Slide 1

**Cogroupoids, Hopf algebroids and cohomology**

**Andrew Baker, Glasgow**

***Talk in Glasgow: 2 December, 1998***

**(3/12/1998)**

Slide 2

# 1 Cogroupoids in the category of algebras

Let $\Bbbk$ be a commutative unital ring. Recall that a commutative Hopf algebra $H$ over $\Bbbk$ is a *cogroup object* in $\mathbf{ComAlg}_{\Bbbk}$. Thus $H$ has a coproduct $\psi\colon H \longrightarrow H \otimes H$, a counit $\varepsilon\colon H \longrightarrow \Bbbk$ and an antipode $\chi\colon H \longrightarrow H$ which are morphisms in $\mathbf{ComAlg}_{\Bbbk}$ satisfying the commutative diagrams in which $\varphi\colon H \otimes H \longrightarrow H$ and $\eta\colon \Bbbk \longrightarrow H$ are the product and unit.

$$\begin{array}{ccc} H & \xrightarrow{\psi} & H \otimes H \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{I}\otimes\psi} \\ H \otimes H & \xrightarrow[\psi\otimes \mathrm{I}]{} & H \otimes H \otimes H \end{array}$$

**Slide 3**

$$\begin{array}{ccccc} H\otimes H & \xleftarrow{\psi} & H & \xrightarrow{\psi} & H\otimes H \\ {\scriptstyle \varepsilon\otimes\mathrm{I}}\downarrow & & \downarrow{\scriptstyle \mathrm{I}} & & \downarrow{\scriptstyle \varepsilon\otimes\mathrm{I}} \\ \Bbbk\otimes H & = & H & = & H\otimes\Bbbk \end{array}$$

$$\begin{array}{ccccc} H\otimes H & \xleftarrow{\psi} & H & \xrightarrow{\psi} & H\otimes H \\ {\scriptstyle \chi\otimes\mathrm{I}}\downarrow & & \downarrow{\scriptstyle \eta\varepsilon} & & \downarrow{\scriptstyle \mathrm{I}\otimes\chi} \\ H\otimes H & \xrightarrow[\varphi]{} & H & \xleftarrow[\varphi]{} & H\otimes H \end{array}$$

These are formally the axioms for a group object in $\mathbf{ComAlg}_{\Bbbk}^{\mathrm{op}}$, i.e., a cogroup object in $\mathbf{ComAlg}_{\Bbbk}$.

**Slide 4**

Recall that a groupoid is a small category in which all morphisms are invertible. Thus a group is a groupoid with one object. The structure maps of a groupoid $\mathcal{C}$ and its object set $\mathcal{O}$ are

- a partial product $\mu\colon \mathcal{C}\underset{\mathcal{O}}{\times}\mathcal{C} \longrightarrow \mathcal{C}$;
- an identity $\iota\colon \mathcal{O} \longrightarrow \mathcal{C}$;
- a domain $\delta\colon \mathcal{C} \longrightarrow \mathcal{O}$ and codomain $\kappa\colon \mathcal{C} \longrightarrow \mathcal{O}$;
- an inverse $\chi\colon \mathcal{C} \longrightarrow \mathcal{C}$.

These satisfy the following axioms.

$$\begin{array}{ccc} \mathcal{C}\underset{\mathcal{O}}{\times}\mathcal{C}\underset{\mathcal{O}}{\times}\mathcal{C} & \xrightarrow{\mu\times\mathrm{I}} & \mathcal{C}\underset{\mathcal{O}}{\times}\mathcal{C} \\ {\scriptstyle \mathrm{I}\times\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathcal{C}\underset{\mathcal{O}}{\times}\mathcal{C} & \xrightarrow[\mu]{} & \mathcal{C} \end{array}$$

**Slide 5**

$$
\begin{tikzcd}
\mathcal{C} \underset{\mathcal{O}}{\times} \mathcal{C} \arrow[r, "\mu"] & \mathcal{C} & \mathcal{C} \underset{\mathcal{O}}{\times} \mathcal{C} \arrow[l, "\mu"'] \\
\mathcal{O} \underset{\mathcal{O}}{\times} \mathcal{C} \arrow[u, "\iota\otimes\mathrm{I}"] \arrow[r, equal] & \mathcal{C} \arrow[u, "\mathrm{I}"'] \arrow[r, equal] & \mathcal{C} \underset{\mathcal{O}}{\times} \mathcal{O} \arrow[u, "\mathrm{I}\otimes\iota"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{C} \underset{\mathcal{O}}{\times} \mathcal{C} \arrow[r, "\mu"] & \mathcal{C} & \mathcal{C} \underset{\mathcal{O}}{\times} \mathcal{C} \arrow[l, "\mu"'] \\
\mathcal{C} \times \mathcal{C} \arrow[u, "\chi\otimes\mathrm{I}"] & \mathcal{C} \arrow[u, "\delta\iota=\iota\kappa"'] \arrow[l, "\mathrm{diag}"] \arrow[r, "\mathrm{diag}"'] & \mathcal{C} \times \mathcal{C} \arrow[u, "\mathrm{I}\otimes\chi"']
\end{tikzcd}
$$

**Slide 6**

A cogroupoid object in the category of commutative unital algebras over $\Bbbk$ is called a *Hopf algebroid over* $\Bbbk$.

If $\mathcal{C}$ is a finite groupoid then $\Bbbk^{\mathcal{C}} = \mathrm{Map}^{\mathrm{f}}(\mathcal{C}, \Bbbk)$, the set of maps $\mathcal{C} \longrightarrow \Bbbk$, is a Hopf algebroid. If $\mathcal{O} = \mathrm{Obj}\,\mathcal{C}$, the algebra $\Bbbk^{\mathcal{O}}$ has two distinct actions on $\Bbbk^{\mathcal{C}}$ coming from the domain and codomain maps of $\mathcal{C}$, and we usually view these as left and right actions.

In general, a Hopf algebroid consists of two commutative $\Bbbk$-algebras $H$ and $R$, where $H$ is an $R$-bimodule and in fact a commutative augmented $R$-algebra with respect to each of these structures. It also has a coproduct

$$\psi\colon H \longrightarrow H \underset{R}{\otimes} H$$

in which the tensor product is for bimodules, and is also an algebra homomorphism with respect to the left and right hand $R$-algebra structures. We highlight the rôle of $R$ by writing $(H, R)$.

## 2 Representations of groupoids and comodules over Hopf algebroids

**Slide 7**

Let $\mathcal{C}$ be a groupoid. Then a $\Bbbk$-representation of $\mathcal{C}$ is a functor $\underline{M}\colon \mathcal{C} \rightsquigarrow \mathbf{Mod}_{\Bbbk}$. Thus for each object $x$ of $\mathcal{C}$ there is $\Bbbk$-module $M_x$ and for each morphism $f\colon x \longrightarrow y$ a $\Bbbk$-isomorphism $Mf\colon M_x \longrightarrow M_y$. When $\mathcal{C}$ is a group (i.e., has exactly one object) this is just a $\Bbbk$-representation in the usual sense.

When dealing with the Hopf algebroid $\Bbbk^{\mathcal{C}}$ it is easy to see that $\Bbbk$-modules of the form $\mathrm{Map}^{\mathrm{f}}(\mathcal{C}, V)$ where $V$ is a $\Bbbk$-module are injective $\Bbbk^{\mathcal{C}}$-comodules at least when $\Bbbk$ is a field and then $\mathrm{Map}^{\mathrm{f}}(\mathcal{C}, V) \cong \Bbbk^{\mathcal{C}} \otimes V$. Here a comodule $M$ over a Hopf algebroid $(H, R)$ has to possess a coaction $\gamma\colon M \longrightarrow H \otimes_R M$ which is coassociative and counital.

**Slide 8**

Motivated by this we see that a comodules of the form $H \otimes_R W$ where $W$ is a left $R$-module are relatively injective for extensions of $H$-comodules $0 \longrightarrow L \longrightarrow M$ which split as $R$-modules. This allows us to define $\mathrm{Ext}^*_{(H,R)}(P, M)$ by using such relatively injective resolutions of $M$.

The theory of representations of groupoids and their cohomology seems not to have been extensively studied by algebraists. However, topologists have made many calculations.

Slide 9

## 3 Cohomological equivalences

A functor $F\colon \mathcal{C}_2 \rightsquigarrow \mathcal{C}_1$ induces a pullback functor $F^*$ on representations. Dually, on Hopf algebroids a morphism $\varphi\colon (H_1, R_1) \longrightarrow (H_2, R_2)$ induces

$$R_2 \underset{R_1}{\otimes} M \xrightarrow{\mathrm{Id}\otimes\gamma} R_2 \underset{R_1}{\otimes} H_1 \underset{R_1}{\otimes} M = R_2 \underset{R_1}{\otimes} H_1 \underset{R_1}{\otimes} R_2 \underset{R_1}{\otimes} M \longrightarrow H_2 \underset{R_1}{\otimes} M$$

and thus induces a $(H_2, R_2)$-comodule structure on $R_2 \otimes_{R_1} M$. This induces a map on cohomology

$$\mathrm{Ext}^*_{(H_1,R_1)}(P, M) \xrightarrow{\varphi^*} \mathrm{Ext}^*_{(H_2,R_2)}(R_2 \underset{R_1}{\otimes} P, R_2 \underset{R_1}{\otimes} M).$$

There is a notion of two such functors $F$, $G$ being equivalent and such functors induce equivalent maps in Ext. Such a notion also makes sense for Hopf algebroids.

Slide 10

In particular, a *split groupoid* $\mathcal{C}$ is one for which there is a group $G$ such that for every object $x$, the totality of morphisms out of $x$ is identified with $G$. Then $\mathcal{C} = \mathcal{O} \rtimes G$. It turns out that there is a canonical identification

$$\mathrm{Ext}_{\Bbbk^{\mathcal{C}}}(\Bbbk^{\mathcal{O}},\ ) = \mathrm{Ext}_{\Bbbk^{G}}(\Bbbk,\ ).$$

Similarly, a Hopf algebroid of the form $(H, R) = (S \otimes H', S \otimes R')$ where $S$ is a $\Bbbk$-subalgebra of $R$ with suitable conditions on the coaction also has

$$\mathrm{Ext}_H(R,\ ) = \mathrm{Ext}_{H'}(R',\ ).$$

## 4 A new result

Let $(H, R)$ be a $\Bbbk$-Hopf algebroid and $f\colon R \longrightarrow S$ a $\Bbbk$-algebra homomorphism. We can define a new algebra $H_f = S \otimes_R H \otimes_R S$

and $(H_f, S)$ is a Hopf algebroid in a natural way. Moreover there is an obvious morphism of Hopf algebroids $\varphi_f\colon (H,R) \longrightarrow (H_f, S)$. Furthermore, given a comodule $M$ over $H$ there is a functorial construction $f^*M = S \otimes_R M$ of a $H_f$-comodule compatible with $\varphi_f$.

**Slide 11**

**Theorem 4.1 (Baker–Würgler/Hopkins/Hovey–Sadofsky)** *If $f\colon R \longrightarrow S$ is faithfully flat then the induced map*

$$\mathrm{Ext}^*_{(H_f,S)}(S, f^*M) \longrightarrow \mathrm{Ext}^*_{(H,R)}(R, M)$$

*is an isomorphism.*

This appears to include all such known 'change of rings' results on cohomology of groupoids that have previously appeared in topology!

**Slide 12**

# 5 Applications

Let $L = \mathbb{Z}[x_i : i \geqslant 1]$, $LB = L[b_i : i \geqslant 1]$. Then $(LB, L)$ has a natural $\mathbb{Z}$-Hopf algebroid structure, where the coproduct is partly given by

$$\sum_i \psi(b_i) T^{i+1} = \sum_r 1 \otimes b_r \left( \sum_s b_s \otimes 1\, T^{s+1} \right)^{r+1}$$

with $b_0 = 1$. Then there is a semi-direct splitting as above. On localising at a prime $p$, $LB_{(p)} = S \otimes_V VT \otimes_V S$ where

$$V = \mathbb{Z}_{(p)}[v_i : i \geqslant 1], \quad S = V[s_i : i \neq p^r - 1], \quad VT = V[t_i : i \geqslant 1].$$

Then

$$\mathrm{Ext}^*_{(LB,L)}(L, \mathbb{Q} \otimes M) \cong \mathrm{Ext}^0_{(LB,L)}(L, \mathbb{Q} \otimes M),$$
$$\mathrm{Ext}^*_{(LB,L)}(L, M_{(p)}) = \mathrm{Ext}^*_{(VT,V)}(V, M_{(p)}).$$

Slide 13

For each $n \geqslant 0$, there is an invariant prime ideal $I_n = (p, v_1, \ldots, v_{n-1}) \triangleleft V$ and $v_n^{-1}V/I_n$ is a comodule over $VT$.

**Theorem 5.1 (Morava/Miller–Ravenel)** *If* $K(n) = v_n^{-1}V/(v_i : 0 \leqslant i \neq n)$ *and* $\Sigma(n) = K(n) \otimes_V VT \otimes_V K(n)$*, then*

$$\mathrm{Ext}^*_{(VT,L)}(V, v_n^{-1}V/I_n) \cong \mathrm{Ext}^*_{(\Sigma(n),K(n))}(K(n), K(n)).$$

In proving this we replace $(VT, L)$ by $(\Gamma(n), v_n^{-1}V/I_n)$ where $\Gamma(n) = v_n^{-1}VT/I_n$ and then

$$\mathrm{Ext}^*_{(VT,L)}(V, v_n^{-1}V/I_n) \cong \mathrm{Ext}^*_{(\Gamma(n),v_n^{-1}V/I_n)}(v_n^{-1}V/I_n, v_n^{-1}V/I_n).$$

$(\Sigma(n), K(n))$ is actually a Hopf algebra over $K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$. In fact, there is a grading on everything and $K(n)$ is a graded field.

Slide 14

## References


- A. Baker & U. Würgler, Liftings of formal group laws and the Artinian completion of $v_n^{-1}BP$, Proc. Camb. Phil. Soc. **106** (1989), 511–30.
- A. Baker, On the Adams $E_2$-term for elliptic cohomology, Glasgow University Mathematics Department preprint 97/15.
- M. J. Hopkins, Hopf-algebroids and a new proof of the Morava–Miller–Ravenel change of rings theorem, preprint.
- M. Hovey & H. Sadofsky, Invertible spectra in the $E(n)$-local stable homotopy category, preprint.
- H. R. Miller & D. C. Ravenel, Morava stabilizer algebras and localization of Novikov's $E_2$-term, Duke Math. J. **44** (1977), 433–47.