ON THE ADAMS E_2-TERM FOR ELLIPTIC COHOMOLOGY

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Abstract. We investigate the E_2-term of Adams spectral sequence based on elliptic homology. The main results describe this E_2-term from a 'chromatic' perspective.

At a prime $p > 3$, the Bousfield class of $E\ell\ell$ is the same as that of $K(0) \vee K(1) \vee K(2)$. Using delicate facts due to Katz (which also play a major rôle in work on the structure $E\ell\ell_*E\ell\ell$ by Clarke & Johnson, the author and Laures) as well as our description of supersingular elliptic cohomology in terms of $K(2)$-theory, we show that the E_2-term is chromatically of length 2 and totally determined by the 0, 1 and 2 columns of the usual chromatic spectral sequence for BP. We apply our results to recover results of [7,13] and indeed extend them to completely determine this Adams E_2-term.

In the Appendix we reprove Katz's result and a generalisation which allows a similar analysis of the chromatic spectral sequence for the E_2-term of the Adams spectral sequence based on $E(2)$. This approach is also of use in connection with the more general case associated to $E(n)$ for $n > 2$.

Introduction.

Although the main driving force behind the development of elliptic cohomology undoubtedly lies in its geometric significance, it also has considerable interest to algebraic topologists. Versions of elliptic cohomology defined using Landweber's Exact Functor Theorem turn out to have a rich enough internal structure to make their analysis worthwhile. For example, their stable operation algebras contain Hecke operators and the dual cooperation algebras are arithmetically interesting. Moreover, their reductions modulo a prime p reflect the theory of 'ordinary' reductions of elliptic curves and the theory of mod p modular forms due to Swinnerton-Dyer and Serre, as well as the theory of supersingular reductions. Such reductions and localisations are related to v_1 and v_2-periodicity in the associated cohomology theories, which suggests that a geometrically defined version of elliptic cohomology should have the potential to capture both types of periodicity.

Work of Clarke & Johnson [7], the author [4] and Laures [13] has put the structure of the cooperation algebra $E\ell\ell_*E\ell\ell$ on a firm footing, and attention has turned to applications in stable homotopy theory, particularly the Adams spectral sequence based on elliptic (co)homology. It is clear from these papers that interesting connections exist between algebraic or homotopy theoretic questions about the E_2-term

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of this spectral sequence and arithmetic questions about modular forms. The main goal of the present work is to describe this E_2-term from a 'chromatic' perspective along the lines pioneered by Miller, Ravenel & Wilson [15]. In a companion paper to the present [5], we use Hecke operations to compute the 1-line, thus providing a direct homotopy theoretic application of the operations constructed in [1,2].

Since elliptic (co)homology is essentially defined to be 'v_2-periodic' at each prime p, it turns out as might be expected, that at p this E_2-term has only rational, v_1 and v_2-periodic constituents, thus it is chromatically 'of length 2'. It is easy to see that the Bousfield class of $E\ell\ell_{(p)}$ is the same as that of $K(0) \vee K(1) \vee K(2)$, which also fits well with our algebraic results. In proving our main results, we make use of a delicate result due to Katz [11] (which also plays a major rôle in work on $E\ell\ell_*E\ell\ell$) as well as our description of supersingular elliptic cohomology in terms of $K(2)$-theory. We also give a 'change of rings' result for the cohomology of a Hopf algebroid which was suggested by a technique used in [6]; a similar result was earlier discovered by Hopkins [9,10].

We apply our results to recover results of [7,13] as well as extending them to completely determine the relevant Adams E_2-term in terms of the algebraic results of [15]. However, our main aim is to give structural results rather than present detailed calculations such as those found in the work of Laures.

In the Appendix, we give a new proof of Katz's result and a generalisation result which allows a similar analysis of the chromatic spectral sequence for the E_2-term of the Adams spectral sequence based on $E(2)$. Hovey & Sadofsky [10] have used similar techniques to analyse the Adams E_2-term based on the spectrum $E(n)$ of Johnson & Wilson; Theorem A.3 gives a more explicit version of their Theorem 3.6 which they attribute to Lazard; their use of faithful flatness in proving such results was influenced by Hopkins [9].

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§1 Recollections on elliptic cohomology.

We refer to our earlier papers [1,3,4] for notation and other background material.

We take the coefficient ring of elliptic cohomology to be the graded ring of meromorphic modular forms of level 1 with q-coefficients in $\mathbb{Z}[1/6]$,

$$E\ell\ell_* = \mathbb{Z}[1/6][Q, R, \Delta^{-1}]_*,$$

where

$$Q = E_4, \ R = E_6, \quad \text{and} \quad \Delta = \frac{1}{1728}(Q^3 - R^2),$$

with weights 4, 6 and 12 respectively, and hence topological degrees 8, 12 and 24. Elliptic cohomology is then defined to be the functor on finite CW complexes given by

$$E\ell\ell^*(\ ) = E\ell\ell^* \underset{MU_*}{\otimes} MU^*(\ ),$$

where $E\ell\ell^n = E\ell\ell_{-n}$ and the tensor product is formed using the level 1 genus $MU_* \longrightarrow E\ell\ell_*$ of [2].

For a prime $p > 3$, we discussed the reductions $E\ell\ell_*/(p)$ and $E\ell\ell_*/(p, A)$ in [1,3], where

$$A = E_{p-1} \in (E\ell\ell_*)_{(p)}$$

is the *Hasse invariant* (at the prime p). In particular we showed that there is an associated cohomology theory

$$(E\ell\ell/(p,A))^*(\ )$$

with coefficient ring $E\ell\ell_*/(p,A)$ and moreover there is a natural multiplicative equivalence of functors

$$(E\ell\ell/(p,A))^*(\ ) \cong E\ell\ell_*/(p,A) \underset{K(2)_*}{\otimes} K(2)^*(\ ),$$

where $E\ell\ell_*/(p,A)$ is a finite dimensional algebra over $K(2)_*$. The latter fact uses the existence of a suitable genus $MU_* \longrightarrow E\ell\ell_*/(p,A)$ factoring through $K(2)_*$ and constructed in [3].

§2 Katz's ring of divided congruences and a topological interpretation.

In [11], Katz defines the *ring of divided congruences* amongst modular forms; a more general discussion can be found in [8]. Katz's work is carried out in a p-adic setting, however, it is straightforward to formulate a version defined over the p-local integers, or even over $\mathbb{Z}[1/6]$. Topological implications of this work are discussed by the present author in [4] and by Laures in [13]. In this section we consider some further aspects of Katz's construction which will be used later. We will work locally at a prime $p > 3$. We set $K = KU_{(p)}$, the p-local K-theory spectrum; then $K_* = \mathbb{Z}_{(p)}[t, t^{-1}]$ where $t \in K_2$ is the Bott periodicity element.

The ring DC_p is defined to be the subring of $\mathbb{Q}[[q]]$ consisting of all (inhomogeneous) sums of q-expansions of modular forms of level 1 with q-coefficients in $\mathbb{Z}_{(p)}$. Thus any inhomogeneous polynomial in Q and R over $\mathbb{Z}_{(p)}$ lies in DC_p as does the important element $(E_{p-1} - 1)/p$. An explicit set of algebra generators for DC_p is described in [11]. We will also consider the localisation

$$\mathrm{DC}_p[\Delta^{-1}] \subset \mathbb{Q}((q)),$$

where Δ denotes the discriminant function which has q-expansion

$$\Delta(q) = q \prod_{1 \leqslant n} (1 - q^n)^{24}.$$

DC_p admits an action of the group of p-local units $\mathbb{Z}_{(p)}^\times$ by ring automorphisms for which

$$\alpha \cdot \sum_i F_i(q) = \sum_i \alpha^i F_i(q),$$

where F_i is of weight i; this action extends to $\mathrm{DC}_p[\Delta^{-1}]$. On reduction modulo p^n, we obtain continuous actions of the groups of p-adic units $\mathbb{Z}_p^\times$ on the rings $\mathrm{DC}_p/(p^n)$ and $\mathrm{DC}_p[\Delta^{-1}]/(p^n)$. We remark that from a topological perspective, this action is a consequence of the existence of the stable Adams operations in $K_*E\ell\ell$, the p-local K-homology of the elliptic spectrum $E\ell\ell$.

Now set $\mathrm{D} = \mathrm{DC}_p/(p)$ and $\mathrm{D}_k = \mathrm{D}^{1+p^k\mathbb{Z}_p} \subset \mathrm{D}$, the subring of elements invariant under the closed subgroup $1 + p^k\mathbb{Z}_p \subset \mathbb{Z}_p^\times$. There is an exhaustive filtration of subalgebras

$$\mathrm{D}_1 \subset \mathrm{D}_2 \subset \cdots \subset \mathrm{D}_k \subset \cdots \subset \mathrm{D}$$

in which each subextension $\mathrm{D}_k \subset \mathrm{D}_{k+1}$ is an *Artin–Schreier* extension with Galois group $\mathbb{Z}/p$ (see §7 for a brief exposition of Artin–Schreier theory). Moreover, we may identify D_1 with the homomorphic image of the (ungraded) ring $\mathbb{F}_p[Q,R]$. By a result of Swinnerton-Dyer and Serre [18,19] (see also [12]) which predates and indeed motivated Katz's work,

$$\mathrm{D}_1 = \mathbb{F}_p[Q,R]/(A-1),$$

where $A = E_{p-1}$ has q-expansion $A(q) \equiv 1 \pmod{p}$. Similarly, there is an exhaustive filtration of subalgebras

$$\mathrm{D}_1[\Delta^{-1}] \subset \mathrm{D}_2[\Delta^{-1}] \subset \cdots \subset \mathrm{D}_k[\Delta^{-1}] \subset \cdots \subset \mathrm{D}[\Delta^{-1}]$$

in which

$$\mathrm{D}_1[\Delta^{-1}] = \mathbb{F}_p[Q,R,\Delta^{-1}]/(A-1).$$

The following is a useful consequence of Katz's observation that each of the subextensions is in fact a free module.

Proposition 2.1. *For $k < \ell \leqslant \infty$, each of the ring extensions $\mathrm{D}_k \subset \mathrm{D}_\ell$ and $\mathrm{D}_k[\Delta^{-1}] \subset \mathrm{D}_\ell[\Delta^{-1}]$ is faithfully flat and in fact a free module.*

The stable K-theory Adams operations ψ^r $(r \in \mathbb{Z}_{(p)}^{\times})$ satisfy

$$\psi^r(t^k) = r^k t^k.$$

Katz identifies DC_p as the universal ring for elliptic curves in Weierstraß form over $\mathbb{Z}_{(p)}$-algebras with a given strict isomorphism between the canonical formal group law and the multiplicative one. By standard topological considerations the graded version of the latter ring is naturally isomorphic to

$$K_* \underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} E\ell\ell_* \cong K_*E\ell\ell.$$

Hence, Katz's result gives the following topological result.

Theorem 2.2. *There are isomorphisms of rings*

$$\mathrm{DC}_p[\Delta^{-1}] \cong K_0E\ell\ell$$

and graded rings

$$\mathrm{DC}_p[\Delta^{-1}][t,t^{-1}] \cong K_*E\ell\ell.$$

The q-expansion map is related to Miller's elliptic character $E\ell\ell_* \to K_*E\ell\ell$, which is given by

$$F \longmapsto t^{\operatorname{wt} F} F(q).$$

Notice that the reduction $E\ell\ell_*/(p) \to K_*E\ell\ell/(p)$ is a monomorphism for which

$$F \pmod{p} \longmapsto t^{\operatorname{wt} F} F(q) \pmod{p}$$

and in particular,

$$A \longmapsto t^{p-1} \pmod{p}.$$

In this situation we have

Proposition 2.3. *For $k < \ell \leqslant \infty$, the ring extensions $\mathrm{D}_k[t, t^{-1}] \subset \mathrm{D}_\ell[t, t^{-1}]$ and $\mathrm{D}_k[\Delta^{-1}][t, t^{-1}] \subset \mathrm{D}_\ell[\Delta^{-1}][t, t^{-1}]$ are faithfully flat and in fact free modules.*

Corollary 2.4. *The extension $A^{-1}E\ell\ell_*/(p) \to K_*E\ell\ell/(p)$ is faithfully flat and in fact a free $A^{-1}E\ell\ell_*/(p)$-module.*

§3 Some change of rings results.

In this section we give two 'change of rings' results for the cohomology of Hopf algebroids. After a first version of this paper was circulated we became aware of the preprint of Hovey and Sadofsky [10] in which a result similar to our Proposition 3.2 appears together with a disguised form of our Theorem A.3. It turns out that 3.2 either alone or in combination with A.3 is extremely powerful and seems to imply most of the known change of rings results. Since our proofs and formulations differ somewhat from those in [10] we give them in full. We will assume familiarity with basic details and notation on Hopf algebroids such as may be found in Ravenel's book [16].

Let (A, Γ) be a Hopf algebroid over a ring $\Bbbk$, and let B be a flat commutative A-algebra under each of the two $\Bbbk$-algebra homomorphisms $f, g\colon A \longrightarrow B$. Then there are Hopf algebroids (B, Σ_f) and (B, Σ_g) with

$$\Sigma_f = B \underset{f}{\otimes} \Gamma \underset{f}{\otimes} B, \quad \Sigma_g = B \underset{g}{\otimes} \Gamma \underset{g}{\otimes} B,$$

where the tensor products are taken over A using the homomorphisms f and g. Given a left Γ-comodule M, we can form the left Σ_f-comodule

$$f^*M = B \underset{f}{\otimes} M,$$

and the left Σ_g-comodule

$$g^*M = B \underset{g}{\otimes} M.$$

Similar constructions exist for a right Γ-comodule.

Proposition 3.1. *If there is a $\Bbbk$-algebra homomorphism $H\colon \Gamma \longrightarrow B$ such that*

$$H \circ \eta_L = f, \quad H \circ \eta_R = g,$$

then the Hopf algebroids (B, Σ_f) and (B, Σ_g) are naturally equivalent. Hence there is a natural isomorphism

$$\mathrm{Ext}^*_{\Sigma_f}(f^*M, f^*N) \cong \mathrm{Ext}^*_{\Sigma_g}(g^*M, g^*N)$$

for any pair of left (or right) Γ-comodules M and N.

Proof. See [16], Lemma 6.1.5. □

In our second change of rings result, the key idea of using faithful flatness is due to Würgler and was used in [6]; a similar result was also obtained by Hopkins and is given with a different proof in [10].

Proposition 3.2. *If the algebra extension $f\colon A \longrightarrow B$ is faithfully flat, then for any Γ-comodule N there is a natural isomorphism*

$$\mathrm{Ext}^*_{\Sigma_f}(B, f^*N) \cong \mathrm{Ext}^*_\Gamma(A, N).$$

Proof. This proof replaces the published one which had some errors and omissions pointed out by Drew Heard.

For general results on Hopf algebroids and their homological algebra, we refer the reader to [16], especially Appendix A1. Given a Hopf algebroid (A, Γ), we assume that Γ is flat as a left or equivalently as a right A-module so that we can do homological algebra with Γ-comodules. In particular, we recall that for a left A-module W,

$$A \underset{\Gamma}{\square} (\Gamma \otimes_A W) \cong W,$$

by [16], lemma A1.1.16.

We assume that N is a left Γ-comodule, the case where it is a right comodule being similar. Recall from [16] that

$$\mathrm{Ext}^*_\Gamma(A, N) \cong \mathrm{Cotor}^*_\Gamma(A, N),$$

where $\mathrm{Cotor}^*_\Gamma(A, \ )$ consists of the derived functors of the cotensor product functor $A \square_\Gamma (\ )$ on the category of left Γ-comodules. Considering Γ as a right Γ-comodule we set

$$\widetilde{\Gamma} = \Gamma \underset{A}{\otimes} B$$

which has an obvious right Σ-comodule structure, where $\Sigma = \Sigma_f = B \otimes_A \Gamma \otimes_A B$. First we note that

$$\widetilde{\Gamma} \underset{\Sigma}{\square} B = A.$$

To see this, first observe that the exact sequence defining $\widetilde{\Gamma} \square_\Sigma B$,

$$0 \longrightarrow \widetilde{\Gamma} \underset{\Sigma}{\square} B \to \widetilde{\Gamma} \underset{B}{\otimes} B \to \widetilde{\Gamma} \underset{B}{\otimes} \Sigma \underset{B}{\otimes} B, \tag{Seq-1}$$

is a sequence of left A-modules, and the left unit $A \longrightarrow \Gamma$ factors through $\widetilde{\Gamma} \square_\Sigma B$ and gives rise to a commutative diagram of left A-modules of the following form.

$$\begin{array}{ccccccc}
 & & A & \longrightarrow & \widetilde{\Gamma} \otimes_B B & \longrightarrow & \widetilde{\Gamma} \otimes_B \Sigma \otimes_B B \\
 & & \big\downarrow & & = \big\downarrow & & = \big\downarrow \\
0 & \longrightarrow & \widetilde{\Gamma} \square_\Sigma B & \longrightarrow & \widetilde{\Gamma} \otimes_B B & \longrightarrow & \widetilde{\Gamma} \otimes_B \Sigma \otimes_B B
\end{array}$$

Tensoring with the flat A-module B gives the commutative diagram

$$\begin{array}{ccccccc}
 & & B & \longrightarrow & B \otimes_A (\widetilde{\Gamma} \otimes_B B) & \longrightarrow & B \otimes_A (\widetilde{\Gamma} \otimes_B \Sigma \otimes_B B) \\
 & & \cong \big\downarrow & & \cong \big\downarrow & & \cong \big\downarrow \\
0 & \longrightarrow & \Sigma \square_\Sigma B & \longrightarrow & \Sigma \otimes_B B & \longrightarrow & \Sigma \otimes_B \Sigma \otimes_B B
\end{array}$$

where the lower row is the exact sequence defining

$$B \underset{A}{\otimes} (\Gamma \underset{B}{\otimes} B \underset{\Sigma}{\square} B) \cong \Sigma \underset{\Sigma}{\square} B \cong B.$$

By faithful flatness of B over A, the sequence

$$0 \longrightarrow A \to \widetilde{\Gamma} \underset{B}{\otimes} B \to \widetilde{\Gamma} \underset{B}{\otimes} \Sigma \underset{B}{\otimes} B$$

is exact and equivalent to (Seq-1), while the map

$$A \cong A \underset{A}{\otimes} \widetilde{\Gamma} \underset{\Sigma}{\square} B \to B \underset{A}{\otimes} \widetilde{\Gamma} \underset{\Sigma}{\square} B \cong B$$

agrees with f.

Using the isomorphism

$$B \underset{A}{\otimes} (V \underset{A}{\otimes} \widetilde{\Gamma}) \cong V \underset{A}{\otimes} B \underset{A}{\otimes} \widetilde{\Gamma},$$

This argument can be generalised to show that for any right A-module U,

$$(U \underset{A}{\otimes} \widetilde{\Gamma}) \underset{\Sigma}{\square} B \cong U.$$

We also note that for any left A-module V, $B \otimes_A \Gamma \otimes_A V$ is a left Σ-comodule and

$$B \underset{\Sigma}{\square} (B \underset{A}{\otimes} \Gamma \underset{A}{\otimes} V) \cong V.$$

Now let

(Seq-2) $$0 \longrightarrow N \to J^*$$

be a relatively injective resolution of N by left Γ-comodules, where

$$J^s = \Gamma \underset{A}{\otimes} \widetilde{J}^s$$

for suitable left A-modules $\widetilde{J}^s$. Then $\mathrm{Cotor}^*_\Gamma(A, N)$ is the cohomology of the complex

$$A \underset{\Gamma}{\square} (\Gamma \underset{A}{\otimes} \widetilde{J}^*) \cong \widetilde{J}^*.$$

Tensoring with B we obtain a resolution

$$0 \longrightarrow B \underset{A}{\otimes} N \to B \underset{A}{\otimes} \Gamma \underset{A}{\otimes} \widetilde{J}^*$$

of left Σ-comodules and $\mathrm{Cotor}^*_\Sigma(B, B \otimes_A N)$ is the cohomology of the complex

$$B \underset{\Sigma}{\square} (B \underset{A}{\otimes} \Gamma \underset{A}{\otimes} \widetilde{J}^*) \cong \widetilde{J}^*.$$

So we have

$$\mathrm{Cotor}^*_\Sigma(B, B \underset{A}{\otimes} N) \cong \mathrm{Cotor}^*_\Gamma(A, N)$$

as claimed. $\square$

§4 A chromatic resolution.

Consider the exact sequence of $E\ell\ell_*E\ell\ell$ comodules

$$0 \to E\ell\ell_*/(p) \to A^{-1}E\ell\ell_*/(p) \to E\ell\ell_*/(p, A^\infty) \to 0$$

in which

$$E\ell\ell_*/(p, A^\infty) = \operatorname*{colim}_{\substack{\longrightarrow \\ n}} E\ell\ell_*/(p, A^n),$$

where the colimit is taken over the evident inclusions of comodules

$$E\ell\ell_*/(p, A^n) \to E\ell\ell_*/(p, A^{n+1}).$$

On applying the functor

$$\mathrm{Ext}^{*,*}(\ ) = \mathrm{Ext}^{*,*}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*,\ )$$

we obtain a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Ext}^{s-1,*}(E\ell\ell_*/(p, A^\infty)) \to & \\ \mathrm{Ext}^{s,*}(E\ell\ell_*/(p)) \to \mathrm{Ext}^{s,*}(A^{-1}E\ell\ell_*/(p)) \to \mathrm{Ext}^{s,*}(E\ell\ell_*/(p, A^\infty)) \to & \\ \mathrm{Ext}^{s+1,*}(E\ell\ell_*/(p)) \to \cdots & \end{aligned}$$

which we will now analyse.

First we remark that the quotient ring $E\ell\ell_*/(p, A)$ can be related to $K(2)_*$ using results of [3], in which we showed that there is a composite of ring homomorphisms

$$\theta\colon MU_* \to BP_* \to v_2^{-1}BP_*/(p, v_1) \to K(2)_* \to E\ell\ell_*/(p, A)$$

inducing a formal group law of height 2 and strictly isomorphic to the canonical one. This amounts to the existence of a ring homomorphism

$$MU_*MU \to E\ell\ell_*/(p, A)$$

whose precomposition with the left unit is the reduction of the elliptic genus $\varphi\colon MU_* \to E\ell\ell_*/(p, A)$ while that with the right unit is θ. It is worth remarking that $BP_* \subset (MU_*)_{(p)}$ and

$$\varphi(v_1) \equiv A \pmod{p}.$$

In the next result and after we follow [15] in using the somewhat misleading notation

$$K(n)_*K(n) = K(n)_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*,$$

which does *not* agree with the self-homology $K(n)_*(K(n))$!

Setting

$$E\ell\ell_*E\ell\ell/(p, A) \cong E\ell\ell_*/(p, A) \underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} E\ell\ell_*/(p, A),$$

we also make use of the following isomorphism which is a special case of [14] Proposition 1.3d:

$$\mathrm{Ext}^{*,*}_{E\ell\ell_*E\ell\ell}(E\ell\ell_*, E\ell\ell_*/(p, A)) \cong \mathrm{Ext}^{*,*}_{E\ell\ell_*E\ell\ell/(p,A)}(E\ell\ell_*/(p, A), E\ell\ell_*/(p, A)).$$

Theorem 4.1. *There is a natural isomorphism*

$$\mathrm{Ext}^{*,*}(E\ell\ell_*/(p,A)) \cong \mathrm{Ext}^{*,*}_{K(2)_*K(2)}(K(2)_*, K(2)_*).$$

Proof. This follows from Propositions 3.1 and 3.2, since by [3] we know that $E\ell\ell_*/(p,A)$ is a product of finite field extensions of $K(2)_*$, hence is a faithfully flat extension. □

Theorem 4.2. *For* $1 \leqslant n \leqslant \infty$,

$$\begin{aligned}\mathrm{Ext}^{*,*}(E\ell\ell_*/(p,A^n)) &\cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p,v_1^n)) \\ &\cong \mathrm{Ext}^{*,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p,v_1^n)).\end{aligned}$$

Proof. From [14] we have the fundamental result

$$\mathrm{Ext}^{*,*}_{K(2)_*K(2)}(K(2)_*, K(2)_*) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p,v_1)).$$

By considering the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \longrightarrow & v_2^{-1}BP_*/(p,v_1^n) & \longrightarrow & v_2^{-1}BP_*/(p,v_1^{n+1}) & \longrightarrow & v_2^{-1}BP_*/(p,v_1) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & E\ell\ell_*/(p,A^n) & \longrightarrow & E\ell\ell_*/(p,A^{n+1}) & \longrightarrow & E\ell\ell_*/(p,A) & \longrightarrow 0 \end{array}$$

and applying Ext over the Hopf algebroids (BP_*, BP_*BP) and $(E\ell\ell_*, E\ell\ell_*E\ell\ell)$ respectively to the two rows, we obtain long exact sequences of Ext groups with a map between them. Diagram chasing allows an easy induction on n that shows

$$\mathrm{Ext}^{*,*}(E\ell\ell_*/(p,A^n)) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p,v_1^n))$$

for $n < \infty$, and the infinite case follows by taking colimits over the finite cases. For the second isomorphism see [14]. □

By a similar argument with powers of p, v_1 and A, then taking appropriate colimits, we can establish

Theorem 4.3. *For* $1 \leqslant m \leqslant \infty$,

$$\begin{aligned}\mathrm{Ext}^{*,*}(E\ell\ell_*/(p^m,A^\infty)) &\cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p^m,v_1^\infty)) \\ &\cong \mathrm{Ext}^{*,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p^m,v_1^\infty)).\end{aligned}$$

We now describe the Ext group $\mathrm{Ext}(A^{-1}E\ell\ell_*/(p))$. From Katz's work, $\mathrm{D}[\Delta^{-1}]$ is a flat extension of $\mathbb{F}_p[Q,R,\Delta^{-1}]/(A-1)$. By Corollary 2.4, $K_*E\ell\ell/(p)$ is free over $A^{-1}E\ell\ell_*/(p)$. Furthermore, by Katz's description there is a ring homomorphism $MU_*MU \longrightarrow K_*E\ell\ell$ extending the natural homomorphisms $MU_* \longrightarrow K_*$ and $MU_* \longrightarrow E\ell\ell_*$. We may even p-typify the multiplicative group law and deduce the existence of a homomorphism

$$K(1)_*MU \to K_*E\ell\ell/(p).$$

Thus we have

$$\begin{aligned}K_*E\ell\ell/(p) \underset{MU_*}{\otimes} MU_*MU \underset{MU_*}{\otimes} K_*E\ell\ell/(p) \\ \cong K_*E\ell\ell/(p) \underset{K(1)_*}{\otimes} K(1)_*K(1) \underset{K(1)_*}{\otimes} K_*E\ell\ell/(p).\end{aligned}$$

which gives an equivalence of Hopf algebroids. By Propositions 3.1 and 3.2 we have

Theorem 4.4. *There is a natural isomorphism*

$$\begin{aligned}\mathrm{Ext}^{*,*}(A^{-1}E\ell\ell_*/(p)) &\cong \mathrm{Ext}^{*,*}_{K(1)_*K(1)}(K(1)_*, K(1)_*)\\ &\cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_1^{-1}BP_*/(p)).\end{aligned}$$

More generally, we have

Theorem 4.5. *For* $1 \leqslant m \leqslant \infty$,

$$\begin{aligned}\mathrm{Ext}^{*,*}(A^{-1}E\ell\ell_*/(p^m)) &\cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_1^{-1}BP_*/(p^m))\\ &\cong \mathrm{Ext}^{s,*}_{E(1)_*E(1)}(E(1)_*, E(1)_*/(p^m)).\end{aligned}$$

§5 The elliptic Adams E_2-term.

In this section we analyse the elliptic Adams E_2-term from the point of view of a chromatic spectral sequence analogous to that of Miller, Ravenel and Wilson for BP-theory [15,16]. In the present context, the chromatic spectral sequence is trivial from the E_2-term on, however, as always it provides a helpful conceptual framework.

Using the exact sequence

$$0 \longrightarrow (E\ell\ell_*)_{(p)} \to E\ell\ell_*\mathbb{Q} \to E\ell\ell_*/(p^\infty) \longrightarrow 0$$

we can reduce the calculation of $\mathrm{Ext}^{*,*}((E\ell\ell_*)_{(p)})$ to that of $\mathrm{Ext}^{*,*}(E\ell\ell_*/(p^\infty))$ since by a standard argument,

$$\mathrm{Ext}^{*,*}(E\ell\ell_*\mathbb{Q}) = \mathbb{Q} \quad \text{(in bidegree } (0,0)).$$

Thus we may concentrate on the exact sequence

$$0 \longrightarrow E\ell\ell_*/(p^\infty) \to A^{-1}E\ell\ell_*/(p^\infty) \to E\ell\ell_*/(p^\infty, A^\infty) \longrightarrow 0.$$

By work of Adams and Baird as well as Miller, Ravenel and Wilson [15], it is known that for $s > 1$,

$$\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, v_1^{-1}BP_*/(p^\infty)) \cong \mathrm{Ext}^{s,*}_{\mathrm{E}(1)_*E(1)}(E(1)_*, E(1)_*/(p^\infty)) = 0.$$

From the long exact sequence obtained by applying Ext to the above exact sequence, for $s \geqslant 3$ we have

$$\mathrm{Ext}^{s,*}(E\ell\ell_*/(p^\infty)) \cong \mathrm{Ext}^{s-1,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p^m, v_1^\infty)),$$

together with the exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Ext}^{0,*}(E\ell\ell_*/(p^\infty)) &\to \mathrm{Ext}^{0,*}_{\mathrm{E}(1)_*E(1)}(E(1)_*, E(1)_*/(p^\infty))\\ \to \mathrm{Ext}^{0,*}_{E(2)_*E(2)}&(E(2)_*, E(2)_*/(p^\infty, v_1^\infty)) \to \mathrm{Ext}^{1,*}(E\ell\ell_*/(p^\infty))\\ \to \mathrm{Ext}^{1,*}_{E(1)_*E(1)}&(E(1)_*, E(1)_*/(p^\infty)) \to \mathrm{Ext}^{1,*}_{E(2)_*E(2)}(E(2)_*, E(2)_*/(p^\infty, v_1^\infty))\\ &\to \mathrm{Ext}^{2,*}(E\ell\ell_*/(p^\infty)) \longrightarrow 0.\end{aligned}$$

From [15] it is known that

$$\begin{aligned}\operatorname{Ext}^{1,*}_{E(1)_*E(1)}(E(1)_*, E(1)_*/(p^\infty)) &\cong \operatorname{Ext}^{1,*}_{BP_*BP}(BP_*, v_1^{-1}BP_*/(p^\infty))\\ &\cong \mathbb{Z}/p^\infty \quad \text{(in bidegree } (1,0)).\end{aligned}$$

From [15] Theorem 4.2b, generators of this group are given by

$$y_k = \left[\sum_{1\leqslant r\leqslant k} \frac{(-1)^r(v_1^{-1}t_1)^r}{rp^{k+1-r}}\right]$$

in the standard cobar and chromatic notation, where y_k has order p^k and satisfies $py_{k+1} = y_k$. Under the natural map

$$\operatorname{Ext}^{1,*}_{BP_*BP}(BP_*, v_1^{-1}BP_*/(p^\infty)) \to \operatorname{Ext}^{1,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p^\infty, v_1^\infty))$$

it is easily verified that

$$y_1 \longmapsto \left[\frac{-t_1}{pv_1}\right]$$

which is the (non-zero) element coming from

$$[-t_1] \in \operatorname{Ext}^{1,*}_{K(2)_*K(2)}(K(2)_*, K(2)) \cong \operatorname{Ext}^{1,*}_{BP_*BP}(BP_*, v_2^{-1}BP_*/(p, v_1)).$$

Thus we have

Proposition 5.1. *For $n \in \mathbb{Z}$, the natural coboundary map is an epimorphism*

$$\operatorname{Ext}^{0,n}(E\ell\ell_*/(p^\infty, A^\infty)) \to \operatorname{Ext}^{1,n}(E\ell\ell_*/(p^\infty)).$$

§**6 Some consequences.**

First we rederive and extend some results originally due to Clarke & Johnson [7] and Laures [13]. Recall that the p-adic ordinal of $n \in \mathbb{Z}$ is defined to be

$$\nu_p(n) = \max\{k : p^k \mid n\}.$$

Theorem 6.1. *The group $\operatorname{Ext}^{0,k}(E\ell\ell_*/(p^\infty))$ is trivial unless $2(p-1) \mid k$. For $n \in \mathbb{Z}$,*

$$\operatorname{Ext}^{0,2n(p-1)}(E\ell\ell_*/(p^\infty)) \cong \begin{cases} \mathbb{Z}/p^{\nu_p(n)+1} & \text{if } n > 0,\\ \mathbb{Z}/p^\infty & \text{if } n = 0,\\ 0 & \text{otherwise,}\end{cases}$$

and moreover, when $n > 0$, a generator of the cyclic group is provided by

$$\frac{B_{n(p-1)}}{n(p-1)} E_{n(p-1)} \pmod{\mathbb{Z}_{(p)}}$$

where B_{2k} denotes the kth Bernoulli number. For $n = 0$, the cyclic subgroup of order p^r is generated by

$$\frac{1}{p^r} \pmod{\mathbb{Z}_{(p)}}.$$

For the A-periodic Ext groups we have the following result which implies the last Theorem. This result is originally due to Miller, Ravenel & Wilson [15] in the context of the BP-Adams E$_2$-term, but we give it here in the language of elliptic cohomology and modular forms.

Theorem 6.2. *The group* $\mathrm{Ext}^{0,k}(A^{-1}E\ell\ell_*/(p^\infty))$ *is trivial unless* $2(p-1) \mid k$*. For* $n \in \mathbb{Z}$*,*

$$\mathrm{Ext}^{0,2n(p-1)}(A^{-1}E\ell\ell_*/(p^\infty)) \cong \begin{cases} \mathbb{Z}/p^{\nu_p(|n|)+1} & \textit{if } n \neq 0, \\ \mathbb{Z}/p^\infty & \textit{if } n = 0, \\ 0 & \textit{otherwise,} \end{cases}$$

and moreover, for $n < 0$*, a generator is provided by*

$$\frac{B_{n(p-1)}}{n(p-1)} A^{-2n} E_{n(p-1)} \equiv \frac{B_{n(p-1)}}{n(p-1)} A^{-n} \pmod{\mathbb{Z}_{(p)}}.$$

Proof. Under the natural embedding

$$E\ell\ell_* \to K_*E\ell\ell \cong \mathrm{DC}_p[t, t^{-1}],$$

we have for $n > 0$,

$$\begin{aligned} E_{n(p-1)} &\longmapsto E_{n(p-1)}(q) t^{n(p-1)} \\ &= \left(1 + \frac{n(p-1)}{B_{n(p-1)}} \sum_{r \geqslant 1} \sigma_{2n-1}(r) q^r \right) t^{n(p-1)} \\ &\equiv t^{n(p-1)} \pmod{p}, \end{aligned}$$

by standard results on p-adic valuations of Bernoulli numbers. Since

$$A(q) = E_{(p-1)}(q) \equiv 1 \pmod{p},$$

we have

$$A^n \equiv t^{n(p-1)} \pmod{p^{\nu_p(n)+1}}.$$

Using the description of $E\ell\ell_*E\ell\ell$ in [4] or the arguments of [7] and [13], we find that the stated elements do indeed lie in Ext^0, and under our isomorphism

$$\mathrm{Ext}^{0,*}(A^{-1}E\ell\ell_*/(p^\infty)) \to \mathrm{Ext}^{0,*}_{\mathrm{E}(1)_*E(1)}(E(1)_*, E(1)_*/(p^\infty))$$

we see that

$$\begin{aligned} \frac{B_{n(p-1)}}{n(p-1)} E_{n(p-1)} &\longmapsto \frac{B_{n(p-1)}}{n(p-1)} v_1^n, \\ \frac{B_{n(p-1)}}{n(p-1)} A^{-2n} E_{n(p-1)} &\longmapsto \frac{B_{n(p-1)}}{n(p-1)} v_1^{-n}. \end{aligned}$$

Thus these elements are generators.

Under the natural map

$$\mathrm{Ext}^{0,*}(A^{-1}E\ell\ell_*/(p^\infty)) \to \mathrm{Ext}^{0,*}(E\ell\ell_*/(p^\infty, A^\infty))$$

it is easy to see that for $m \geqslant 0$ and $n \geqslant 1$,

$$\frac{A^{-m}}{p^n} \longmapsto \frac{1}{p^n A^m} \neq 0,$$

hence none of these elements come from $\mathrm{Ext}^{0,*}(E\ell\ell_*/(p^\infty))$. □

We refer the reader to [13] for further calculations, in particular the identification of representatives for the β family of homotopy elements.

Appendix: A proof of Katz's result and a generalisation.

In this section we give an algebraic result reproving and generalising that of Katz described in §2, and which may be used to deduce results on the chromatic spectral sequence based on $E(n)$ along the lines of [10]. We begin by describing an algebraic result whose content is essentially the same as those aspects of Katz's work required in earlier sections.

Let R_* be an $\mathbb{F}_p$-algebra and $\varphi\colon MU_* \to R_*$ a genus inducing the formal group law F over R_* whose p-series has the form

$$[p]_F X \equiv uX^p \pmod{X^{p+1}}$$

for a unit $u \in R_*$. Consider the R_*-algebra

$$R_*K = R_* \underset{MU_*}{\otimes} MU_*MU \underset{MU}{\otimes} K_*,$$

where the tensor products are formed using the genus φ and the Todd genus $MU_* \to K_*$. Since we are working p-locally we can p-typify to formal group law F over R_* and replace the multiplicative group law over K_* by that induced from the standard inclusion $E(1)_* \to K_*$. We can then make the identification

$$R_*K = R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} E(1)_* \underset{E(1)_*}{\otimes} K_*,$$

where $K_* = E(1)_*[t]/(t^{p-1} - v_1)$. We can work with the subring

$$\begin{aligned} L_* &= R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} E(1)_* \\ &= R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(1)_*. \end{aligned}$$

It is worth remarking that under the p-typical genus $BP_* \to L_*$, v_1 maps to u. We will often denote the images of elements of BP_* in L_* by the same name, for example writing v_1 for u.

The following result and its consequence provide a useful optic through which to view 2.2, 2.3 and 2.4.

Theorem A.1. *The ring L_* is a free R_*-module. Moreover, there is an exhaustive filtration of subalgebras*

$$R_* = L_*^{(1)} \subset L_*^{(2)} \subset \cdots \subset L_*^{(k)} \subset \cdots \subset L_*$$

in which each extension $L_*^{(k)} \to L_*^{(k+1)}$ is a free $L_*^{(k)}$-module and a Galois extension of Artin–Schreier type with Galois group $\mathbb{Z}/p$.

There is an action of the group of p-adic units $\mathbb{Z}_p^\times$ under which

$$L_*^{(k)} = L_*^{1+p^k\mathbb{Z}_p},$$

the fixed point set of the subgroup $1 + p^k\mathbb{Z}_p \subset \mathbb{Z}_p^\times$.

Corollary A.2. *The extension $R_* \to R_*K$ is faithfully flat and in fact a free R_*-module.*

Before stating and proving an even more general result, we digress to discuss Artin–Schreier theory, which is treated more fully in [17] for example.

Let A be a commutative algebra over the finite field $\mathbb{F}_q$ where $q = p^d$. For $a \in A$, associated to the separable polynomial $Q(X) = X^q - X - a \in A[X]$ is an extension of $\mathbb{F}_q$-algebras $A \longrightarrow B = A[X]/(Q(X))$. Then B is a free (hence faithfully flat) module over A with basis consisting of the powers x^j $(j = 0, \dots, q-1)$ of the residue class of X. Moreover, there is a 'Galois group' $\mathrm{Gal}(B/A)$ consisting of the A-algebra automorphisms of B. There is an isomorphism $\mathbb{F}_q \cong \mathrm{Gal}(B/A)$ under which $u \longleftrightarrow \varphi_u$ where $\varphi_u(x) = x + u$. We call such an extension of algebras an *Artin–Schreier extension for the polynomial $Q(X)$ with Galois group $\mathbb{F}_q$*. In the case where $A = \Bbbk$, a field, and $Q(X)$ is irreducible (equivalently, has no roots in $\Bbbk$), then B is a Galois extension in the classical sense with Galois group $\mathbb{F}_q$.

The following result may be used to rederive the change of rings results in [10], including those of Morava which provide major input to the chromatic spectral sequence. Theorem A.1 corresponds to the case $n = 1$ and the identification $\mathbb{S}_1 = 1 + p\mathbb{Z}_p$.

Theorem A.3. *Let R_* be an algebra over $\mathbb{F}_{p^n} \otimes BP_*$ which is annihilated by I_n and in which there exists a unit u satisfying*

$$v_n = u^{(p^n-1)/(p-1)}.$$

Then the ring

$$R_*K(n) = R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*$$

is a free R_*-module. Moreover, there is an exhaustive filtration of subalgebras

$$R_* = R_*K(n)^{(1)} \subset R_*K(n)^{(2)} \subset \cdots \subset R_*K(n)^{(k)} \subset \cdots \subset R_*K(n)$$

in which each extension $R_*K(n)^{(k)} \to R_*K(n)^{(k+1)}$ is a free $R_*K(n)^{(k)}$-module and a Galois extension of Artin–Schreier type with Galois group $\mathbb{F}_{p^n}$.

There is an action of the Morava stabiliser group $\mathbb{S}_n$ under which

$$R_*K(n)^{(k)} = R_*K(n)_*^{\mathbb{S}_n^{[k]}},$$

the fixed point set of the closed subgroup

$$\mathbb{S}_n^{[k]} = \left\{ 1 + \sum_{k \leqslant r} \alpha_r S^r : \forall r,\, \alpha_r^{p^n} = \alpha_r \right\} \subseteq \mathbb{S}_n.$$

Proof. Tensoring the free BP_*-module BP_*BP with the BP_*-algebra R_* gives a free R_*-algebra

$$R_*BP = R_* \underset{BP_*}{\otimes} BP_*BP = R_*[t_j : j \geqslant 0],$$

where the t_j are the usual generators for BP_*BP. Notice that R_*BP is also a BP_*-algebra under the map $\eta_R\colon BP_* \longrightarrow R_*BP$ induced from the right unit in BP_*BP. We may exploit the latter structure to form

$$R_*K(n) = R_*BP/(\eta_R(v_{n_j}) : j \geqslant 1)$$

and since $\eta_R(v_n) = v_n = \eta_L(v_n)$ in R_*BP, this is easily seen to be isomorphic to

$$R_*BP \underset{BP_*}{\otimes} K(n) \cong R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*.$$

Define

$$R_*K(n)^{(k)} = \operatorname{im} R_*[t_j : 1 \leqslant j \leqslant k-1] \to R_*K(n),$$

where the map is obtained by restricting the natural map

$$R_*BP \to R_*K(n).$$

Clearly $R_*K(n)^{(1)} = R_*$ and the $R_*K(n)^{(k)}$ exhaust $R_*K(n)$.

The relations satisfied by the images of the t_j in $R_*K(n)$ are consequences of the fact that the elements $\eta_R(v_{n+j})$ $(j \geqslant 1)$ are zero. Using standard formulæ for the coaction in BP_*BP and working in the R_*-algebra $R_*K(n)$, the relations are found to generated from a sequence of the following form:

$$v_n t_j^{p^n} - v_n^{p^j} t_j + w_j = 0 \quad (j \geqslant 1),$$

where $w_j \in R_{2(p^{n+j}-1)}K(n)^{(j)}$. Since $v_n = u^{(p^n-1)/(p-1)}$ for some unit u, we can rewrite these relations in the form

$$(u^{-(p^j-1)/(p-1)} t_j)^{p^n} - (u^{-(p^j-1)/(p-1)} t_j) + u^{-(p^{n+j}-1)/(p-1)} w_j = 0 \quad (j \geqslant 1).$$

Thus, for each $j \geqslant 1$, the extension $R_*K(n)^{(j)} \to R_*K(n)^{(j+1)}$ is an Artin–Schreier extension for the separable polynomial

$$X^{p^n} - X + u^{-(p^{n+j}-1)/(p-1)} w_j,$$

since the element $t_j \in R_*K(n)^{(j+1)}$ generates this algebra over $R_*K(n)^{(j)}$ and is a root of this polynomial.

The action of the Morava stabilizer group follows from the standard fact (see [16]) that each $\alpha \in \mathbb{S}_n$ can be realised as an element

$$\psi^\alpha \in \operatorname{Hom}_{K(n)_*}(K(n)_*K(n), \mathbb{F}_{p^n} \otimes K(n)_*).$$

We let α act on $R_*K(n)$ by forming the following composite defined in terms of the obvious 'coproduct' in which all unlabelled arrows represent what should be obvious maps:

$$\begin{aligned}
R_*K(n) = R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_* &\xrightarrow{1\otimes\mathrm{coprod}\otimes 1} \\
R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_* &\longrightarrow \\
R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*K(n) \underset{K(n)_*}{\otimes} K(n)_* &\xrightarrow{\cong} \\
R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*K(n) &\xrightarrow{1\otimes 1\otimes\psi^\alpha} \\
R_* \underset{BP_*}{\otimes} BP_*BP \underset{K(n)_*}{\otimes} K(n)_* &\longrightarrow R_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} K(n)_*.
\end{aligned}$$

Here we have treated elements of $\mathbb{F}_{p^n}$ as scalars with respect to the tensor products in moving them to the leading copy of R_*. This defines a continuous action of $\mathbb{S}_n$ on the discrete module $R_*K(n)$ and it is easily verified that each $R_*K(n)^{(k)}$ is indeed the stated fixed point set. $\square$

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