HECKE OPERATIONS AND THE ADAMS E_2-TERM BASED ON ELLIPTIC COHOMOLOGY

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Abstract. Hecke operators are used to investigate part of the E_2-term of the Adams spectral sequence based on elliptic homology. The main result is a derivation of Ext^1 which combines use of classical Hecke operators and p-adic Hecke operators due to Serre.

Introduction.

Elliptic cohomology (and its dual homology theory) potentially offers a setting in which v_2-periodic phenomena might be studied from a geometric (rather than purely homotopy theoretic) perspective. Hence it is important to investigate the limits of what might be achieved this way. The Adams spectral sequence based on elliptic (co)homology provides the appropriate framework for studying stable homotopy. Clarke & Johnson [7] and Laures [10], essentially determined the v_1-periodic part of the E_2-term of this spectral sequence for spheres. Here we rederive this result using stable operations related to the classical Hecke operators which were originally constructed in [2,3] and discussed further in [5,6]. Hitherto, these operations appear to have lacked serious topological applications.

In stating our main result we use notation for Ext groups found in Adams [1] and Switzer [13]. In particular,

$$\mathrm{m}(n) = \operatorname{denom} \frac{B_n}{n},$$

where B_n denotes the nth Bernoulli number. We will prove

Theorem.

$$\mathrm{Ext}^{1,2n}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*) \cong \begin{cases} \mathbb{Z}[1/6]/\mathrm{m}(n) & \text{if } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Our proof of this is modelled on one previously used in proving the analogous result in K-theory,

$$\mathrm{Ext}^{1,2n}_{KU_*KU}(KU_*, KU_*) \cong \mathbb{Z}/\mathrm{m}(|n|) \quad \text{if } n \neq 0.$$

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This utilises carefully selected (stable) Adams operations to give bounds on the orders of elements. In our case, we first use Hecke operations to show that only holomorphic modular forms and hence nonnegative degrees can possibly yield nonvanishing groups, then we use Adams operations in elliptic homology to bound the orders and realise these bounds with Eisenstein functions. Finally we use operations in a p-adic version of elliptic cohomology to show that these indeed exhaust Ext^1.

§1 Hecke operations and cooperations.

In [3] we showed that on finite CW complexes and spectra, there are families of stable operations

$$\mathrm{T}_n, \psi^n_{E\ell\ell}\colon E\ell\ell^*(\ ) \longrightarrow E\ell\ell[1/n]^*(\ ).$$

which give rise to operations on the dual homology functors

$$\mathrm{T}_n, \psi^n_{E\ell\ell}\colon E\ell\ell_*(\ ) \longrightarrow E\ell\ell[1/n]_*(\ ).$$

In [5] we described the cooperation algebra $E\ell\ell_*E\ell\ell$ which has the structure of a Hopf algebroid over $\mathbb{Z}[1/6]$. We now explain the relationship between comodule structures over $E\ell\ell_*E\ell\ell$ and actions of Hecke operations. This discussion involves some reworking of our earlier description which we leave to the interested reader.

First recall from [4] the ring $E\ell\ell_*^{\Gamma(n)}$ consisting of all modular forms for the (normal) congruence subgroup

$$\Gamma(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv \mathrm{I}_2 \pmod{n}\} \lhd \mathrm{SL}_2(\mathbb{Z}),$$

which are meromorphic at each cusp and have all their q-coefficients in $\mathbb{Z}[1/6n, \zeta_n]$ where $\zeta_n = e^{2\pi i/n}$. We may view $E\ell\ell_*^{\Gamma(n)}$ as a ring of functions on the space of modular points $(L, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is the ordered pair

$$\boldsymbol{\alpha} = \left(\frac{\omega_1}{n} + L, \frac{\omega_2}{n} + L\right)$$

of basis vectors for the free $\mathbb{Z}/n$-module $(1/n)L/L$ with $\{\omega_1, \omega_2\} \subset L$ an oriented basis of L. The extension (of integral domains) $E\ell\ell_*^{\Gamma(n)}/E\ell\ell_*[1/n]$ is Galois with group $\mathrm{SL}_2(\mathbb{Z}/n) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(n)$. Following [4,5] we interpret $E\ell\ell_*^{\Gamma(n)}$ as a ring of functions on the space $\mathcal{V}(n) = \mathcal{V}/\Gamma(n)$ which provides an analytic model for the space of modular points mentioned above, admitting an analytic principal fibre bundle

$$\mathcal{V}(n) \longrightarrow \mathcal{V}/\mathrm{SL}_2(\mathbb{Z}) = \mathcal{L}$$

with group $\mathrm{SL}_2(\mathbb{Z}/n)$.

For each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(n)$$

(the set of all 2×2 integer matrices of determinant n), there is a map

$$\mathcal{V}(n) \xrightarrow{[A]} \mathcal{L}$$

sending each modular point $(L, \boldsymbol{\alpha})$ to the lattice

$$\left\langle \frac{a\omega_1 + c\omega_2}{n}, \frac{c\omega_1 + d\omega_2}{n} \right\rangle$$

and only depending on the coset $A\,\mathrm{SL}_2(\mathbb{Z}) \in \mathrm{M}_2(n)/\,\mathrm{SL}_2(\mathbb{Z})$. The map $[A]$ induces a ring homomorphism

$$[A]^*\colon E\ell\ell_* \longrightarrow E\ell\ell_*^{\Gamma(n)}$$

extending to an $E\ell\ell_*$-algebra homomorphism

$$[A]_*\colon E\ell\ell_*E\ell\ell \longrightarrow E\ell\ell_*^{\Gamma(n)}.$$

In particular, the matrix $n\,\mathrm{I}_2 \in \mathrm{M}_2(n^2)$ provides homomorphisms

$$\begin{aligned}
[n\,\mathrm{I}_2]^*\colon E\ell\ell_* &\longrightarrow E\ell\ell_*^{\Gamma(n^2)},\\
[n\,\mathrm{I}_2]_*\colon E\ell\ell_*E\ell\ell &\longrightarrow E\ell\ell_*^{\Gamma(n^2)}.
\end{aligned}$$

From [5], we know that there are $E\ell\ell_*$-linear maps

$$\mathrm{T}_{n\,*}, \psi^n_{E\ell\ell\,*}\colon E\ell\ell_*E\ell\ell \longrightarrow E\ell\ell_*[1/n]$$

which may be used to define stable operations

$$\mathrm{T}_n, \psi^n_{E\ell\ell}\colon E\ell\ell^*(\ ) \longrightarrow E\ell\ell^*(\ )[1/n]$$

in elliptic cohomology on finite CW complexes and in the dual homology theories. We have the defining formulæ

$$\begin{aligned}
\mathrm{T}_{n\,*} &= \frac{1}{n}\sum_{A:\mathrm{M}_2(n)/\,\mathrm{SL}_2(\mathbb{Z})}[A]_*,\\
\psi^n_{E\ell\ell\,*} &= [n\,\mathrm{I}_2]_*,
\end{aligned}$$

where the notation '$A\colon \mathrm{M}_2(n)/\,\mathrm{SL}_2(\mathbb{Z})$' indicates that we sum over a complete set of representatives of the left cosets of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{M}_2(n)$. Although by definition these take values in $E\ell\ell_*^{\Gamma(n)}$ and $E\ell\ell_*^{\Gamma(n^2)}$, they turn out to have images in $E\ell\ell[1/n]_*$ and we view them as giving $E\ell\ell_*$-linear maps

$$\begin{aligned}
\mathrm{T}_{n\,*}\colon E\ell\ell_*E\ell\ell &\longrightarrow E\ell\ell[1/n]_*,\\
\psi^n_{E\ell\ell\,*}\colon E\ell\ell_*E\ell\ell &\longrightarrow E\ell\ell[1/n]_*.
\end{aligned}$$

Let M_* be a *right* comodule over $E\ell\ell_*E\ell\ell$ with coproduct

$$\gamma\colon M_* \longrightarrow M_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*E\ell\ell.$$

Each operation $\Theta = \mathrm{T}_n$ or $\psi^n_{E\ell\ell}$ is obtained as a composite of the form

$$M_* \xrightarrow{\gamma} M_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*E\ell\ell \xrightarrow{\Theta_*} M_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*[1/n] \cong M_*[1/n].$$

Proposition 1.1. *Let $x \in \mathrm{Ext}^{0,k}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, M_*)$. Then for $r > 0$ and a prime ℓ, we have*

$$\begin{aligned} \mathrm{T}_\ell\, x &= (1+\ell^{-1})x, \\ \psi^r_{E\ell\ell} x &= x. \end{aligned}$$

Proof. By definition, $\psi^r_{E\ell\ell}$ has the coaction as a factor, and this sends x to $x \otimes 1$. Similarly, the operation T_ℓ is $1/\ell$ times a sum of multiplicative operations, which must all act trivially on x since they have the coaction as a factor. $\square$

§2 Hecke operators and the Adams 1-line.

It is easy to show that

$$\mathrm{Ext}^{r,s}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } (r,s) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

By considering the exact sequence of $E\ell\ell_*E\ell\ell$-comodules

$$0 \longrightarrow E\ell\ell_* \to E\ell\ell_*\mathbb{Q} \to E\ell\ell_*\mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

and its derived long exact sequence in Ext, we obtain an exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \to \mathrm{Ext}^{0,s}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*\mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^{1,s}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*) \longrightarrow 0.$$

Thus to compute $\mathrm{Ext}^{1,s}$ it suffices to compute

$$\mathrm{Ext}^{0,s}(E\ell\ell_*\mathbb{Q}/\mathbb{Z}) = \mathrm{Ext}^{0,s}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*\mathbb{Q}/\mathbb{Z}).$$

We will write elements of $E\ell\ell_*\mathbb{Q}/\mathbb{Z}$ in the form

$$\frac{F}{d} = (1/d)F \pmod{\mathbb{Z}[1/6]},$$

where $F \in E\ell\ell_*$ and $0 < d \in \mathbb{Z}$ is not divisible by 2 or 3. The q-expansion gives a series

$$\frac{F(q)}{d} = (1/d)F(q) \pmod{\mathbb{Z}[1/6]((q))}.$$

Note that $\dfrac{F}{d} = 0$ if and only if $\dfrac{F(q)}{d} = 0$. We begin with the following 'Preparation Lemma'.

Lemma 2.1. *Let $F \in E\ell\ell_{2k}$ and suppose that*

$$\frac{F(q)}{d} = \frac{\sum_{r_0 \leqslant r} a_r q^r}{d} \pmod{\mathbb{Z}[1/6]((q))}.$$

Then there is an $F' \in E\ell\ell_{2k}$ such that

$$F'(q) = \sum_{r_0 \leqslant r} a'_r q^r$$

and

$$\frac{F'(q)}{d} = \frac{F(q)}{d} \pmod{\mathbb{Z}[1/6]((q))}.$$

Proof. Suppose that

$$F(q) = \sum_{s_0 \leqslant s} a_s q^s$$

for $s_0 < r_0$. Then $a_{s_0}/d \in \mathbb{Z}[1/6]$. We may choose a modular form $Q^a R^b \Delta^c \in E\ell\ell_{2k}$ with the property that

$$Q(q)^a R(q)^b \Delta(q)^c = q^{s_0} + \text{ higher degree terms.}$$

Then $F - a_{s_0} Q^a R^b \Delta^c$ has q-expansion of the form $\sum_{s_0+1 \leqslant s} b_s q^s$ and also satisfies

$$\frac{F - a_{s_0} Q^a R^b \Delta^c}{d} = \frac{F}{d} \pmod{\mathbb{Z}[1/6]((q))}.$$

Repeating this process we eventually obtain the desired F'. □

Now let $\dfrac{F}{d} \in \mathrm{Ext}^{0,2k}(E\ell\ell_* \mathbb{Q}/\mathbb{Z})$. By the Preparation Lemma 2.1, we can assume that

$$F(q) = \sum_{r_0 \leqslant r} a_r q^r,$$
$$\frac{a_{r_0}}{d} \neq 0 \pmod{\mathbb{Z}[1/6]}.$$

By Dirichlet's Theorem on primes in arithmetic progressions, there is a prime ℓ satisfying $\ell + 1 \equiv 0 \pmod{d}$.

Suppose that $r_0 < 0$. Then from [3],

$$(\mathrm{T}_\ell\, F)(q) = \sum_{r_0 \leqslant r\ell} a_{r\ell} q^r + \ell^{k-1} \sum_{r_0 \leqslant r} a_r q^{r\ell}$$

and so

$$\mathrm{T}_\ell \frac{F}{d} = \frac{(\mathrm{T}_\ell\, F)(q)}{d} \neq 0,$$

since the leading q-expansion term is $\dfrac{\ell^{k-1} a_{r_0} q^{r_0 \ell}}{d}$. Proposition 1.1 implies that

$$\mathrm{T}_\ell \frac{F}{d} = (1 + \ell^{-1}) \frac{F}{d} \equiv 0 \pmod{\mathbb{Z}[1/6]((q))},$$

giving a contradiction.

We can assume that F is holomorphic at the cusp and so $k \geqslant 0$. In the case $k = 0$ the only holomorphic modular forms of weight zero are constants which all lie in $\mathrm{Ext}^{0,0}(E\ell\ell_* \mathbb{Q}/\mathbb{Z})$, thus we see that this group is isomorphic to $\mathbb{Q}/\mathbb{Z}[1/6]$ with generators the elements $\dfrac{1}{d}$ for $6 \nmid d$.

Now we may apply the operations $\psi^t_{E\ell\ell}$ where t is prime to d. We have

$$\psi^t_{E\ell\ell}\frac{F}{d}=\frac{t^kF}{d}$$

and the standard argument familiar from K-theory shows that $d \mid \mathrm{m}(k)$ since

$$\mathrm{m}(k)=\gcd\{t^k-1:(d,t)=1\}.$$

In particular, since we are inverting the prime 2, we can only have a non-zero group if k is even. For even $k \geqslant 4$, the Eisenstein function E_k, characterised by its q-expansion

$$E_k(q)=1-\frac{2k}{B_k}\sum_{r\geqslant 1}\sigma_{k-1}(r)q^r,$$

is a modular form of weight k with rational q-expansion coefficients. By well known properties of the Bernoulli number B_k, E_k gives rise to an element $(B_k/k)E_k \in E\ell\ell_{2k}\mathbb{Q}$ which in turn yields an element of $E\ell\ell_{2k}\mathbb{Q}/\mathbb{Z}$ of the form $\dfrac{E'_k}{\mathrm{m}(k)}$ with $E'_k \in E\ell\ell_{2k}$. Moreover,

$$E'_k(q)\equiv \operatorname{numer}\frac{B_k}{k}E_k(q) \pmod{\mathrm{m}(k)},$$

and so all of the Hecke operators T_n, $\psi^n_{E\ell\ell}$ with $(n,\mathrm{m}(k))=1$ annihilate $\dfrac{E'_k}{\mathrm{m}(k)}$. Indeed, this element lies in $\mathrm{Ext}^{0,2k}(E\ell\ell_*\mathbb{Q}/\mathbb{Z})$ as noted in [7,10]. Thus if $k>0$, $\mathrm{Ext}^{0,2k}(E\ell\ell_*\mathbb{Q}/\mathbb{Z})$ contains a summand isomorphic to $\mathbb{Z}[1/6]/\mathrm{m}(k)$.

§3 p-adic Hecke operators.

In order to show that we have captured all of $\mathrm{Ext}^{0,2k}(E\ell\ell_*\mathbb{Q}/\mathbb{Z})$ when $k>0$ is even, we will make use of a further modification of elliptic cohomology and its operations described in [2].

From [2] we recall the following construction, which we discuss with modified notation. Let $p>3$ be a prime and $A=E_{p-1}\in E\ell\ell[1]_{2(p-1)}$ be the $(p-1)$st Eisenstein function, which agrees modulo p with the so called *Hasse invariant* of the p-reduction of the universal Weierstraß cubic. Setting $Q=E_4$, $R=E_6$ and $\Delta=(Q^3-R^2)/1728$, we define graded rings

$$\begin{aligned}e\ell\ell_*&=\mathbb{Z}_{(p)}[Q,R],\\ e\ell\ell[1]_*&=\mathbb{Z}_{(p)}[Q,R,A^{-1}],\\ E\ell\ell[1]_*&=\mathbb{Z}_{(p)}[Q,R,A^{-1},\Delta^{-1}]=(E\ell\ell_*)_{(p)}[A^{-1}].\end{aligned}$$

The ring $e\ell\ell_*$ is the subring of $E\ell\ell_*$ consisting of modular forms holomorphic at infinity, and the level 1 elliptic genus $MU_*\longrightarrow E\ell\ell_*$ takes values in $e\ell\ell_*$. The induced genus $MU_*\longrightarrow e\ell\ell[1]_*$ makes $e\ell\ell[1]_*$ into an algebra over MU_* satisfying the conditions of the Landweber Exact Functor Theorem, hence there are multiplicative homology and cohomology theories

$$\begin{aligned}e\ell\ell[1]_*(\ )&=e\ell\ell[1]_*\underset{MU_*}{\otimes}MU_*(\ ),\\ e\ell\ell[1]^*(\ )&=e\ell\ell[1]^*\underset{MU_*}{\otimes}MU^*(\ ),\end{aligned}$$

with the latter defined on finite CW complexes or spectra. On localizing with respect to powers of Δ, we find

$$\begin{aligned} e\ell\ell[1]_*(\ )[\Delta^{-1}] &= E\ell\ell[1]_* \underset{MU_*}{\otimes} MU_*(\ ) \\ &\cong E\ell\ell[1]_* \underset{E\ell\ell_*}{\otimes} E\ell\ell_*(\ ), \\ e\ell\ell[1]^*(\ )[\Delta^{-1}] &= E\ell\ell[1]^* \underset{MU_*}{\otimes} MU^*(\ ) \\ &\cong E\ell\ell[1]_* \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ), \end{aligned}$$

at least on finite CW complexes and spectra.

We now form a sort of p-adic completion of $e\ell\ell[1]^*(\ )$ and its dual homology theory. For $n \geqslant 1$, there is a q-expansion modulo p^n,

$$\varepsilon_n \colon e\ell\ell[1]_{2k} \longrightarrow \mathbb{Z}[[q]]/(p^n).$$

Following [2] which in turn depends on results of [11,12] (see also [8]), we know that for $F \in e\ell\ell[1]_{2r}$ and $G \in e\ell\ell[1]_{2s}$, then

$$\varepsilon_n(F) = \varepsilon_n(G) \quad \Longrightarrow \quad s - r = t(p-1)p^{n-1} \text{ for some } t \in \mathbb{Z}.$$

In particular, we may use this to define an equivalence relation on $e\ell\ell[1]_*/(p^n)$, giving a graded object $e\ell\ell[1]/p^n_\bullet$ indexed on the finite group $\mathbb{Z}/2(p-1)p^{n-1}$. We then set

$$\widehat{e\ell\ell[1]}_\bullet = \varprojlim_n e\ell\ell[1]/p^n_\bullet$$

which is naturally graded on the profinite group $\mathbb{Z}/2(p-1) \times \mathbb{Z}_p$; apart from the factor of 2, this grading group is sometimes usefully interpreted as being the p-adic units $\mathbb{Z}_p^\times$. Note that for each $n > 0$, there are natural maps

$$\begin{aligned} e\ell\ell_{2k}/(p^n) &\longrightarrow e\ell\ell[1]_{2k}/(p^n), \\ e\ell\ell_{2k} &\longrightarrow e\ell\ell[1]_{2k}. \end{aligned}$$

There is an additive operation of degree 0,

$$\mathrm{U}_p \colon \widehat{e\ell\ell[1]}_\bullet \longrightarrow \widehat{e\ell\ell[1]}_\bullet,$$

defined on q-expansions by

$$(\mathrm{U}_p\, F)(q) = \sum_n a_{np} q^n,$$

where $F(q) = \sum_n a_n q^n$. By [2], U_p extends to a stable cohomology operation

$$\mathrm{U}_p \colon \widehat{e\ell\ell[1]}^\bullet(\ ) \longrightarrow \widehat{e\ell\ell[1]}^\bullet(\ ).$$

This operation is induced from an element of

$$\mathrm{Hom}_{e\ell\ell[1]_*}(e\ell\ell[1]_\bullet \widehat{e\ell\ell[1]}, \widehat{e\ell\ell[1]}_\bullet)$$

where

$$e\ell\ell[1]_{\bullet}\widehat{e\ell\ell[1]} = e\ell\ell[1]_{*}e\ell\ell[1] \underset{e\ell\ell[1]_{*}}{\otimes} \widehat{e\ell\ell[1]}_{\bullet}.$$

Let M_* be a right $e\ell\ell[1]_*e\ell\ell[1]$-comodule. Then the extended module

$$M_{\bullet} = M_* \underset{e\ell\ell[1]_{*}}{\otimes} \widehat{e\ell\ell[1]}_{\bullet}$$

is also graded on $\mathbb{Z}/2(p-1) \times \mathbb{Z}_p$. For $x \in M_*$, we will denote $x \otimes 1$ by x. The coaction γ can be used to define the operation U_p on $M_{\bullet}$, namely as the composite

$$\begin{aligned} M_{\bullet} \xrightarrow{\gamma} M_* \underset{e\ell\ell[1]_{*}}{\otimes} e\ell\ell[1]_*e\ell\ell[1] \underset{e\ell\ell[1]_{*}}{\otimes} \widehat{e\ell\ell[1]}_{\bullet} &= M_* \underset{e\ell\ell[1]_{*}}{\otimes} e\ell\ell[1]_{\bullet}\widehat{e\ell\ell[1]} \\ &\xrightarrow{\mathrm{Id}\otimes \mathrm{U}_p} M_* \underset{e\ell\ell[1]_{*}}{\otimes} \widehat{e\ell\ell[1]}_{\bullet} = M_{\bullet}. \end{aligned}$$

We now return to our discussion of $\mathrm{Ext}^{0,2k}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*\mathbb{Q}/\mathbb{Z}[1/6])$. We have to show that an element of the form $\dfrac{F}{d}$ where $F(q) = \sum_{r_0 \leqslant r} a_r q^r$ for some $r_0 > 0$ must be zero. It suffices to factorize d into its prime power factors and verify this for denominators of such form; indeed, it is even sufficient to consider the case where

$$d = p \quad \text{and} \quad a_{r_0} \not\equiv 0 \pmod{p}.$$

Here the coaction agrees with the right unit $\eta_R\colon E\ell\ell_* \longrightarrow E\ell\ell_*E\ell\ell$ and it is easy to verify that for an element $F \in E\ell\ell_*$ with holomorphic q-expansion

$$\eta_R(F) \in e\ell\ell_* \underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} e\ell\ell_*.$$

Hence

$$F \in \mathrm{Ext}^{0,2k}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p^n)) \Longrightarrow F \in \mathrm{Ext}^{0,2k}_{e\ell\ell[1]_*e\ell\ell[1]}(e\ell\ell[1]_*, e\ell\ell[1]_*/(p^n)).$$

In particular we have $\mathrm{U}_p F = F$ for such F.

Proposition 3.1. *Let $F \in e\ell\ell[1]_{2(p-1)t}$ be a cusp form satisfying*

$$\begin{aligned} \mathrm{T}_\ell F &\equiv (1+\ell^{-1})F \pmod{p} \quad \text{for every prime } \ell \neq p, \\ \mathrm{U}_p F &\equiv F \pmod{p}. \end{aligned}$$

Then $F \equiv 0 \pmod{p}$.

Proof. Let F have q-expansion $F(q) = \sum_{1\leqslant r} a_r q^r$ where $a_r \in \mathbb{Z}_{(p)}$. By the first assumption on F, we have for each prime $\ell \neq p$,

$$\mathrm{T}_\ell F \equiv \frac{(1+\ell)}{\ell} F \equiv (1+\ell^{-1})F \pmod{p},$$

implying the following system of congruences modulo p:

$$\begin{aligned} a_{n\ell} &\equiv (1+\ell^{-1})a_n & &\text{if } \ell \nmid n, \\ a_{n\ell} + \ell^{-1}a_{n/\ell} \equiv a_{n\ell} + \ell^{(p-1)t-1}a_{n/\ell} &\equiv (1+\ell^{-1})a_n & &\text{if } \ell \mid n. \end{aligned}$$

By the second assumption, the following congruences are also true:

$$a_{np} \equiv a_n \quad (n > 0).$$

As remarked in Serre [12] §2.3, Lemme 4, the general solution of these congruences can be shown by induction on n to have the form

$$a_n \equiv \sigma_{p-2}(n) a_1 \pmod{p},$$

where

$$\sigma_{p-2}(n) = \sum_{m|n} m^{p-2} \equiv \sum_{m|n} m^{-1} \pmod{p}.$$

Thus we must have

$$F(q) \equiv a_1 \varphi_p(q)$$

for some $a_1 \in \mathbb{Z}_{(p)}$ and

$$\varphi_p(q) = \sum_{1 \leqslant n} \sigma_{p-2}(n) q^n.$$

But it is an important fact that $\varphi_p(q)$ is not the reduction modulo p of the q-expansion of a modular form over $\mathbb{Z}_{(p)}$, as is proved by Serre in [11] §2.2 Lemme, see also Lang's account in [9]. Thus $a_1 \equiv 0$ and hence $F \equiv 0$ modulo p. □

This result completes the proof of the Theorem.

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