ELLIPTIC GENERA OF LEVEL N AND ELLIPTIC COHOMOLOGY

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(Version 17 (published version): 13/7/2001)

Abstract. Elliptic genera of level N have been defined by F. Hirzebruch, generalising the earlier notion of elliptic genus due to S. Ochanine. We show that there are corresponding elliptic cohomology theories which are naturally associated to such genera and that these are obtained from the level 1 case by algebraic extension of the coefficient rings from level 1 to level N modular forms.

Introduction.

In [8], F. Hirzebruch has introduced *elliptic genera of level* N which are (multiplicative) genera

$$\rho_{\boldsymbol{\alpha}}\colon MU_* \longrightarrow \mathbb{Z}[1/N, \zeta_{N}]((q_{N})).$$

Here $\zeta_{N} = e^{2\pi\,\mathrm{i}/N}$, $q = e^{2\pi\,\mathrm{i}\,\tau}$ for $\tau \in \mathfrak{H}$ (the upper half plane), $q_{N} = e^{2\pi\,\mathrm{i}\,\tau/N}$, and $\boldsymbol{\alpha} \in (1/N)L_\tau/L_\tau \subseteq \mathbb{C}/L_\tau$ is required to have order N as an element of the torus $\mathbb{C}/L_\tau$ associated to the lattice $L_\tau = \langle\tau, 1\rangle \subseteq \mathbb{C}$. For any ring R, we denote by $R((X)) = R[[X]][X^{-1}]$ the ring of Laurent series in X over R with finitely many negative degree terms. The main purpose of the present work is to fit such genera into the framework of elliptic cohomology in a manner which generalises the original level 2 constructions.

When earlier versions of this paper were written the author was unaware of the work of J.-L. Brylinski [4], who constructs higher level theories in many respects similar to ours, although he does not invert N in his level N theory. However, he does make use of deep facts from the theory of moduli schemes and in some cases this allows him to prove stronger results than ours, which only use standard facts from the theory of complex cobordism comodules. We hope to investigate further the precise relationships between these approaches in future

1991 *Mathematics Subject Classification.* 55N22 14L05 10D05.

Key words and phrases. Elliptic cohomology, level N structure, Weil pairing.

The author would like to acknowledge the support of the Science and Engineering Research Council and the Max-Planck-Institute, Bonn.

This paper appeared in Jour. Lond. Math. Soc. **49** (1994), 581–93.

work. It is also worth mentioning the recent work of J. Francke [7] as of interest as a companion approach to these matters.

As basic references on elliptic cohomology, we use [2], [3], [11] and [12]. We also refer to [1] and [14] for essential ideas on formal groups and their relationship to complex oriented cohomology theories. For details of 'level N' structures and the Weil pairing we cite [9], [10] and [16].

I would like to thank F. Clarke, F. Hirzebruch, M. Hovey, P. Landweber, J. Morava and S. Ochanine for their help and encouragement whilst this work was undertaken.

§1 Modular forms for congruence subgroups of level N.

We begin by considering the level N congruence subgroup $\Gamma_1(N)$ of $\mathrm{SL}_2(\mathbb{Z})$,

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

for any $N \geqslant 2$ and set $\Gamma_1(1) = \mathrm{SL}_2(\mathbb{Z})$. Let $L \subseteq \mathbb{C}$ be a lattice. We follow [10] in introducing the notion of a *modular point for* $\Gamma_1(N)$, $(L, \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \in \mathbb{C}/L$ has order N. Given a basis $\{\omega_1, \omega_2\}$ of L and the identification of $\mathrm{SL}(L)$ with $\mathrm{SL}_2(\mathbb{Z})$ using this basis, the induced action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}/L$ gives rise to a stabilizer group

$$\Gamma_1(\boldsymbol{\alpha}) = \mathrm{Stab}(\boldsymbol{\alpha}) \subseteq \mathrm{SL}_2(\mathbb{Z})$$

which is conjugate to $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Let $\mathcal{L}$ denote the set of all lattices L in $\mathbb{C}$ and let $\mathcal{M}(\Gamma_1(N))$ be the set of all modular points for $\Gamma_1(N)$; this set can be given the structure of a two dimensional complex analytic manifold which is a finite covering of $\mathcal{L}$. A function $F\colon \mathcal{M}(\Gamma_1(N)) \longrightarrow \mathbb{C}$ is called a *modular function for* $\Gamma_1(N)$ *on* $\mathcal{M}(\Gamma_1(N))$ *of weight* $k \in \mathbb{Z}$ if for all $\lambda \in \mathbb{C}^\times$,

$$F(\lambda L, \lambda\boldsymbol{\alpha}) = \lambda^{-k} F(L, \boldsymbol{\alpha}).$$

For any $M = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we also have the notion of a modular function $f\colon \mathfrak{H} \longrightarrow \mathbb{C}$ for the subgroup $M\Gamma_1(N)M^{-1} \subseteq \mathrm{SL}_2(\mathbb{Z})$, where $\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{im}\tau > 0\}$. We say that f is *a modular function for* $M\Gamma_1(N)M^{-1}$ *on* $\mathfrak{H}$ *of weight* k if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \forall \tau \in \mathfrak{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M\Gamma_1(N)M^{-1}.$$

We will now explain the relationship between these two notions of modular functions. Given a lattice $L \subseteq \mathbb{C}$, then we can choose a basis for L, say $\{\omega_1, \omega_2\}$, for which

$$\tau = \frac{\omega_1}{\omega_2} \in \mathfrak{H}.$$

We will refer to such a basis as *an oriented basis* for L. Now suppose that $f\colon \mathfrak{H} \longrightarrow \mathbb{C}$ is a modular function for $M\Gamma_1(N)M^{-1}$ on $\mathfrak{H}$ of weight k. We can construct a modular function for $\Gamma_1(N)$ on $\mathcal{M}(\Gamma_1(N))$ of weight k as follows. For a modular point $(L, \boldsymbol{\alpha}) \in \mathcal{M}(\Gamma_1(N))$, where $\boldsymbol{\alpha} = (\alpha/N) + L$ with $\alpha \in L$, choose an oriented basis $\{\omega_1, \omega_2\}$ for L so that

$$\alpha \equiv -c'\omega_1 + a'\omega_2 \mod N,$$

and then define

$$F(L, \boldsymbol{\alpha}) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right).$$

This is clearly well defined and modular. Conversely, given an F, we can define an f by setting

$$f(\tau) = F\left(\langle \tau, 1 \rangle, \frac{-c'\tau + a'}{N}\right).$$

In fact, this shows that there is an equivalence between modular functions F for $\Gamma_1(N)$ on $\mathcal{M}(\Gamma_1(N))$ and collections of functions $\left\{\left(f|_{[M]_k} \colon \mathfrak{H} \longrightarrow \mathbb{C}\right) : M \in \mathrm{SL}_2(\mathbb{Z})\right\}$ where

$$f|_{[M]_k}(\tau) = (c'\tau + d')^{-k} f\left(\frac{a'\tau + b'}{c'\tau + d'}\right)$$

is a modular form of weight k for $M\Gamma_1(N)M^{-1}$. Of course it is sufficient to let M range over a complete set of representatives for $\mathrm{SL}_2(\mathbb{Z})/\Gamma_1(N)$. We will pass freely between these types of modular functions when necessary and will say that a pair F, f related as above are *associated* modular functions. Notice that in the case of $N = 1$, these concepts agree with the usual notions of modular functions for $\mathrm{SL}_2(\mathbb{Z})$.

Now suppose that $F \colon \mathcal{M}(\Gamma_1(N)) \longrightarrow \mathbb{C}, f \colon \mathfrak{H} \longrightarrow \mathbb{C}$ are associated modular functions for $\Gamma_1(N)$ and that each of the modular functions $f|_{[M]_k}$ is holomorphic on $\mathfrak{H}$ and meromorphic at $\mathrm{i}\,\infty$, or equivalently has a q-expansion

$$\widetilde{f}|_{[M]_k}(q) = \sum_{-\infty \ll n} a_n^M q_N^n;$$

the collection of all such expansions are the *q-expansions at the cusps*. Then we say that F, f are *modular forms for* $\Gamma_1(N)$. If for some subring $K \subseteq \mathbb{C}$ all of the q-expansions $\widetilde{f}|_{[M]_k}(q)$ have coefficients in the ring $K[\zeta_N]$, then we say that F, f are *defined over* K.

Now let F be a modular form for $M\Gamma_1(N)M^{-1}$ for some $M \in \mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{M}(\Gamma_1(N))$ of weight k. Given a lattice $L \in \mathcal{L}$, we can define a polynomial

$$\Phi_L(X) = \prod_{\substack{\boldsymbol{\alpha} \in \mathbb{C}/L \\ |\boldsymbol{\alpha}| = N}} (X - F(L, \boldsymbol{\alpha})) \in \mathbb{C}[X].$$

Now if $\lambda \in \mathbb{C}^\times$, we have

$$\begin{aligned} \Phi_{\lambda L}(X) &= \prod_{\substack{\boldsymbol{\alpha}' \in \mathbb{C}/\lambda L \\ |\boldsymbol{\alpha}'| = N}} (X - F(\lambda L, \boldsymbol{\alpha}')) \\ &= \prod_{\substack{\boldsymbol{\alpha} \in \mathbb{C}/L \\ |\boldsymbol{\alpha}| = N}} (X - \lambda^{-k} F(L, \boldsymbol{\alpha})). \end{aligned}$$

Hence, if

$$\varPhi_L(X) = \sum_{0\leqslant j\leqslant D_N} H_j(L) X^{D_N-j},$$

where $D_N = \deg \varPhi$ (and is independent of L), then H_j satisfies

$$H_j(\lambda L) = \lambda^{-jk} H_j(L),$$

and so is in fact a modular function of L of weight jk for $\mathrm{SL}_2(\mathbb{Z})$ rather than just for $M\Gamma_1(N)M^{-1}$. If now the f associated to F is holomorphic, so is the h_j associated to H_j. Thus we have proved the following algebraic result.

Theorem (1.1). *Let F be a modular form of weight k for $M\Gamma_1(N)M^{-1}$ on $\mathcal{M}(\Gamma_1(N))$; then F is a solution of a functional equation of the form*

$$\varPhi(X) = \sum_{0\leqslant j\leqslant D_N} H_j\, X^{D_N-j} = 0,$$

where the coefficients are modular forms for $SL_2(\mathbb{Z})$ on $\mathcal{L}$, with H_j of weight jk.

Now let $E\ell\ell[1/N]_* = \mathbb{Z}[1/6N][E_4, E_6, \Delta^{-1}]$ be the ring of all modular forms for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ which are holomorphic on $\mathfrak{H}$, meromorphic at $\mathrm{i}\,\infty$ and have q-expansion coefficients in $\mathbb{Z}[1/6N]$. This is the localisation of the ring $E\ell\ell_* = \mathbb{Z}[1/6][E_4, E_6, \Delta^{-1}]$ of [3] with respect to the multiplicative set of powers of N. In the above, the modular form Δ is the discriminant function satisfying the relation

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$$

and E_{2k} denotes the Eisenstein function of weight $2k$. The grading on $E\ell\ell[1/N]_*$ is such that $E\ell\ell[1/N]_{2k}$ correspond to weight k. We can also define $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$ to be the graded ring of modular forms of integer weight for $M\Gamma_1(N)M^{-1}$ which are meromorphic at each cusp, and are defined over the ring $\mathbb{Z}[1/6N]$, i.e., they have all their q-expansions in the ring $\mathbb{Z}[1/6N, \zeta_{_N}]((q_{_N}))$. As in [3], we set the topological grading to be twice the weight for elements of this ring.

Clearly we can regard a modular form for $\mathrm{SL}_2(\mathbb{Z})$ as one for $\Gamma_1(N)$ by forgetting $\boldsymbol{\alpha}$ in each modular point $(L, \boldsymbol{\alpha})$; hence

$$E\ell\ell[1/N]_* \subseteq E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$$

and is a subring.

Given $M_1, M_2 \in \mathrm{SL}_2(\mathbb{Z})$, let $M = M_2 M_1^{-1}$ and consider the isomorphism

$$(\ )^M \colon E\ell\ell_*^{M_1\Gamma_1(N)M_1^{-1}} \longrightarrow E\ell\ell_*^{M_2\Gamma_1(N)M_2^{-1}};$$

$$f \longmapsto f|_{[M^{-1}]_k} \quad \forall\, f \in E\ell\ell_{2k}^{M_1\Gamma_1(N)M_1^{-1}}$$

induced from conjugation by M. This isomorphism clearly fixes $E\ell\ell[1/N]_*$, and so is an isomorphism of $E\ell\ell[1/N]_*$ algebras. By Theorem (1.1), $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$ is a finitely generated algebraic extension of $E\ell\ell[1/N]_*$. We could now appeal to Galois theory for the existence of a minimal Galois extension of $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$ over $E\ell\ell[1/N]_*$ (i.e., *a splitting ring*) and embed all of the other algebras of the form $E\ell\ell_*^{M'\Gamma_1(N)M'^{-1}}$ in it. However, we can construct this splitting ring as yet another ring of modular forms, this time for the congruence subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

which is normal in $\mathrm{SL}_2(\mathbb{Z})$.

We follow [10] in defining a *modular point for* $\Gamma(N)$ to be a pair of form $(L, \{\omega_1/N \bmod L, \omega_2/N \bmod L\})$, where $L \subseteq \mathbb{C}$ is a lattice and the pair $\{\omega_1, \omega_2\}$ is an oriented basis for L (equivalently, $\{\omega_1/N \bmod L, \omega_2/N \bmod L\}$ is an oriented basis for the free $\mathbb{Z}/N$ module $(1/N)L/L$). Now the standard action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}/L$ determined by any such basis $\{\omega_1, \omega_2\}$ in turn induces an action on the set $\mathcal{M}(\Gamma(N))$ of all modular points $(L, \{\omega_1'/N \bmod L, \omega_2'/N \bmod L\})$. With respect to this action, we have

$$\mathrm{Stab}(L, \{\omega_1/N \bmod L, \omega_2/N \bmod L\}) = \Gamma(N).$$

By analogy with our earlier definition, we can define *a modular function for* $\Gamma(N)$ *on* $\mathcal{M}(\Gamma(N))$ *of weight* k*,* $F\colon \mathcal{M}(\Gamma(N)) \to \mathbb{C}$*,* to be a function satisfying

$$F(\lambda L, \{\lambda\omega_1/N \bmod L, \lambda\omega_2/N \bmod L\}) = \lambda^{-k} F(L, \{\omega_1/N \bmod L, \omega_2/N \bmod L\})$$

for any $\lambda \in \mathbb{C}^\times$. Similarly, we can define *a modular function for* $\Gamma(N)$ *on* $\mathfrak{H}$ *of weight* k*,* $f\colon \mathfrak{H} \to \mathbb{C}$*,* to be a function satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \forall \tau \in \mathfrak{H}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N).$$

The relationship between these is provided by setting

$$f(\tau) = F(\langle \tau, 1\rangle, \{\tau, 1\})$$

and

$$F(L, \{\omega_1/N \bmod L, \omega_2/N \bmod L\}) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right).$$

Similarly, we can modify our earlier definitions of *modular forms* by insisting that such a function f be holomorphic on $\mathfrak{H}$, holomorphic or meromorphic at each cusp and has q-expansions of the form

$$\widetilde{f}|_{[M]_k}(q) = \sum a_n^M q_N^n$$

for each $M \in \mathrm{SL}_2(\mathbb{Z})$. If we have all of the coefficients a_n^M in $K[\zeta_N]$ for some subring $K \subseteq \mathbb{C}$, then we say that f is *defined over* K. In particular we have the graded $\mathbb{Z}[1/6N]$ algebra of integer weight modular forms for $\Gamma(N)$ defined over $\mathbb{Z}[1/6N]$, which we will denote by $E\ell\ell_*^{\Gamma(N)}$. By the argument of Theorem (1.1), this is an algebraic extension of $E\ell\ell[1/N]_*$. The following result describes this extension more fully.

Theorem (1.2). *The ring $E\ell\ell_*^{\Gamma(N)}$ is a finitely generated Galois extension of $E\ell\ell[1/N]_*$ with Galois group $SL_2(\mathbb{Z})/\Gamma(N) \cong SL_2(\mathbb{Z}/N)$. The subgroup $\Gamma_1(N)/\Gamma(N) \subseteq SL_2(\mathbb{Z})/\Gamma(N)$ has fixed ring $E\ell\ell_*^{\Gamma_1(N)}$.*

Proof. We will require the theory of integral extensions of (graded) rings; all the necessary results are to be found in Lang's book [13], especially Chapter IX. The ring $E\ell\ell[1/N]_*$ is a Noetherian entire factorial ring, hence $E\ell\ell_*^{\Gamma(N)}$ is the integral closure of $E\ell\ell[1/N]_*$ in the field of fractions of $E\ell\ell_*^{\Gamma(N)}$. Now for any modular form F for $\Gamma(N)$ we have a polynomial

$$\prod_{\{\omega_1/N \bmod L, \omega_2/N \bmod L\}} (X - \lambda^{-k} F(L, \{\omega_1/N \bmod L, \omega_2/N \bmod L\}))$$

where the product is over all oriented bases of $(1/N)L/L$ and which lies in $E\ell\ell[1/N]_*[X]$ by the same reasoning as in the case of $\Gamma_1(N)$ above. This has degree equal to the number of elements of $\mathrm{SL}_2(\mathbb{Z}/N)$, say D'_N. Hence every element of $E\ell\ell_*^{\Gamma(N)}$ is of degree at most D'_N. The following result of [13,Chapter VIII Lemma 1] implies that the field of fractions of $E\ell\ell_*^{\Gamma(N)}$ is of degree at most D'_N over that of $E\ell\ell[1/N]_*$.

Lemma (1.3). *Let $\Bbbk$ be a field and $E/\Bbbk$ be a separable algebraic extension. Suppose that for some n, every element $\alpha \in E$ has $\deg_{\Bbbk} \alpha \leqslant n$. Then E is a finite extension and $[E : \Bbbk] \leqslant n$.*

Thus we see that this extension must be Galois of degree D'_N, since the elements of the quotient group $\mathrm{SL}_2(\mathbb{Z}/N) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(N)$ act as distinct automorphisms. Hence, the ring $E\ell\ell_*^{\Gamma(N)}$ is a finitely generated extension of $E\ell\ell[1/N]_*$ with automorphism group $\mathrm{SL}_2(\mathbb{Z}/N)$. □

Remark: Notice that we do not claim that $E\ell\ell_*^{\Gamma(N)}$ is a free module over $E\ell\ell[1/N]_*$, although since $E\ell\ell[1/N]_*$ is Noetherian, it is finitely generated. In [6,Theorem 8.4], Eichler and Zagier prove a related result for the extension $(E\ell\ell_*^{\Gamma(N)} \otimes \mathbb{Q})/(E\ell\ell_* \otimes \mathbb{Q})$; in this case the result is free of rank D'_N. A similar result holds for $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$. In [4], Brylinski has in effect shown that these are faithfully flat extensions of $E\ell\ell_*$, but he makes use of deep results from the theory of moduli schemes, whereas our approach only uses standard machinery from algebraic topology.

Now take $\tau \in \mathfrak{H}$ and consider the lattice $L_\tau = \langle \tau, 1 \rangle$. Let $\boldsymbol{\alpha} \in \mathbb{C}/L_\tau$ be a point of order N, hence the pair $(L_\tau, \boldsymbol{\alpha})$ is a modular point for $\Gamma_1(N)$. Consider the lattice $L_{\boldsymbol{\alpha}}^{\perp} = L_\tau + \langle \alpha/N \rangle \subseteq \mathbb{C}$ generated by L_τ together with any element of the coset $\boldsymbol{\alpha} = \alpha/N + L_\tau$, where $\alpha \in L_\tau$. We have that $L_\tau \subseteq L_{\boldsymbol{\alpha}}^{\perp}$ and $[L_{\boldsymbol{\alpha}}^{\perp} : L_\tau] = N$. Now for any holomorphic modular function $f \colon \mathfrak{H} \longrightarrow \mathbb{C}$ for $\mathrm{SL}_2(\mathbb{Z})$ (of weight k say, and with associated function F on lattices), we have a q-expansion

$$\widetilde{f}(q) = \sum_{n=-\infty}^{\infty} a_n q^n \quad \text{where } q = e^{2\pi\,\mathrm{i}\,\tau}.$$

If f is meromorphic at infinity, then this sum is for $n \gg -\infty$ only. Associated to $L_{\boldsymbol{\alpha}}^{\perp}$ we have a (non-unique) $\tau' = \tau_{\boldsymbol{\alpha}} \in \mathfrak{H}$ for which $L_{\boldsymbol{\alpha}}^{\perp} = \lambda_{\boldsymbol{\alpha}} L_{\tau'}$ and $\lambda_{\boldsymbol{\alpha}} \in \mathbb{C}^{\times}$. Then evaluating f at τ' we obtain

$$\widetilde{f}(q') = \sum_{n=-\infty}^{\infty} a_n {q'}^n \quad \text{where } q' = e^{2\pi\,\mathrm{i}\,\tau'}.$$

As

$$\frac{\alpha}{N} = \frac{r\tau + s}{N} \in \left\langle \frac{\tau}{N}, \frac{1}{N} \right\rangle$$

for suitable integers r, s, we have

$$q' = \zeta_{N}^{s} q_{N}^{r}.$$

If we suppose that the q-expansion has coefficients in a subring $K \subseteq \mathbb{C}$, then we also have

$$\widetilde{f}(q') \in K((q')) \subseteq K[\zeta_{N}]((q_{N})).$$

If now $M_{\boldsymbol{\alpha}} \in \mathrm{Stab}(\boldsymbol{\alpha})$, then it is easily verified that the function $f(\tau')$ is modular with respect to $M_{\boldsymbol{\alpha}}^{-1}\Gamma_1(N)M_{\boldsymbol{\alpha}}$. Notice that if we chose τ'' in place of τ', we would have a relationship of the form

$$\tau'' = \frac{a\tau' + b}{c\tau' + d} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

and hence setting $q'' = e^{2\pi\,\mathrm{i}\,\tau''}$, we have

$$\widetilde{f}(q'') = (c\tau' + d)^k \widetilde{f}(q').$$

All of this amounts to the fact that the associated function

$$(L, \boldsymbol{\alpha}) \longmapsto F(L_{\boldsymbol{\alpha}}^{\perp})$$

is a function on $\mathcal{M}(\Gamma_1(N))$ which is a weight k modular function for $\Gamma_1(N)$.

§2 The Weil pairing and elliptic genera of level N.

Now let $(L, \boldsymbol{\alpha})$ be a modular point for $\Gamma_1(N)$. Then following [16] we can define *the Weil pairing*

$$\mathrm{e}_N \colon (1/N)L/L \times (1/N)L/L \longrightarrow \mu_N$$

where $(1/N)L/L \subseteq \mathbb{C}/L$ is the subgroup of points of order dividing N, and $\mu_N = \langle \zeta_N \rangle \subseteq \mathbb{C}^{\times}$ is the subgroup of N th roots of unity.

Proposition (2.1). *The Weil pairing is a non-degenerate, bilinear, skew symmetric pairing. Moreover, if $\{\omega_1, \omega_2\}$ is an oriented basis for L, then we have*

$$\mathrm{e}_N\left(\frac{\omega_1}{N} \bmod L, \frac{\omega_2}{N} \bmod L\right) = \zeta_{_N}.$$

This is proved in [16] and we review the construction. The reader should also compare this with the construction of [8].

Assume that $N \geqslant 2$; then there is a unique elliptic function $\mathcal{G}\colon \mathbb{C}/L \longrightarrow \mathbb{C}$ with divisor $\operatorname{div}(\mathcal{G}) = N(0) - N(\boldsymbol{\alpha})$ and whose Taylor expansion about 0 begins with $(2\pi\,\mathrm{i}\,z/N)^N$. We can express the function $\mathcal{G}(z)$ in the form

$$\mathcal{G}(z) = \left(\frac{\sigma(z, L)}{N\sigma(z-\alpha, L)}\right)^N, \tag{2.2}$$

where $\sigma(z, L)$ denotes the *Weierstrass sigma function for the lattice* L, and α is a representative of $\boldsymbol{\alpha} \in \mathbb{C}/L$. In particular, for a lattice of the form $\langle\tau, 1\rangle$ where $\tau \in \mathfrak{H}$, we have the following well known product expansion:

$$2\pi\,\mathrm{i}\,\sigma(z, \langle\tau, 1\rangle) = e^{-G_2(\tau)z^2}(e^{\pi\,\mathrm{i}\,z} - e^{-\pi\,\mathrm{i}\,z}) \prod_{n\geqslant 1} \frac{\left(e^{2\pi\,\mathrm{i}\,z} - q^n\right)\left(e^{-2\pi\,\mathrm{i}\,z} - q^n\right)}{(1-q^n)^2},$$

where

$$G_2(\tau) = -\frac{(2\pi\,\mathrm{i})^2}{24} E_2(q)$$

and

$$E_2(q) = 1 - 24\sum_{r\geqslant 1} \sigma_1(r) q^r,$$

with the latter being a function of τ which is periodic but is not a modular form. A convenient reference for this is [18]. Now introducing the variable

$$w = (e^{2\pi\,\mathrm{i}\,z} - 1),$$

we find that

$$2\pi\,\mathrm{i}\,\sigma(z, \langle\tau, 1\rangle) \in \mathbb{Q}[[q, w]].$$

Using this together with the fact that $\alpha \in (1/N)L$, we see that for a lattice of the form $L = \langle\tau, 1\rangle$,

$$\mathcal{G}(z) \in \mathbb{Z}[1/6N, \zeta_N][[q_N, w]]. \tag{2.3}$$

Indeed, the latter has a product expansion of the form

$$\mathcal{G}(z) = \frac{1}{N^N}(e^{2\pi\,\mathrm{i}\,z}-1)^N\left(\frac{1-q_\alpha}{e^{2\pi\,\mathrm{i}\,z}-q_\alpha}\right)^N \times \prod_{n\geqslant 1}\left(\frac{(e^{2\pi\,\mathrm{i}\,z}-q^n)}{(e^{2\pi\,\mathrm{i}\,z}-q^nq_\alpha)}\frac{(e^{2\pi\,\mathrm{i}\,z}-q^{-n})}{(e^{2\pi\,\mathrm{i}\,z}-q^{-n}q_\alpha)}\frac{(1-q^n)}{(1-q^nq_\alpha)}\frac{(1-q^{-n})}{(1-q^{-n}q_\alpha)}\right)^N, \tag{2.4}$$

where

$$q_\alpha = q_N^{r'}\zeta_N^{s'}.$$

Now for any point $\boldsymbol{\beta}\in\mathbb{C}/L$ for which $N\boldsymbol{\beta}=\boldsymbol{\alpha}$, we have a second elliptic function (depending on $\boldsymbol{\beta}$) $\mathcal{F}\colon \mathbb{C}/L'\longrightarrow\mathbb{C}$ where L' is a certain lattice for which $L\subseteq L'\subseteq(1/N)L$, and satisfying

$$\operatorname{div}(\mathcal{F}) = \sum_{\boldsymbol{\gamma}\in(1/N)L/L'}(\boldsymbol{\gamma})-(\boldsymbol{\gamma}+\boldsymbol{\beta}'),$$

and

$$(\mathcal{F}(z))^N = \mathcal{G}(Nz),$$

where $\boldsymbol{\beta}'\in\mathbb{C}/L'$ is the image of $\boldsymbol{\beta}\in\mathbb{C}/L$ under the natural map of tori. We can uniquely specify $\mathcal{F}$ by requiring that its Taylor expansion about 0 begins with the term $2\pi\,\mathrm{i}\,z$, and thus making use of (2.2) and (2.3) we deduce that

$$\mathcal{F}(z) = \left(\frac{\sigma(Nz,L)}{N\sigma(Nz-\alpha,L)}\right) \in \mathbb{Z}[1/N,\zeta_N][[q_N,w]]. \tag{2.5}$$

We now define

$$\mathrm{e}_N(\boldsymbol{\gamma},\boldsymbol{\alpha}) = \frac{\mathcal{F}(z+\gamma)}{\mathcal{F}(z)} \quad \text{for } \boldsymbol{\gamma}=\gamma+L\in(1/N)L/L.$$

The lattice L' is characterised by the fact that the quotient $L'/L\subseteq\mathbb{C}/L$ is the orthogonal complement of the subgroup of $(1/N)L/L$ generated by $\boldsymbol{\alpha}$; it is actually equal to the lattice generated by L together with a representative β of $\boldsymbol{\beta}=\beta+L$. By the skew symmetry and non-degeneracy of e_N we have, in the notation of §1, $L'=L_{\boldsymbol{\alpha}}^{\perp}=L+\langle\alpha/N\rangle$ where $\boldsymbol{\alpha}=\alpha/N+L$.

The construction of the level N genus in [8] is closely related to this. In fact, Hirzebruch's function f is elliptic with respect to the lattice $2\pi\,\mathrm{i}\,NL_{\boldsymbol{\alpha}}^{\perp}$ and we have

$$\mathcal{F}(z) = f(Nz).$$

Thus the two constructions for $\mathcal{F}$ are essentially equivalent.

Now viewing $\mathcal{F}$ as elliptic with respect to $L_{\boldsymbol{\alpha}}^{\perp}$, we obtain an expansion of the form

$$\mathcal{F}(z) = 2\pi\,\mathrm{i}\,z + \sum_{k\geqslant 1} H_k(L_{\boldsymbol{\alpha}}^{\perp})(2\pi\,\mathrm{i}\,z)^k \tag{2.6}$$

where the H_k are modular forms of weight k which we evaluate at the lattice $L_{\boldsymbol{\alpha}}^{\perp}$. If we write $L_{\boldsymbol{\alpha}}^{\perp}$ in the form $\lambda_{\boldsymbol{\alpha}}\langle\tau',1\rangle$ with $\tau'\in\mathfrak{H}$, then we have

$$\begin{aligned}\mathcal{F}(z) &= 2\pi\,\mathrm{i}\,z + \sum_{k\geqslant 1} H_k(\lambda_{\boldsymbol{\alpha}}\langle\tau',1\rangle)(2\pi\,\mathrm{i}\,z)^k\\ &= 2\pi\,\mathrm{i}\,z + \sum_{k\geqslant 1} \lambda_{\boldsymbol{\alpha}}^{-k} H_k(\langle\tau',1\rangle)(2\pi\,\mathrm{i}\,z)^k\\ &= 2\pi\,\mathrm{i}\,z + \sum_{k\geqslant 1} \lambda_{\boldsymbol{\alpha}}^{-k} h_k(\tau')(2\pi\,\mathrm{i}\,z)^k\end{aligned}$$

where $h_k(\tau) = H_k(\langle\tau,1\rangle)$ is the modular form on $\mathfrak{H}$ associated to H_k. Notice that we have

$$q' = e^{2\pi\,\mathrm{i}\,\tau'} = \zeta_N^{r'} q^{s'/N}$$

for integers r', s' such that

$$\tau' = \frac{r'\tau + s'}{N}.$$

Thus by (2.3), the q-expansion of each $h_k(\tau')$ is an element of the ring $\mathbb{Z}[1/6N,\zeta_N][[q_N]]$. This implies that when expressed as a function of w as above, we find that

$$\mathcal{F}(z) = \widetilde{\mathcal{F}}(w) \in\in \mathbb{Z}[1/6N,\zeta_N][[q_N,w]]. \tag{2.7}$$

§3 Elliptic cohomology and level N genera.

Recall from §1 the graded ring of modular forms for $\Gamma_1(N)$, $E\ell\ell_*^{\Gamma_1(N)}$. Then the chain of subrings

$$E\ell\ell_* \subseteq E\ell\ell[1/N]_* \subseteq E\ell\ell_*^{\Gamma_1(N)}$$

allows us to define a functor on the homotopy category of finite CW complexes,

$$\begin{aligned}(E\ell\ell^{\Gamma_1(N)})^*(\ ) &= E\ell\ell_*^{\Gamma_1(N)} \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ )\\ &\cong E\ell\ell_*^{\Gamma_1(N)} \underset{MU_*}{\otimes} MU^*(\ )\end{aligned}$$

where we use the genus $MU_* \xrightarrow{\varphi_{E\ell\ell}} E\ell\ell_* \hookrightarrow E\ell\ell_*^{\Gamma_1(N)}$ induced from the canonical level 1 genus $\varphi_{E\ell\ell}\colon MU_* \to E\ell\ell_*$ of [3] and the inclusion of $E\ell\ell_*$ into $E\ell\ell_*^{\Gamma_1(N)}$; the functor

$E\ell\ell^*(\ )$ is the level 1 version of elliptic cohomology also described in [3]. Similarly, we have the functor

$$\begin{aligned}(E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ ) &= E\ell\ell_*^{M\Gamma_1(N)M^{-1}} \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\cong E\ell\ell_*^{M\Gamma_1(N)M^{-1}} \underset{MU_*}{\otimes} MU^*(\ )\end{aligned}$$

whenever $M \in \mathrm{SL}_2(\mathbb{Z})$.

We may now appeal to Landweber's Exact Functor Theorem to show that that these functors are cohomology theories, thus generalising the constructions used in [3,11,12]. The argument relies upon the fact that for each prime $p > 3$ for which $p \nmid N$, the sequence p, v_1, v_2 in $E\ell\ell_*$ remains regular in the extension ring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$. From [12] we know that in the quotient ring $E\ell\ell_*/(p, v_1)$, the class of v_2 is a unit and so it suffices to verify regularity for the sequence p, v_1.

To do this, notice that the quotient $E\ell\ell_*/(p)$ is an integral domain (in fact it is a principal ideal domain in a graded sense) and thus if the residue class of v_1 were annihilated by the class of u say, the constant term w say, in the minimal polynomial of u would also annihilate v_1 modulo p. But this could only happen if w and hence u were 0 modulo p. It is perhaps worth remarking that $v_1 \equiv E_{p-1} \pmod p$, and so this is not usually a prime element in $E\ell\ell_*/(p)$ (see [2] for more on this).

We have thus obtained our promised cohomology theories.

Proposition (3.1). *For each $M \in SL_2(\mathbb{Z})$, the functor $(E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ )$ is a multiplicative, complex oriented cohomology theory on finite CW complexes.*

Now the conjugation map from §1,

$$(\ )^M \colon M_1\Gamma_1(N)M_1^{-1} \longrightarrow M_2\Gamma_1(N)M_2^{-1};\ A^M = MAM^{-1}$$

where $M_2 = MM_1$, fixes $E\ell\ell[1/N]_*$ and therefore induces a multiplicative natural isomorphism of functors

$$(E\ell\ell^{M_1\Gamma_1(N)M_1^{-1}})^*(\ ) \underset{\cong}{\xrightarrow{(\ )^M}} (E\ell\ell^{M_2\Gamma_1(N)M_2^{-1}})^*(\ ). \tag{3.2}$$

In a similar way, the natural embedding $E\ell\ell_* \longrightarrow E\ell\ell_*^{\Gamma(N)}$ allows us to construct the functor

$$\begin{aligned}(E\ell\ell^{\Gamma(N)})^*(\ ) &= E\ell\ell_*^{\Gamma(N)} \underset{E\ell\ell_*}{\otimes} E\ell\ell^*(\ ) \\ &\cong E\ell\ell_*^{\Gamma(N)} \underset{MU_*}{\otimes} MU^*(\ )\end{aligned}$$

on finite CW complexes, where the second identification makes use of the homomorphism $MU_* \longrightarrow E\ell\ell_*$ which classifies the 'standard' elliptic cohomology orientation pushed into $E\ell\ell_*^{\Gamma(N)}$. Using the same approach as for Proposition (3.1), we obtain the next result with the aid of Landweber's Exact Functor Theorem.

Proposition (3.3). *The functor* $(E\ell\ell^{\Gamma(N)})^*(\ )$ *is a multiplicative complex oriented cohomology theory on the category of finite CW complexes.*

These theories are related by multiplicative natural transformations of the form

$$E\ell\ell^*(\ ) \to (E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ ) \to (E\ell\ell^{\Gamma(N)})^*(\ ),$$

which are induced from the natural embeddings

$$E\ell\ell_* \to E\ell\ell_*^{M\Gamma_1(N)M^{-1}} \to E\ell\ell_*^{\Gamma(N)}.$$

The natural complex orientation $x^{E\ell\ell} \in E\ell\ell^2(\mathbb{C}\mathrm{P}^\infty)$ (see [1]) for the cohomology theory $(E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ )$ is that induced from that of $E\ell\ell^*(\ )$ by the defining multiplicative natural transformation

$$E\ell\ell^*(\ ) \to (E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ ).$$

We wish to show that there is a second choice of orientation $y^{E\ell\ell} \in E\ell\ell^2(\mathbb{C}\mathrm{P}^\infty)$ which is expressible as a power series $\theta_M(x^{E\ell\ell})$ in $x^{E\ell\ell}$, whose coefficients lie in the ring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$, which begins with the term $x^{E\ell\ell}$ and is also related to Hirzebruch's level N genus. This is equivalent to the existence of a strict isomorphism between the formal group laws associated to the corresponding genera. To establish this we need some algebraic results.

From Equation (2.6), we have the power series in Z

$$\Phi(Z) = \mathcal{F}(Z/2\pi\,\mathrm{i}) = Z + \sum_{k\geqslant 1} H_k(L_{\boldsymbol{\alpha}}^{\perp})(Z)^k$$

whose coefficients lie in the ring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}} \otimes \mathbb{Q} \subseteq E\ell\ell_*^{\Gamma(N)} \otimes \mathbb{Q}$; we can take this series to be the exponential for a formal group law $F_N(X,Y)$ defined over this ring (note that this depends upon the original choice of $\boldsymbol{\alpha}$). The next result shows that this formal group law is actually defined over the subring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$ which we can view as a (non-graded) subring of $\mathbb{Z}[1/6N, \zeta_N]((q_N))$.

Proposition (3.4). *The power series* $\widetilde{\mathcal{F}}(w)$ *is a strict isomorphism from the multiplicative formal group law to* $F_N(X,Y)$*, all of whose coefficients lie in the ring* $\mathbb{Z}[1/6N, \zeta_N]((q_N))$*, and thus* $F_N(X,Y)$ *is defined over the ring* $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$.

Finally, we see that over the ring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$, the formal group law $F_N(X,Y)$ is strictly isomorphic to the canonical level 1 formal group law $F^{E\ell\ell}(X,Y)$ discussed in [3]; this uses the theory of Tate curves as described in the latter reference, which provides us with the following fact. Recall that over any ring the multiplicative formal group law is defined by

$$\widehat{G}_m(X,Y) = X + Y + XY.$$

Proposition (3.5). *Over the ring $\mathbb{Z}[1/6]((q)) \subseteq \mathbb{Z}[1/6N,\zeta_N]((q_N))$, the formal group law $F^{E\ell\ell}$ is strictly isomorphic to the multiplicative formal group law $\widehat{G}_m(X,Y)$.*

Using the isomorphisms of Propositions (3.4) and (3.5), we obtain an isomorphism between the formal group laws $F_N(X,Y)$ and $F^{E\ell\ell}(X,Y)$, which is visibly defined over the ring $\mathbb{Z}[1/6N,\zeta_N]((q_N))$. However, both of these formal group laws are defined over the subring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$, and hence the isomorphism between them has coefficients which are rational polynomials in their coefficients; this means that our isomorphism has coefficients in the intersection

$$\mathbb{Z}[1/6N,\zeta_N]((q)) \cap E\ell\ell_*^{M\Gamma_1(N)M^{-1}} \otimes \mathbb{Q},$$

which is equal to the ring $E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$. Thus we have established the following crucial result relating these two formal group laws.

Theorem (3.6). *The formal group laws $F^{E\ell\ell}(X,Y)$ and $F_N(X,Y)$ defined over the ring $(E\ell\ell^{M\Gamma_1(N)M^{-1}})_*$ are strictly isomorphic.*

If we now let θ_M denote the (unique) strict isomorphism $F^{E\ell\ell} \xrightarrow{\cong} F_N$, we have our new choice of orientation in the cohomology theory $(E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ )$. We remark that the real reason for inverting N in defining the latter cohomology theory is to ensure that such an isomorphism exists, and that the corresponding genera define the same multiplicative cohomology theory. It would be interesting to investigate the situation without inverting N (this is closely related to Brylinski's work [4]).

We can use the level N genus $\rho_{\boldsymbol{\alpha}}\colon MU_* \longrightarrow E\ell\ell_*^{M\Gamma_1(N)M^{-1}}$ to give an alternative definition of the cohomology theory $(E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ )$. Namely we form the tensor product functor

$$\left(E\ell\ell_*^{M\Gamma_1(N)M^{-1}}\right)_{\rho_{\boldsymbol{\alpha}}} \underset{MU_*}{\otimes} MU^*(\ ),$$

where $(\ )_{\rho_{\boldsymbol{\alpha}}}$ indicates that we use the MU_* module structure on the left hand factor arising from $\rho_{\boldsymbol{\alpha}}$. The strict isomorphism θ_M now allows us to define a multiplicative natural isomorphism

$$\left(E\ell\ell_*^{M\Gamma_1(N)M^{-1}}\right)_{\rho_{\boldsymbol{\alpha}}} \underset{MU_*}{\otimes} MU^*(\ ) \cong (E\ell\ell^{M\Gamma_1(N)M^{-1}})^*(\ ). \tag{3.7}$$

We could prove that this new functor is a cohomology theory by directly appealling to Landweber's Exact Functor Theorem, however, this would also make use of the existence of the strict isomorphism θ_M, together with the fact that Landweber's criteria are invariant under change of formal group law by a strict isomorphism.

§4 The case of level 2.

The case where $N = 2$ corresponds to the original version of elliptic cohomology (see [11] and [12]). We remark that $\Gamma_1(2) = \Gamma_0(2)$. It is known that the extension $E\ell\ell_*^{\Gamma_1(2)}$ is algebraic over $E\ell\ell_*$ and generated as a ring by a root of the Weierstrass cubic polynomial

$$4X^3 - \frac{1}{12}E_4X + \frac{1}{216}E_6.$$

The remaining quadratic term in this polynomial splits over $E\ell\ell_*^{\Gamma(2)}$ and its roots generate this extension of $E\ell\ell_*^{\Gamma_1(2)}$. In this case, the extension $E\ell\ell_*^{\Gamma(2)}$ is a free module over $E\ell\ell_*$ of rank 6. Thus the cohomology theory $(E\ell\ell^{\Gamma(2)})^*(\ )$ is naturally isomorphic to a direct sum of six copies of $E\ell\ell^*(\ )$ as module theories over the latter.

§5 Reduction modulo a prime ideal.

In this section we pursue a line similar to that in [2] and consider the theories obtained from our level N theories by reduction modulo prime ideals of the coefficient rings. We will assume the reader to be familiar with the methods of [2]. We will only discuss the case of $\Gamma(N)$ since the others are similar.

First let p be a prime which does not divide $6N$. Then the ideal $(p) \triangleleft E\ell\ell[1/N]_*$ is prime (in a graded sense) and the reduction modulo (p) gives

$$E\ell\ell[1/N]_*/(p) = \mathbb{F}_p[E_4, E_6, \Delta^{-1}]$$

which is a (graded) principal ideal domain. Using a mod p version of Landweber's Exact Functor Theorem [17], this can be shown to be the coefficient ring of a multiplicative cohomology theory of the form

$$(E\ell\ell/p)^*(\ ) = E\ell\ell[1/N]_*/(p) \underset{MU_*/(p)}{\otimes} MU/p^*(\ ),$$

where MU/p is the mod p reduction of the spectrum MU, known to be a module spectrum over MU.

We can similarly consider the ideal $(p) \triangleleft E\ell\ell_*^{\Gamma(N)}$; however, this need not be prime and so we can take a prime π dividing p. Then the reduction $E\ell\ell_*^{\Gamma(N)}/(\pi)$ is also the coefficient ring of a multiplicative cohomology theory $(E\ell\ell^{\Gamma(N)}/\pi)^*(\ )$. However, because of the possibility of ramification, the degree D''_N of the extension $(E\ell\ell_*^{\Gamma(N)}/(\pi))/(E\ell\ell[1/N]_*/(p))$ need only be a divisor of the degree D'_N of the extension $E\ell\ell_*^{\Gamma(N)}/E\ell\ell[1/N]_*$. Notice that as our underlying ring $E\ell\ell[1/N]_*/(p)$ is a principal ideal domain, we can establish that $E\ell\ell_*^{\Gamma(N)}/(\pi)$ is free of rank D''_N over $E\ell\ell[1/N]_*/(p)$.

By Brown's Representability Theorem we can represent both of the above theories by ring spectra $E\ell\ell/p$ and $E\ell\ell^{\Gamma(N)}/\pi$. We now have the following topological result.

Theorem (5.1). *There is a natural isomorphism of multiplicative cohomology theories on finite CW complexes*

$$(E\ell\ell^{\Gamma(N)}/\pi)^*(\ ) \cong E\ell\ell_*^{\Gamma(N)}/(\pi) \underset{E\ell\ell[1/N]_*/(p)}{\otimes} (E\ell\ell/p)^*(\ ).$$

Moreover, there is a splitting of $E\ell\ell/p$ module spectra

$$E\ell\ell^{\Gamma(N)}/\pi \simeq \bigvee_a \Sigma^{d(a)} E\ell\ell/p$$

which can in fact be given the structure of an equivalence of algebra spectra over $E\ell\ell/p$, where a indexes the wedge summands and d is a suitable numerical function.

We can also consider a prime ideal $\mathfrak{p} \lhd E\ell\ell[1/N]_*$ which contains the ideal (p, E_{p-1}) together with another $\mathfrak{P} \lhd E\ell\ell_*^{\Gamma(N)}$ containing $\mathfrak{p}$. From [2] we see that there is a cohomology theory $(E\ell\ell/\mathfrak{p})^*(\ )$ with the quotient graded field $E\ell\ell[1/N]_*/\mathfrak{p}$ as coefficient ring; also, by (flat) extension of the coefficient ring of the latter theory, we can obtain an extended theory $(E\ell\ell^{\Gamma(N)}/\mathfrak{P})^*(\ )$, which has $E\ell\ell_*^{\Gamma(N)}/\mathfrak{P}$ as its coefficient ring. We obtain the following analogue of Theorem (5.1).

Theorem (5.2). *There is a natural isomorphism of multiplicative cohomology theories on finite CW complexes*

$$(E\ell\ell^{\Gamma(N)}/\mathfrak{P})^*(\ ) \cong E\ell\ell_*^{\Gamma(N)}/\mathfrak{P} \underset{E\ell\ell[1/N]_*/\mathfrak{p}}{\otimes} (E\ell\ell/\mathfrak{p})^*(\ ).$$

Moreover, there is a splitting of $E\ell\ell/\mathfrak{p}$ module spectra

$$E\ell\ell^{\Gamma(N)}/\mathfrak{P} \simeq \bigvee_{a'} \Sigma^{d'(a')} E\ell\ell/\mathfrak{p}$$

which can in fact be given the structure of an equivalence of algebra spectra over $E\ell\ell/\mathfrak{p}$, where a' indexes the wedge summands and d' is a suitable numerical function.

Concluding remarks.

As remarked before, our construction of level N theories is only partially satisfactory. Although our methods are perhaps more elementary than those of Brylinski (at least from a topologist's perspective), our results are weaker in as much as we have inverted N from the start. It would certainly be of interest to clarify the basic properties of these higher level elliptic cohomology theories, especially considered locally at primes dividing N. Of course, our whole approach (and indeed that of Brylinski) lacks a coherent geometric underpinning. More recent work of the author on stable (co)operations in elliptic (co)homology has highlighted the rôle of higher level phenomena even in the level 1 context, and we suspect that a complete geometrically based understanding of these matters will involve all levels simultaneously, as has been suggested by G. Segal.

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