DIFFERENTIAL EQUATIONS IN DIVIDED POWER ALGEBRAS, RECURRENCE RELATIONS AND FORMAL GROUPS

ANDREW BAKER

Introduction

These notes are intended to provide an account of the study of certain classes of differential equations in divided power algebras over commutative rings with unity. As a particular example, we have in mind translation invariant operators with respect to a formal group law over such a ring. A biproduct of our approach is that we simultaneously solve differential equations and equivalent recurrence relations. Even in the simplest situation (corresponding to the additive formal group law and linear recurrences) our approach gives the well known classical solutions very cleanly in a way which does not require special arguments for rings with torsion.

Most of our results are surely known. However, the present approach making systematic use of divided powers to avoid problems with division in the case where the underlying ring has torsion may be novel and worth further consideration, at least for pedagogical purposes.

These notes owe much in the way of motivation and viewpoint to discussions with Francis Clarke, Bob Odoni, Nigel Ray and Richard Steiner, all of whom I would like to thank.

1. Divided power algebras

Let R be a commutative ring with unity. Let $\Gamma_R(X)$ denote the *algebra of divided powers on X over R*. This is the free R-module on generators $\gamma_n(X)$ (frequently denoted γ_n when no ambiguity results) for $n \geqslant 0$, with the product given by

$$\gamma_m\gamma_n = \binom{m+n}{m}\gamma_{m+n}.$$

Thus $\gamma_0 = 1$ is the unit for $\Gamma_R(X)$. In fact, $\Gamma_R(X)$ is a Hopf algebra with structure maps

$$\text{(1.1)} \qquad \text{(coproduct)} \quad \psi\colon \Gamma_R(X) \longrightarrow \Gamma_R(X) \underset{R}{\otimes} \Gamma_R(X); \quad \gamma_n \longmapsto \sum_{0\leqslant k\leqslant n} \gamma_k \otimes \gamma_{n-k},$$

$$\text{(1.2)} \qquad \text{(antipode)} \quad \chi\colon \Gamma_R(X) \longrightarrow \Gamma_R(X); \quad \gamma_n \longmapsto (-1)^n\gamma_n,$$

$$\text{(1.3)} \qquad \text{(augmentation)} \quad \varepsilon\colon \Gamma_R(X) \longrightarrow \Gamma_R(X); \quad \gamma_n \longmapsto \begin{cases} 1 & \text{if } n=0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that there is always a homomorphism of R-algebras $R[X] \longrightarrow \Gamma_R(X)$ given by

$$\sum_k r_k X^k \longmapsto \sum_k k! r_k \gamma_k(X)$$

which is even a homomorphism of Hopf algebras if the domain is given the coproduct

$$X \longmapsto X\otimes 1 + 1\otimes X$$

and antipode

$$X \longmapsto -X.$$

Glasgow University Mathematics Department preprint no. 95/19 14/12/2007 version 6.

If R has no $\mathbb{Z}$-torsion this is an embedding; indeed if R is a $\mathbb{Q}$-algebra then we can write $\gamma_n(X) = X^n/n!$, and we then have for the coproduct,

$$\psi(\gamma_n(X)) = \gamma_n(X \otimes 1 + 1 \otimes X). \tag{1.4}$$

We will actually require the formal completion of $\Gamma_R(X)$. To define this, we filter $\Gamma_R(X)$ by setting $\Gamma_R^k(X)$ to be the R submodule generated by the γ_n with $n \geqslant k$. So we have

$$0 = \Gamma_R^\infty(X) \subseteq \cdots \subseteq \Gamma_R^{n+1}(X) \subseteq \Gamma_R^n(X) \subseteq \cdots \subseteq \Gamma_R^0(X) = \Gamma_R(X).$$

Then the product respects this filtration in the sense that

$$\Gamma_R^m(X) \times \Gamma_R^n(X) \longrightarrow \Gamma_R^{m+n}(X).$$

We can *complete* $\Gamma_R(X)$ with respect to this filtration by forming the inverse limit

$$\widehat{\Gamma}_R(X) = \lim_{\substack{\longleftarrow \\ k}} \Gamma_R(X)/\Gamma_R^k(X).$$

Elements of $\widehat{\Gamma}_R(X)$ can be expressed as infinite sums of the form

$$\sum_{0 \leqslant k} a_k \gamma_k(X), \quad \text{for } a_k \in R.$$

Thus $\widehat{\Gamma}_R(X)$ is free as a topological R-module on the topological basis $\{\gamma_k(X)\}_{0\leqslant k}$. Notice that $\widehat{\Gamma}_R(X)$ is a complete topological R-algebra. In fact it is also a topological Hopf algebra. Moreover, $\Gamma_R(X)$ is a dense Hopf subalgebra. Also, any element of the form $f(X) = \sum_{0\leqslant k} a_k \gamma_k$ where a_0 is a unit in R has a multiplicative inverse. To see this, notice that if we consider a sequence $(b_k)_{0\leqslant k}$ of unknowns, then we can inductively solve the infinite collection of equations determined by requiring that

$$\left(\sum_{0 \leqslant k} b_k \gamma_k(X) \right) f(X) = 1.$$

The coefficient of $\gamma_k(X)$ is then seen to be

$$\sum_{0 \leqslant j \leqslant k} \binom{k}{j} a_j c_{k-j},$$

and thus, since $\binom{k}{0} a_0$ is a unit in R, we can express c_k in terms of c_j for $0 \leqslant j < k$, allowing a recursive solution for c_k.

2. Differential operators on divided power algebras

We next discuss *differential operators* on the rings $\Gamma_R(X)$ and $\widehat{\Gamma}_R(X)$. First we consider the simplest case. Let D denote the R derivation given by

$$\mathrm{D}(\gamma_n(X)) = \gamma_{n-1}(X)$$

acting on either of these rings. We can extend this to give an action of the polynomial ring $R[\mathrm{D}]$ on $\Gamma_R(X)$ and $\widehat{\Gamma}_R(X)$. Liebnitz's formula gives

$$\mathrm{D}^n(f(X)g(X)) = \sum_{0 \leqslant j \leqslant n} \binom{n}{j} \mathrm{D}^j f(X) \mathrm{D}^{n-j} g(X)$$

as usual, and we can interpret this as defining a coproduct $R[\mathrm{D}] \longrightarrow R[\mathrm{D}] \otimes_R R[\mathrm{D}]$ given by

$$\mathrm{D}^n \longmapsto \sum_{0 \leqslant j \leqslant n} \binom{n}{j} \mathrm{D}^j \otimes \mathrm{D}^{n-j}.$$

The sequence $(\mathrm{D}^n)_{0\leqslant n}$ is 'dual' to the sequence $(\gamma_n(X))_{0\leqslant n}$, where we use the R-linear pairings

$$\begin{aligned} R[\mathrm{D}] \underset{R}{\otimes} \Gamma_R(X) &\longrightarrow R; \\ \mathrm{D}^n \otimes f(X) &\xmapsto{\langle\ ,\ \rangle} \varepsilon(\mathrm{D}^n f(X)), \\ R[\mathrm{D}] \underset{R}{\otimes} \widehat{\Gamma}_R(X) &\longrightarrow R; \\ \mathrm{D}^n \otimes f(X) &\xmapsto{\langle\ ,\ \rangle} \varepsilon(\mathrm{D}^n f(X)). \end{aligned}$$

Then

$$\langle \mathrm{D}^n, \gamma_k(X)\rangle = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

We generalize this as follows. Let $h(X) \in \widehat{\Gamma}_R(X)$ be a fixed element for which $\left\langle \mathrm{D}^0, h(X)\right\rangle = 0$ and $\langle \mathrm{D}, h(X)\rangle = 1$; hence the element $h'(X) = \mathrm{D}_h(X)$ is invertible. We define the operator D_h by

$$\mathrm{D}_h f(X) = h'(X)^{-1}\mathrm{D}f(X).$$

We can extend this to an action of the polynomial ring $R[\mathrm{D}_h]$ on $\widehat{\Gamma}_R(X)$ as before, but this time the Liebnitz formula does not apply.

3. Ordinary differential equations and linear recurrence relations

Now observe that there is a one to one correspondence between sequences $(a_n)_{0\leqslant n}$ in R and elements of $\widehat{\Gamma}_R(X)$, namely

$$(a_n)_{0\leqslant n} \longleftrightarrow \sum_{0\leqslant n} a_n\gamma_n(X).$$

This is multiplicative if we agree to define multiplication of sequences by the rule

$$(a_m)_{0\leqslant m} * (b_n)_{0\leqslant n} = \left(\sum_{0\leqslant k\leqslant n} \binom{n}{k} a_k b_{n-k}\right)_{0\leqslant n}$$

rather than by the usual Cauchy product formula.

Now observe that the action of the operator D on $\widehat{\Gamma}_R(X)$ is transported into the shift operator

$$\mathrm{S}\colon (a_n)_{0\leqslant n} \longmapsto (a_{n+1})_{0\leqslant n}.$$

Notice that $\ker \mathrm{D}$ and $\ker \mathrm{S}$ have rank 1 as R-modules, and are generated by the series $\gamma_0(X) = 1$ and the sequence $(\delta_{n,0})_{0\leqslant n}$ where $\delta_{r,s}$ is the Kronecker delta function.

Let us consider the parallel questions of the solution of the *ordinary differential equation*

(ODE) $$\left(\sum_{0\leqslant k\leqslant d} c_{d-k}\mathrm{D}^k\right) f(X) = 0$$

and the *recurrence relation*

(LRR) $$\sum_{0\leqslant k\leqslant d} c_{d-k}a_{n+k} = 0.$$

Here we set $f(X) = \sum_{0\leqslant n} a_n\gamma_n(X)$, $c_k \in R$, and we also assume that $c_0 = 1$.

Assumption: R is an integral domain.

We will sketch the solution of (ODE), from which the solution of (LRR) follows.

Let K be the field of fractions of R, K^{alg} be its algebraic closure and S denote the ring of integers in K^{alg} with respect to R. We may factorize the polynomial $\sum_{0\leqslant k\leqslant d} c_{d-k}T^k$ over the ring S, and hence write

$$\sum_{0\leqslant k\leqslant d} c_{d-k}T^k = \prod_s (T - \lambda_s)^{m(s)},$$

where the λ_s are the distinct roots and $m(s) > 0$ are their multiplicities. It suffices to deal with the case where there is a single root having multiplicity m, since it easily verified that

$$\ker(\mathrm{D} - \lambda_s)^{m(s)} \cap \ker(\mathrm{D} - \lambda_{s'})^{m(s')} = \{0\}$$

if $s \neq s'$. Thus we need to solve the equation

$$\text{(ODE1)} \qquad (\mathrm{D} - \lambda)^m f(X) = 0.$$

Observe that the series

$$\sum_{0 \leqslant n} \lambda^n \gamma_n(X) \in \widehat{\Gamma}_R(X),$$

which we *define* to be $e^{\lambda X}$, satisfies

$$e^{\lambda X} \mathrm{D}\left(e^{-\lambda X} g(X)\right) = (\mathrm{D} - \lambda) g(X),$$

and so to solve (ODE1), we are reduced to solving $\mathrm{D}^n f(X) = 0$. This is easily seen to have as its solutions all elements in $\widehat{\Gamma}_R(X)$ of the form

$$b_0 + b_1\gamma_1 + \cdots + b_{n-1}\gamma_{n-1} \quad \text{for } b_k \in R,$$

and hence the solutions of (ODE1) are all elements of the form

$$(b_0 + b_1\gamma_1 + \cdots + b_{n-1}\gamma_{n-1})\, e^{\lambda X} = \sum_{0 \leqslant k} \left(\sum_{0 \leqslant r \leqslant k} \binom{k}{r} b_{k-r} \lambda^r \right) \gamma_k.$$

Of course, this is exactly what the classical theory of ODE's gives! The general solution for (ODE) is now

$$\sum_s \left(b_{s,0} + b_{s,1}\gamma_1 + \cdots + b_{s,m(s)-1}\gamma_{m(s)-1}\right) e^{\lambda_s X}$$
$$= \sum_s \sum_{0 \leqslant k} \left(\sum_{0 \leqslant r \leqslant \min\{k, m(s)\}} \binom{k}{r} b_{s,k-r} \lambda_s^r \right) \gamma_k.$$

Returning to sequences, we obtain as the general solution of (LRR),

$$a_n = \sum_{s,\, 0 \leqslant r \leqslant \min\{n, m(s)\}} \sum \binom{n}{r} b_{s,n-r} \lambda_s^r.$$

If R is an algebra over the rational numbers $\mathbb{Q}$, the binomial coefficients $\binom{n}{r}$ are polynomial functions of n. However, over a general ring R, these are not 'polynomial' functions because of the denominator in

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

However, our method applies to give the general solution regardless of the ring.

4. More general differential equations and the associated recurrence relations

We now consider the more general 'twisted' differential operator D_h defined in Section 2, where we assume that $h(X) = X + \cdots$. We need to find a good description for the 'iterated anti-derivative' (with respect to D_h) of a constant.

Proposition 4.1. *Let $h(X) = \sum_{k \geqslant 1} a_k \gamma_n(X) \in \widehat{\Gamma}_R(X)$ with $a_1 = 1$. Then there is a element $g_n(X) = \sum_{k \geqslant n} b_k \gamma_k(X) \in \widehat{\Gamma}_R(X)$ with the property that*

$$h(X)^n = n!\, g_n(X).$$

Moreover, we can assume that $\mathrm{D}_h^n g_n(X) = 1$.

We denote the unique element $g_n(X) \in \widehat{\Gamma}_R(X)$ for which $\mathrm{D}_h^n g_n(X) = 1$ by $\gamma_n(h(X))$. If R has no $\mathbb{Z}$ torsion then of course

$$g_n(X) = \frac{h(X)^n}{n!}.$$

To prove this Proposition, we follow a route suggested by R. Steiner. We use a sequence of lemmas.

Lemma 4.2. *For natural numbers $m, n > 0$, the product*

$$\frac{(mn)!}{n!(m!)^n}$$

is an integer.

Proof. This is the number of partitions of a set of mn distinct objects into n pairwise disjoint subsets, each having m distinct elements, where we disregard the order of the subsets. Indeed, the denominator is the order of the wreath product subgroup

$$\Sigma_n \wr \Sigma_m = \Sigma_n \ltimes (\Sigma_m)^n \subseteq \Sigma_{mn},$$

which is the stabilizer of such a partition. □

Lemma 4.3. *In the ring $\Gamma_{\mathbb{Q}}(X)$ we have the identity*

$$\gamma_n(\gamma_m(X)) = \frac{(mn)!}{n!(m!)^n}\gamma_{mn}(X)$$

and hence the left hand side is an element of the subring $\Gamma_{\mathbb{Z}}(X) \subseteq \Gamma_{\mathbb{Q}}(X)$.

Proof. An easy calculation shows that

$$\gamma_n(\gamma_m(X)) = \frac{\gamma_m(X)^n}{n!}$$

is equal to the right hand side, and from Lemma 4.2 we deduce the second statement. □

Corollary 4.4. *For any commutative ring with unity R, we can make the definition*

$$\gamma_n(\gamma_m(X)) = \frac{(mn)!}{n!(m!)^n}\gamma_{mn}(X) \in \Gamma_R(X)$$

for all $m, n > 0$. More generally, if $\alpha \in \Gamma_R(X)$ has zero augmentation (i.e., $\varepsilon(\alpha) = 0$ then we can define

$$\gamma_n(\alpha) = \sum_{k \geqslant 1} \sum_{\substack{\sum_{1\leqslant j \leqslant d} r_j = n \\ 0 \leqslant r_j}} a_1^{r_1} \cdots a_d^{r_d} \gamma_{r_1}(\gamma_1(X)) \cdots \gamma_{r_d}(\gamma_d(X))$$

where $\alpha = \sum_{1 \leqslant k \leqslant d} a_k \gamma_k(X)$ with $a_k \in R$.

Proof. We make use of Equation 1.4 to repeatedly expand linear combinations with aid of the formula

$$\gamma_m(\beta + \gamma) = \sum_{0 \leqslant k \leqslant m} \gamma_k(\beta) \otimes \gamma_{m-k}(\gamma)$$

and Induction on d, to establish the result. □

Corollary 4.5. *For $h(X) \in \widehat{\Gamma}_R(X)$ with $\varepsilon(h(X)) = 0$, we can extend the definition of the operation $\gamma_n(\ )$ on $\Gamma_R(X)$ of Corollary 4.4 to $\widehat{\Gamma}_R(X)$ by setting*

$$\gamma_n(h(X)) = \lim_{d \to \infty} \gamma_n(h_d(X)),$$

where $h(X) = \sum_{1 \leqslant k} a_k \gamma_k(X)$ and $h_d(X) = \sum_{1 \leqslant k \leqslant d} a_k \gamma_k(X)$.

The limit in this definition is taken with respect to the standard notion of convergence in the inverse limit defining $\widehat{\Gamma}_R(X)$ in Section 1. This means that for natural numbers r, s we have

$$\gamma_n(h_r(X)) - \gamma_n(h_s(X)) \in \Gamma_R^{t_{r,s}}(X)$$

where $t_{r,s} \to \infty$ as $r, s \to \infty$.

We can now consider differential equations defined using D_h in place of D. This time we find that the general solution of

$$\mathrm{D}_h^n f(x) = 0$$

is the set of elements of $\widehat{\Gamma}_R(X)$ having the form

$$b_0 + b_1\gamma_1(h(X)) + \cdots + b_{n-1}\gamma_{n-1}(h(X)) \quad \text{for } b_k \in R.$$

Moreover, the general solution of $(\mathrm{D} - \lambda)f(X) = 0$ is

$$f(X) = e^{\lambda h(X)}.$$

Thus the general solution of

$$(\mathrm{D}_h - \lambda)^n f(X) = 0 \tag{4.1}$$

has to be of the form

$$(b_0 + b_1\gamma_1(h(X)) + \cdots + b_{n-1}\gamma_{n-1}(h(X)))\, e^{\lambda h(X)}.$$

More generally still, for a polynomial $\sum_{0\leqslant r\leqslant d} c_{d-r}T^r \in R[T]$ with $c_0 = 1$, we have the equation

$$\sum_{0\leqslant r\leqslant d} c_{d-r}\mathrm{D}_h^r f(X) = 0. \tag{4.2}$$

If R is an integral domain, then we may factorize this polynomial in some extension ring as in Section 3,

$$\sum_{0\leqslant r\leqslant d} c_{d-r}T^r = \prod_s (T - \lambda_s)^{m(s)}.$$

The general solution of (4.2) is thus

$$\sum_s \left(b_{s,0} + b_{s,1}\gamma_1(h(X)) + \cdots + b_{s,m(s)-1}\gamma_{m(s)-1}(h(X))\right) e^{\lambda_s h(X)}.$$

Reading off the coefficients a_n of the $\gamma_n(X)$ we obtain a sequence satisfying a non-linear recurrence relation obtained from the coefficients of (4.2) where $f(X) = \sum_{n\geqslant 0} a_n\gamma_n(X)$; however, the length of this recurrence may grow with n, as well as be non-linear.

We illustrate this with an example, where the recurrence relation is non-linear but is of constant length. Consider $\ln(1 + X) \in \mathbb{Q}[[X]]$. Then

$$\ln(1 + X) = \sum_{n\geqslant 1} (-1)^{n-1}(n-1)!\gamma_n(X) \in \widehat{\Gamma}_{\mathbb{Z}}(X).$$

Thus we have the series

$$h(X) = \sum_{n\geqslant 1} (-1)^{n-1}(n-1)!\gamma_n(X) \in \widehat{\Gamma}_R(X)$$

for *any* ring R. Hence $h'(X) = (1 + X)^{-1}$ and so

$$\mathrm{D}_h = (1 + X)\frac{\mathrm{d}}{\mathrm{d}X}.$$

The equation

$$(\mathrm{D}_h - \lambda)f(X) = 0$$

corresponds to the non-linear recurrence relation

$$a_{n+1} + na_n = \lambda a_n.$$

The general solution of this DE is

$$f(X) = b\sum_{n\geqslant 0} \lambda(\lambda - 1)\cdots(\lambda - n + 1)\gamma_n(X)$$

which we symbolically write as $(1+X)^\lambda$. The recurrence relation has general solution

$$a_n = \lambda(\lambda-1)\cdots(\lambda-n+1).$$

More generally, the equation

$$(\mathrm{D}_h - \lambda)^n f(X) = 0$$

has solutions of the form

$$(b_0 + b_1\gamma_1(h(X)) + \cdots + b_{n-1}\gamma_{n-1}(h(X)))\,(1+X)^\lambda.$$

Now in fact it is known that over $\mathbb{Q}$,

$$\ln(1+X)^m = \sum_{k\geqslant m} \frac{m!\mathrm{s}(k,m)X^k}{k!}$$

where $\mathrm{s}(i,j) \in \mathbb{Z}$ denotes a *Stirling number of the first kind* (see e.g., [2]). Hence, over any ring R we have

$$\gamma_m(\ln(1+X)) = \sum_{k\geqslant m} \mathrm{s}(k,m)\gamma_k(X),$$

and so we can read off the general solution of the recurrence relation as in Section 1.

5. Application to formal groups

Let R be a commutative ring with unity. The recall that *a commutative one dimensional formal group law over* R is a power series $F(X,Y) \in R[[X,Y]]$ satisfying the identities

$$\begin{aligned} F(X,Y) &= X + Y + XYG(X,Y) \quad \text{for some } G(X,Y) \in R[[X,Y]]; \\ F(X,F(Y,Z)) &= F(F(X,Y),Z); \\ F(X,0) &= 0 = F(0,X); \\ F(X,Y) &= F(Y,X). \end{aligned}$$

There is a unique series

$$[-1]_F(X) = -X + X^2H(X) \quad \text{for some } H(X) \in R[[X]]$$

which is characterized by the identities

$$F(X,[-1]_F(X)) = 0 = F([-1]_F(X),X).$$

Important examples are provided by

- *The additive formal group law:* $\widehat{G}_{\mathrm{A}}(X,Y) = X + Y$ for any ring R.
- *The multiplicative formal group law:* $\widehat{G}_{\mathrm{M}}(X,Y) = X + Y + XY$ for any ring R.
- *The Euler formal group law:* for any ring R containing an inverse for 2,

$$F^{\mathrm{Euler}}(X,Y) = \frac{X\sqrt{1-Y^4} + Y\sqrt{1-X^4}}{(1+X^2Y^2)}.$$

The latter example is the addition law associated to the elliptic integral

$$\int_0^X \frac{\mathrm{d}z}{\sqrt{1-z^4}}.$$

Now if R has no $\mathbb{Z}$-torsion, then we can embed R into its rationalisation $R \otimes \mathbb{Q}$. Then for any formal group law over such a ring there is a unique power series

$$\log^F(X) = X + \cdots \in R \otimes \mathbb{Q}[[X]]$$

such that

$$\log^F(F(X,Y)) = \log^F(X) + \log^F(Y).$$

This series is called the *logarithm* of F, and its composition inverse $\exp^F(X)$ is called the *exponential* of F. For example,

$$\log^{\widehat{G}_{\mathrm{M}}}(X) = \ln(1+X) \text{ and } \exp^{\widehat{G}_{\mathrm{M}}}(X) = e^X - 1$$

for the multiplicative formal group law, and

$$\log^{F^{\mathrm{Euler}}}(X) = \int_0^X \frac{\mathrm{d}z}{\sqrt{1-z^4}}$$

for the Euler formal group law.

Now in such cases where R is torsion free, it is well known that

$$\log^F(X), \exp^F(X) \in \widehat{\Gamma}_R(X),$$

as can be seen in these examples. The reason is that there is an identity

$$\log^F(X) = \int_0^X \frac{\mathrm{d}z}{F_2(z,0)},$$

where $F_2(X,Y) = \partial F(X,Y)/\partial Y$ (see [1]).

In general there is no notion of logarithm for an arbitrary formal group law in the sense of a power series which linearizes F. However, there *is* always an element of $\widehat{\Gamma}_R(X)$ which does this. To see this we note that the power series $F_2(X,0)$ can be interpreted as an element of $\widehat{\Gamma}_R(X)$ if we replace each power X^n by the expression $n!\gamma_X$. But then we can uniquely integrate $F_2(X,0)^{-1}$ to an element of $\widehat{\Gamma}_R(X)$ with leading term $\gamma_1(X)$. The composition inverse of this series is easily seen to exist as an element of $\widehat{\Gamma}_R(X)$. It is trivial to verify that this does produce a logarithm. This argument could also be carried out for the universal ring of Lazard, which is known to be torsion free (see [1]), and the result mapped into the general case. We will continue to denote the logarithm and exponential of F by $\log^F(X)$ and $\exp^F(X)$ as elements of $\widehat{\Gamma}_R(X)$.

Now suppose that we have fixed upon a formal group law over a ring R, and take $h(X) = \log^F(X)$. Then the operator D_h of Section 2 is *translation invariant* with respect to F, in the sense that

$$[\mathrm{D}_h f(X)]_{X=F(Y,Z)} = [\mathrm{D}_h f(F(X,Z))]_{X=Y}.$$

This follows from the connection between $\log^F(X)$ and $F_2(X,0)$ described above. More generally, the polynomial ring $R[\mathrm{D}_h]$ is the ring of all translation invariant operators on $\widehat{\Gamma}_R(X)$ with respect to F. In [1], the notion of *covariant bialgebra of* F is encountered; there is a Hopf algebra homomorphism from $R[\mathrm{D}_h]$ into the latter which is an embedding if R is torsion free. Thus our earlier discussion reduces in this situation to the study of the solutions of differential equations of the form

$$\Phi f(X) = 0,$$

where $\Phi \in R[\mathrm{D}_h]$ is an arbitrary translation variant operator with respect to F on $\widehat{\Gamma}_R(X)$.

References

[1] M. Hazewinkel, Formal Groups and Applications, Academic Press (1978), New York.
[2] J. Riordan, Combinatorial Identities, Krieger (1979), New York.

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: a.baker@maths.gla.ac.uk
URL: http://www.maths.gla.ac.uk/~ajb