$$\mathrm{E}_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{\widehat{E(J)}}(\widehat{E(J)}_*(X), M_*) \Longrightarrow M^{s+t}(X).$$

Such spectral sequences promise to be of great use in calculations.

§4 Some examples.

We end by considering two examples of cohomology theories which are related to exotic Morava K-theories as discussed in the earlier sections. These are essentially the only known periodic theories which have (or are suspected to have) geometric descriptions, and remarkably they both appear to be naturally related to the original versions of Morava $K(1)$ and $K(2)$ and its liftings, rather than truly exotic versions.

***K*-theory.** Consider the case of complex K-theory localised at a prime p, $K^*(\ ) = KU^*_{(p)}(\ )$. Then reduction modulo p gives a theory $K/p^*(\ )$ satisfying

$$K/p^*(\ ) \cong \bigoplus_{0\leqslant k<(p-1)} K(1)^{*+2k}(\ )$$

as multiplicative theories where we define the product on the direct sum by requiring that there be an isomorphism of rings

$$K/p_* \cong K(1)_*[u]/(u^{p-1} - v_1)$$

when evaluated on a point. Thus

$$K/p \simeq \bigvee_{0\leqslant k<(p-1)} \Sigma^{2k} K(1)$$

as ring spectra, where the wedge is given an appropriate algebra spectrum structure over $K(1)$. Lifting this result gives an equivalence

$$\widehat{K} \simeq \bigvee_{0\leqslant k<(p-1)} \Sigma^{2k} \widehat{E(1)}$$

which is known to arise before p-adic completion.

Elliptic cohomology. Let $E\ell\ell$ be the spectrum representing the version of elliptic cohomology whose coefficient ring is the ring of modular forms for $SL_2(\mathbb{Z})$ meromorphic at infinity (see [2]), localised at a prime $p > 3$. Then in [2] we showed that if E_{p-1} denotes the $(p-1)$ st Eisenstein function, then there is an equivalence of ring spectra

$$E\ell\ell/(p, E_{p-1}) \simeq \bigvee_{\alpha} \Sigma^{2f(\alpha)} K(2)$$

where $E\ell\ell/(p, E_{p-1})$ is the reduction of $E\ell\ell$ modulo the ideal (p, E_{p-1}) in an appropriate sense. This lifts to a splitting of topological ring spectra

$$E\ell\ell\widehat{_{(p,E_{p-1})}} \simeq \bigvee_{\alpha'} \Sigma^{2f'(\alpha')} \widehat{E(2)}.$$

In both cases the wedge is finite, and we need to impose appropriate algebra spectra structures over the bottom summands.

It would be of interest to find "naturally" occurring examples involving truly exotic versions of Morava K-theories. Of course, for the examples given we can take an exotic $K(1)$ or $K(2)$ and use this to impose an exotic multiplication upon either mod p K-theory or elliptic cohomology, but it is then unclear whether the resulting multiplicative theories have any geometric descriptions.

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