RIGHT EIGENVALUES FOR QUATERNIONIC MATRICES: A TOPOLOGICAL APPROACH

ANDREW BAKER

Abstract. We apply the Lefschetz Fixed Point Theorem to show that every square matrix over the quaternions has *right* eigenvalues. We classify them and discuss some of their properties such as an analogue of Jordan canonical form and diagonalization of elements of the compact symplectic group $\mathrm{Sp}(n)$.

Introduction

Hamilton's ring of quaternions $\mathbb{H}$ has long provided a source of (often nontrivial) generalizations of and counterexamples to, results over the real or complex numbers. In particular, linear algebra over $\mathbb{H}$ has been studied by algebraists, topologists and those interested in applications. The recent survey of Zhang [3] provides an overview of some of this work and the present article is in part a commentary on that paper. Our main result is a new topological proof of the existence of *right* eigenvalues using the Lefschetz Fixed Point Theorem which is a counterpart to Wood's topological proof of the existence of *left* eigenvalues in [2, 3]. A proof of our result by more algebraic techniques is given in [3].

1. Matrices over the quaternions

The ring of quaternions $\mathbb{H}$ is the unique finite central division algebra (skew-field) over the field of real numbers $\mathbb{R}$. It contains the field of complex numbers $\mathbb{C}$ as a maximal commutative subfield and is of dimension 4 over $\mathbb{R}$ and 2 over $\mathbb{C}$ viewed as a left or right $\mathbb{C}$-vector space.

In this paper, a right $\mathbb{H}$-module will be called an $\mathbb{H}$-vector space; we may sometimes also refer to a left $\mathbb{H}$-module as a left $\mathbb{H}$-vector space. The theory of bases, dimension and linear transformations applies in a straightforward way to such vector spaces. However, because of the non-commutativity of $\mathbb{H}$, care must be taken when working with coordinates and matrices relative to bases.

For example, given bases $\mathbf{u} = \{u_1, \ldots, u_m\}$ and $\mathbf{v} = \{v_1, \ldots, v_n\}$ and a (right) $\mathbb{H}$-linear transformation $\varphi\colon U \longrightarrow V$, we can write

$$\varphi(u_j) = \sum_{r=1}^{n} v_r a_{rj}$$

and introduce the matrix $A = [a_{ij}]$. Then

$$\varphi(\sum_{j=1}^{m} u_j x_j) = \sum_{j=1}^{m} \sum_{r=1}^{n} v_r a_{rj} x_j,$$

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and writing

$$\varphi(\sum_{j=1}^{m} u_j x_j) = \sum_{i=1}^{n} v_i y_i$$

we have

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

This allows us to identify U and V with the $\mathbb{H}$-vector spaces $\mathbb{H}^m$ and $\mathbb{H}^n$ and the action of φ with the action of A *on the left.*

Now let $U = V$. Notice that φ is a monomorphism if and only if it is an epimorphism; hence if one (or equivalently both) of these conditions are satisfied, then the matrix A has left and right inverses B and C, so

$$BA = I_n, \quad AC = I_n.$$

But then

$$C = (BA)C = B(AC) = B,$$

hence we have a unique two-sided inverse $A^{-1} = B = C$. This gives an apparently more straightforward approach to the discussion of Question 1 on page 22 of [3].

From this point of view, the *right eigenvalue problem* amounts to the existence of a non-zero column vector $\mathbf{x}$ and $\lambda \in \mathbb{H}$ for which $A\mathbf{x} = \mathbf{x}\lambda$. Equivalently, in terms of the linear transformation φ, we are interested in the existence of a non-zero $v \in V$ for which $\varphi(v) = v\lambda$. The *left eigenvalue problem* amounts to the existence of a column vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$, but there seems to be no obvious interpretation of this in terms of the endomorphism φ. The left eigenvalue problem was solved by Wood [2]. In Theorem 2.1, we will give a seemingly new topological proof of the right eigenvalue problem which has previously been tackled by algebraic means involving embeddings into matrix rings over the complex numbers. We will also derive generalizations of canonical forms for complex matrices.

2. Existence and properties of right eigenvalues

In order to investigate the right eigenvalue problem, we observe that it suffices to consider the group of invertible $\mathbb{H}$-linear transformations $V \longrightarrow V$, or equivalently of the group $\mathrm{GL}_n(\mathbb{H})$ of invertible $n \times n$, matrices over $\mathbb{H}$. We view $\mathrm{GL}_n(\mathbb{H})$ as acting on the left of the right $\mathbb{H}$-vector space $\mathbb{H}^n$, whose entries are considered as column vectors. This action induces a left action on the quaternionic projective space

$$\mathbb{H}\mathrm{P}^{n-1} = \mathbb{H}^n_0/\mathbb{H}^\times$$

where

$$\mathbb{H}^n_0 = \{v \in \mathbb{H}^n : v \neq 0\}.$$

Let $A \in \mathrm{GL}_n(\mathbb{H})$. Then $v \in \mathbb{H}^n_0$ is an *eigenvector* of A with *eigenvalue* $\lambda \in \mathbb{H}$ if $Av = v\lambda$; note that $\lambda \neq 0$. Notice also that if $0 \neq \alpha \in \mathbb{H}$, then

$$Av = v\lambda \quad \Longrightarrow \quad Av\alpha = v\alpha(\alpha^{-1}\lambda\alpha), \tag{2.1}$$

so we can sensibly talk about the *eigenline* spanned by an eigenvector v, even though there may be many associated eigenvalues! The existence of an eigenvector v is clearly equivalent to the the existence of a fixed point of A acting on $\mathbb{H}\mathrm{P}^{n-1}$.

Theorem 2.1. *The action of A on $\mathbb{H}\mathrm{P}^{n-1}$ has a fixed point.*

Proof. We will use the Lefschetz Fixed Point Theorem (see Gray [1], theorem 26.42) which states that if A has no fixed points then

$$\operatorname{Tr}_* A^* = \sum_{0\leqslant k\leqslant n-1} (-1)^k \operatorname{Tr}_k A^* = 0.$$

Here $\operatorname{Tr}_k A^*$ denotes the trace of the induced map $A^*\colon H^k(\mathbb{H}\mathrm{P}^{n-1};\mathbb{Q}) \longrightarrow H^k(\mathbb{H}\mathrm{P}^{n-1};\mathbb{Q})$ in rational cohomology. Since

$$H^*(\mathbb{H}\mathrm{P}^m;\mathbb{Q}) = \mathbb{Q}[y]/(y^{m+1})$$

where $y \in H^4(\mathbb{H}\mathrm{P}^m;\mathbb{Q})$, we have for $m = n-1$, and $A^*y = ty$,

$$\operatorname{Tr}_* A^* = 1 + t + \cdots + t^{n-1}.$$

We have the following result.

Proposition 2.2. *If $A \in \mathrm{GL}_n(\mathbb{H})$, then the induced continuous map $A\colon \mathbb{H}\mathrm{P}^{n-1} \longrightarrow \mathbb{H}\mathrm{P}^{n-1}$ satisfies $A^*y = y$ in cohomology. Hence, $\operatorname{Tr}_* A^* > 0$.*

Proof. Since $\mathrm{GL}_n(\mathbb{H})$ is path connected, A is homotopic to the identity map on $\mathbb{H}\mathrm{P}^{n-1}$ and so

$$\operatorname{Tr} A^* = \operatorname{Tr}\mathrm{Id} = n.$$

□

Thus the action of A on $\mathbb{H}\mathrm{P}^{n-1}$ must have a fixed point and hence at least one eigenvalue, which completes the proof of Theorem 2.1. □

We can classify the eigenvalues of A.

Proposition 2.3. *If $r + sw$ is an eigenvalue of A, where $r, s \in \mathbb{R}$, $s \geqslant 0$ and $w \in \mathbb{H}$ is a unit pure quaternion, then all quaternions of the form $r + sw'$ with $w' \in \mathbb{H}$ a unit pure quaternion are also eigenvalues.*

Proof. From Equation (2.1), if $Av = v(r + sw)$ then for each $\alpha \in \mathbb{H}^\times$,

$$Av\alpha = v\alpha(\alpha^{-1}(r + sw)\alpha).$$

But it is easily checked that

$$\{\alpha^{-1}w\alpha : \alpha \in \mathbb{H}^\times\} = \{w' : w' \text{ is a unit pure quaternion}\},$$

these sets both being the unit sphere in the set of pure quaternions. □

We will write $a \sim b$ whenever $a, b \in \mathbb{H}^\times$ are conjugate.

Proposition 2.4. *Suppose that $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues for A, no two of which are conjugate, and let $v_1, \ldots, v_r$ be corresponding eigenvectors. Then $v_1, \ldots, v_r$ are linearly independent.*

Proof. Suppose not. By reordering and discarding some of the λ_j and v_j if necessary, we may assume that

$$v_r = \sum_{j<r} v_j t_j$$

for some $t_j \in \mathbb{H}$ with $v_1, \ldots, v_{r-1}$ linearly independent. Then

$$\begin{aligned} Av_r &= \sum_{j<r} Av_j t_j \\ &= \sum_{j<r} v_j \lambda_j t_j, \end{aligned}$$

implying

$$Av_r - v_r\lambda_r = \sum_{j<r} v_j(\lambda_j t_j - t_j\lambda_r),$$

and so

$$\sum_{j<r} v_j(\lambda_j t_j - t_j\lambda_r) = 0.$$

For non-zero t_j, this gives

$$\lambda_j = t_j^{-1}\lambda_r t_j,$$

implying that $\lambda_j \sim \lambda_r$. Hence, all of the t_j are zero, contradicting the original assumption. □

Corollary 2.5. *If A has n non-conjugate eigenvalues, then it can be diagonalized in the sense that there is a $P \in \mathrm{GL}_n(\mathbb{H})$ for which PAP^{-1} is diagonal.*

Now we consider the compact symplectic group $\mathrm{Sp}(n) \subseteq \mathrm{GL}_n(\mathbb{H})$, consisting of all matrices perserving the standard quaternionic inner product $u \cdot v = u^* v$, where u^* denotes the transpose of the vector consisting of quaternionic conjugates of the entries of u. For $u, v \in \mathbb{H}^n$ and $\alpha, \beta \in \mathbb{H}$, this inner product satisfies

$$(u\alpha) \cdot (v\beta) = \alpha^*(u \cdot v)\beta.$$

It is easy to verify that if λ is an eigenvalue of $A \in \mathrm{Sp}(n)$, then $\lambda^{-1} = \lambda^*$ since $|\lambda|^2 = \lambda^*\lambda = 1$.

Proposition 2.6. *Let $A \in \mathrm{Sp}(n)$ and suppose that λ, λ' are distinct eigenvalues for A which are not conjugate, with corresponding eigenvectors v, v'. Then $v \cdot v' = 0$. Hence, A there exists $S \in \mathrm{Sp}(n)$ such that SAS^{-1} is diagonal.*

Proof. We have

$$v \cdot v' = (Av) \cdot (Av') = \lambda^*(v \cdot v')\lambda'.$$

If $v \cdot v' \neq 0$, then

$$\lambda' = (v \cdot v')^{-1}\lambda(v \cdot v'),$$

contradicting the nonconjugacy assumption. The final statement is proved by induction on n, exactly as is done for the diagonalization of unitary matrices. □

If $\lambda \in \mathbb{H}$, then

$$J^1_\lambda = \{v : (A - I\lambda)v = 0\}$$

is a vector subspace over the *centralizer* of λ,

$$\mathrm{Z}(\lambda) = \{\gamma \in \mathbb{H} : \gamma\lambda = \lambda\gamma\}.$$

In fact,

$$\mathrm{Z}(\lambda) = \begin{cases} \mathbb{H} & \text{if } \lambda \in \mathbb{R}, \\ \mathbb{R}[\lambda] \cong \mathbb{C} & \text{otherwise.} \end{cases}$$

Define the $\mathbb{R}$-vector subspace

$$J^1[\lambda] = \sum_{\alpha \in \mathbb{H}^\times} J^1_{\alpha\lambda\alpha^{-1}}.$$

This is actually an $\mathbb{H}$-vector subspace.

Theorem 2.7. *Let λ be an eigenvalue for A. Then a $\mathrm{Z}(\lambda)$-basis for J^1_λ is an $\mathbb{H}$-basis for $J^1[\lambda]$.*

Proof. If $v \in J^1_{\alpha\lambda\alpha^{-1}}$, then $v\alpha \in J_\lambda$. Hence, it suffices to show that a $\mathrm{Z}(\lambda)$-basis $\{v_1, \ldots, v_r\}$ for J^1_λ is $\mathbb{H}$-linearly independent.

Suppose not. By discarding and renaming elements we may suppose that $\{v_1, \ldots, v_{r-1}\}$ is $\mathbb{H}$-linearly independent and

$$v_r = \sum_{j<r} v_j \alpha_j$$

for some $\alpha_j \in \mathbb{H}$. Then

$$0 = \sum_{j<r} (Av_j\alpha_j - v_j\alpha_j\lambda) = \sum_{j<r} (v_j\lambda\alpha_j - v_j\alpha_j\lambda).$$

By assumption, for each j this gives $\lambda\alpha_j = \alpha_j\lambda$ and so $\alpha_j \in \mathrm{Z}(\lambda)$ which implies $\alpha_j = 0$. $\square$

We can also define the $\mathrm{Z}(\lambda)$-vector subspace

$$J^k_\lambda = \{v : (A - I\lambda)^k v = 0\}.$$

and the $\mathbb{R}$-vector subspace

$$J^k[\lambda] = \sum_{\alpha \in \mathbb{H}^\times} J^k_{\alpha\lambda\alpha^{-1}}.$$

Theorem 2.8. *Let λ be an eigenvalue for A. Then a $\mathrm{Z}(\lambda)$-basis for J^k_λ is an $\mathbb{H}$-basis for $J^k[\lambda]$.*

For large enough k, this defines an analogue of *Jordan canonical form* for A.

References

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Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland.

E-mail address: a.baker@maths.gla.ac.uk