ON THE HOMOLOGY OF REGULAR QUOTIENTS

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Abstract. We construct a free resolution of R/I^s over R where $I \triangleleft R$ is generated by a (finite or infinite) regular sequence. This generalizes the Koszul complex for the case $s = 1$. For $s > 1$, we easily deduce that the algebra structure of $\mathrm{Tor}^R_*(R/I, R/I^s)$ is trivial and the reduction map $R/I^s \longrightarrow R/I^{s-1}$ induces the trivial map of algebras.

Introduction

Let R be a commutative unital ring. We will say that an ideal $I \triangleleft R$ is *regular* if it is generated by a regular sequence $u_1, u_2, \ldots$ which may be finite or infinite. We will call the quotient ring R/I a *regular quotient* of R. All tensor products and homomorphisms will be taken over R unless otherwise indicated.

It is well known, see [8] for example, that there is a *Koszul resolution*

$$\mathbf{K}_* \longrightarrow R/I \to 0,$$

where

$$\mathbf{K}_* = \Lambda_R(e_i : i \geqslant 1)$$

is a differential graded algebra with e_i in degree 1 and differential given by $\mathrm{d}\, e_i = u_i$. The following result is standard, see for example [4, 8].

Proposition 0.1. *If $I \triangleleft R$ is regular, then $\mathbf{K}_*$ provides a free resolution of R/I over R. Moreover, $(\mathbf{K}_*, \mathrm{d})$ is a differential graded R-algebra.*

Corollary 0.2. *As R/I-algebras,*

$$\mathrm{Tor}^R_*(R/I, R/I) = \Lambda_{R/I}(e_i : i \geqslant 1).$$

We will generalize this by defining a family of free resolutions

$$\mathbf{K}(R; I^s)_* \longrightarrow R/I^s \to 0 \quad (s \geqslant 1),$$

which are well related and allow efficient calculation of the R/I-algebra $\mathrm{Tor}^R_*(R/I, R/I^s)$.

The resolution we construct may well be known, however lacking a convenient reference we give the details. Our immediate motivation lies in topological calculations that are part of joint work with A. Jeanneret and A. Lazarev [1, 2], but we believe this algebraic construction may be of wider interest. Our approach to this construction was suggested by derived category ideas and in particular the construction of Cartan-Eilenberg resolutions [3, 8]. Tate's method of killing homology classes [7] seems to be related, as does Smith's work on homological algebra [5], but neither appears to give our result explicitly.

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Notation. Our indexing conventions are predominantly homological (*i.e.*, lower index) as opposed to cohomological, since that is appropriate for the topological applications we have in mind. Consequently, complexes have differentials which *decrease* degrees.

For a complex (C_*, d), we define its *k-fold suspension* $(C[-k]_*, d[-k])$ by

$$C[-k]_n = C_{n-k}, \quad d[-k] = (-1)^k d\colon C_{n-k} \longrightarrow C_{n-k-1}.$$

For an R-module M, we sometimes view M as the complex with

$$M_n = \begin{cases} M & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

1. A resolution for R/I^s

In this section we describe an explicit R-free resolution for R/I^s which allows homological calculations. We begin with a standard result; actually the cited proof applies when I is finitely generated, but the adaption to the general case is straightforward. We will always interpret I^0/I as R/I.

Lemma 1.1 ([4], Theorem 16.2)**.** *For $s \geqslant 0$, I^s/I^{s+1} is a free R/I-module with a basis consisting of the residue classes of the distinct monomials of degree s in the u_i.*

Corollary 1.2. *For $s \geqslant 0$, there is a free resolution of I^s/I^{s+1} over R of the form*

$$\mathbf{Q}^{(s)}_* = \mathbf{K}_* \otimes \mathrm{U}^{(s)} \longrightarrow I^s/I^{s+1} \to 0,$$

where $\mathrm{U}^{(s)}$ is a free R-module on a basis indexed by the distinct monomials of degree s in the generators u_i.

For a sequence $\mathbf{i} = (i_1, \ldots, i_s)$ and its associated monomial $u_{\mathbf{i}} = u_{i_1} \cdots u_{i_s}$, we will denote the corresponding basis element $1 \otimes u_{i_1} \cdots u_{i_s}$ of $\mathbf{K}_* \otimes \mathrm{U}^{(s)}$ by $\widetilde{u}_{\mathbf{i}}$ and more generally $x \otimes \widetilde{u}_{\mathbf{i}}$ by $x\widetilde{u}_{\mathbf{i}}$. We will also denote the differential on $\mathbf{Q}^{(s)}_*$ by $\mathrm{d}^{(s)}_{\mathbf{Q}}$, noting that

$$\mathrm{d}^{(s)}_{\mathbf{Q}}\, x\widetilde{u}_{\mathbf{i}} = (\mathrm{d}\, x)\widetilde{u}_{\mathbf{i}}. \tag{1.1}$$

For $s \geqslant 0$, there is also a map

$$\partial^{(s+1)}\colon \mathbf{Q}^{(s)}_* \longrightarrow \mathbf{Q}^{(s+1)}_*; \quad \partial^{(s+1)} \sum_{\mathbf{i}} y_{\mathbf{i}}\widetilde{u}_{\mathbf{i}} = \sum_{\mathbf{i}} (\mathrm{d}\, y_{\mathbf{i}})\widetilde{u}_{\mathbf{i}},$$

where we interpret the products for $y_{(i_1,\ldots,i_s)} \in \mathbf{K}_*$ according to the formula

$$(\mathrm{d}\, y_{(i_1,\ldots,i_s)})\widetilde{u}_{(i_1,\ldots,i_s)} = \sum_j y_{(i_1,\ldots,i_s),j}\; \widetilde{u}_{(i_1,\ldots,i_s,j)}$$

with

$$\mathrm{d}\, y_{(i_1,\ldots,i_s)} = \sum_{(i_1,\ldots,i_s),j} y_{(i_1,\ldots,i_s),j}\widetilde{u}_j.$$

For $s \geqslant 1$, define

$$\mathbf{K}(R; I^s)_* = \mathbf{Q}^{(0)}_* \oplus \mathbf{Q}^{(1)}_* \oplus \cdots \oplus \mathbf{Q}^{(s-1)}_*,$$

with the differential $\mathrm{d}^{(s)}$ given by

$$\mathrm{d}^{(s)}(x_0, x_1, \ldots, x_{s-1}) = (x'_0, x'_1, \ldots, x'_{s-1}), \tag{1.2}$$

where

$$x'_k = \begin{cases} \mathrm{d}_{\mathbf{Q}}^{(0)}\, x_0 & \text{if } k = 0, \\ \partial^{(k)} x_{k-1} + \mathrm{d}_{\mathbf{Q}}^{(k)}\, x_k & \text{otherwise.} \end{cases}$$

We need to show that $(\mathrm{d}^{(s)})^2 = 0$. This follows from the following easily verified identities which hold for all $r \geqslant 0$:

$$\mathrm{d}_{\mathbf{Q}}^{(r+1)}\, \partial^{(r+1)} + \partial^{(r+1)}\, \mathrm{d}_{\mathbf{Q}}^{(r)} = 0, \tag{1.3}$$

$$\partial^{(r+1)} \partial^{(r)} = 0. \tag{1.4}$$

Then

$$(\mathrm{d}^{(s)})^2(x_0, x_1, \ldots, x_{s-1}) = (x''_0, x''_1, \ldots, x''_{s-1}),$$

where

$$x''_0 = (\mathrm{d}^{(0)})^2 x_0 = 0,$$
$$x''_1 = \partial^{(1)}\, \mathrm{d}_{\mathbf{Q}}^{(0)}\, x_0 + \mathrm{d}^{(1)}\, \partial^{(1)} x_0 + (\mathrm{d}^{(1)})^2 x_1 = 0,$$

while for $2 \leqslant k \leqslant s-1$,

$$x''_k = \partial^{(k)} \partial^{(k-1)} x_{k-2} + \partial^{(k)}\, \mathrm{d}_{\mathbf{Q}}^{(k-1)}\, x_{k-1} + \mathrm{d}_{\mathbf{Q}}^{(k)}\, \partial^{(k)} x_{k-1} + (\mathrm{d}_{\mathbf{Q}}^{(k)})^2 x_k) = 0.$$

There is an augmentation map

$$\varepsilon^{(s)} \colon \mathbf{K}(R; I^s)_0 \longrightarrow R/I^s,$$

namely the R-module homomorphism

$$\varepsilon^{(s)} \left(a_0, \sum_{(i_1)} a_{(i_1)} \widetilde{u}_{(i_1)}, \sum_{(i_1,i_2)} a_{(i_1,i_2)} \widetilde{u}_{(i_1,i_2)}, \ldots, \sum_{(i_1,i_2,\ldots,i_{s-1})} a_{(i_1,i_2,\ldots,i_{s-1})} \widetilde{u}_{(i_1,i_2,\ldots,i_{s-1})} \right)$$
$$= a_0 + \sum_{(i_1)} a_{(i_1)} u_{(i_1)} + \sum_{(i_1,i_2)} a_{(i_1,i_2)} u_{(i_1,i_2)} + \cdots + \sum_{(i_1,i_2,\ldots,i_{s-1})} a_{(i_1,i_2,\ldots,i_{s-1})} u_{(i_1,i_2,\ldots,i_{s-1})},$$

in which the sum $\sum_{(i_1,i_2,\ldots,i_k)}$ is taken over all the distinct monomials $u_{(i_1,i_2,\ldots,i_k)} = u_{i_1} \cdots u_{i_k}$ of degree k and $a_{(i_1,i_2,\ldots,i_k)} \in R$. Then $\varepsilon^{(s)}$ is surjective and in $\mathbf{K}(R; I^s)_0$ we have

$$\operatorname{im} \mathrm{d}^{(s)} \subseteq \ker \varepsilon^{(s)}.$$

On the other hand, suppose that

$$\mathbf{a} = (a_0, \widetilde{a}_1, \ldots, \widetilde{a}_{s-1}) \in \ker \varepsilon^{(s)},$$

where

$$\widetilde{a}_k = \sum_{(i_1,\ldots,i_k)} a_{(i_1,\ldots,i_k)} \widetilde{u}_{(i_1,\ldots,i_k)}.$$

Then writing

$$a_k = \sum_{(i_1,\ldots,i_k)} a_{(i_1,\ldots,i_k)} u_{(i_1,\ldots,i_k)},$$

we find

$$a_0 + a_1 + \cdots + a_{s-1} \in I^s,$$

so $a_0 \in I$. This means that

$$\mathbf{a} \equiv (0, \widetilde{b}_1, \widetilde{a}_2 \ldots, \widetilde{a}_{s-1}) \mod \operatorname{im} \mathrm{d}^{(s)}.$$

Repeating this argument modulo higher powers of I, we find that

$$\mathbf{a} \equiv (0, 0, \ldots, 0, \widetilde{b}_{s-1}) \mod \operatorname{im} \mathrm{d}^{(s)},$$

where

$$\widetilde{b}_{s-1} = \sum_{(i_1,\ldots,i_{s-1})} a_{(i_1,\ldots,i_{s-1})} \widetilde{u}_{(i_1,\ldots,i_{s-1})}$$

and

$$b_{s-1} = \sum_{(i_1,\ldots,i_{s-1})} a_{(i_1,\ldots,i_{s-1})} u_{(i_1,\ldots,i_{s-1})} \in I^s.$$

But taking

$$c = \sum_{(i_1,\ldots,i_{s-1})} a_{(i_1,\ldots,i_{s-1})} \tau_{i_{s-1}} \widetilde{u}_{(i_1,\ldots,i_{s-2})},$$

we find

$$\mathrm{d}^{(s)}(0, \ldots, 0, c) = (0, 0, \ldots, 0, \widetilde{b}_{s-1}).$$

Hence $\mathbf{a} \in \operatorname{im} \mathrm{d}^{(s)}$. This shows that

$$\ker \varepsilon^{(s)} = \operatorname{im} \mathrm{d}^{(s)}.$$

Suppose that $n \geqslant 1$ and

$$\mathbf{x} = (x_0, x_1, \ldots, x_{s-1}) \in \mathbf{K}(R; I^s)_n$$

satisfies $\mathrm{d}^{(s)} \mathbf{x} = 0$. Then $x_0' = 0$ and so by exactness of $\mathbf{Q}_*^{(0)}$,

$$x_0 = \mathrm{d}_{\mathbf{Q}}^{(0)} y_0$$

for some $y_0 \in \mathbf{Q}_{n+1}^{(0)}$. Then

$$\begin{aligned} 0 = x_1' &= \partial^{(1)} \mathrm{d}_{\mathbf{Q}}^{(0)} y_0 + \mathrm{d}_{\mathbf{Q}}^{(1)} x_1 \\ &= \mathrm{d}_{\mathbf{Q}}^{(1)}(-\partial^{(1)} y_0 + x_1), \end{aligned}$$

hence by exactness of $\mathbf{Q}_*^{(1)}$,

$$x_1 = \mathrm{d}_{\mathbf{Q}}^{(1)} y_1 + \partial^{(1)} y_0$$

for some $y_1 \in \mathbf{Q}_{n+1}^{(1)}$. Continuing in this way, eventually we obtain an element

$$(y_0, y_1, \ldots, y_{s-1}) \in \mathbf{K}(R; I^s)_{n+1}$$

for which

$$x_k = \mathrm{d}_{\mathbf{Q}}^{(k)} y_k + \partial^{(k)} y_{k-1} \quad (1 \leqslant k \leqslant s-1).$$

Theorem 1.3. *For $s \geqslant 1$,*

$$\mathbf{K}(R; I^s)_* \xrightarrow{\varepsilon^{(s)}} R/I^s \to 0$$

is a resolution by free R-modules.

The complex $(\mathbf{K}(R; I^s)_*, \mathrm{d}^{(s)})$ has a multiplicative structure coming from the pairings

$$\mathbf{Q}_*^{(p)} \otimes \mathbf{Q}_*^{(q)} \longrightarrow \mathbf{Q}_*^{(p+q)}; \quad (x\widetilde{u}_{(i_1,\ldots,i_p)}) \otimes (y\widetilde{u}_{(j_1,\ldots,j_q)}) \longmapsto (xy)\widetilde{u}_{(i_1,\ldots,i_p,j_1,\ldots,j_q)}.$$

Theorem 1.4. *For $s \geqslant 1$, the complex $(\mathbf{K}(R; I^s)_*, \mathrm{d}^{(s)})$ is a differential graded R-algebra, providing a multiplicative resolution free resolution of R/I^s over R.*

Corollary 1.5. *As an R/I-algebra,*

$$\mathrm{Tor}_*^R(R/I, R/I^s) = \mathrm{H}_*(R/I \otimes \mathbf{K}(R; I^s)_*, 1 \otimes \mathrm{d}^{(s)}).$$

Notice that in the differential graded R/I-algebra $(R/I \otimes \mathbf{K}(R; I^s)_*, 1 \otimes \mathrm{d}^{(s)})$ we have

$$1 \otimes \mathrm{d}^{(s)}(t \otimes (x_0, x_1, \dots, x_{s-1})) = t \otimes (0, \partial^{(1)} x_0, \partial^{(2)} x_1, \dots, \partial^{(s-2)} x_{s-2}). \tag{1.5}$$

We will exploit this in the next section.

2. A spectral sequence

In order to compute $\mathrm{Tor}_*^R(R/I, R/I^s)$ explicitly we will set up a double complex and consider one of the two associated spectral sequences [8]. We begin by defining the double complex $(\mathrm{P}_{*,*}, \mathrm{d}^{\mathrm{h}}, \mathrm{d}^{\mathrm{v}})$ with

$$\begin{aligned} \mathrm{P}_{p,q} &= \mathbf{Q}^{(p)}[-p]_{q+p} (= \mathbf{Q}_q^{(p)} \text{ as } R\text{-modules}), \\ \mathrm{d}^{\mathrm{h}} &= (-1)^p \partial^{(p+1)}[-p] = \partial^{(p+1)}, \\ \mathrm{d}^{\mathrm{v}} &= (-1)^p \, \mathrm{d}_{\mathbf{Q}}^{(p)}[-p] = \mathrm{d}_{\mathbf{Q}}^{(p)}. \end{aligned}$$

Considered as a homomorphism

$$\mathrm{d}^{\mathrm{v}} \mathrm{d}^{\mathrm{h}} + \mathrm{d}^{\mathrm{h}} \mathrm{d}^{\mathrm{v}} \colon \mathrm{P}_{p,q} \longrightarrow \mathrm{P}_{p+1,q+1},$$

we have from Equation (1.3),

$$\mathrm{d}^{\mathrm{v}} \mathrm{d}^{\mathrm{h}} + \mathrm{d}^{\mathrm{h}} \mathrm{d}^{\mathrm{v}} = \mathrm{d}_{\mathbf{Q}}^{(p+1)} \partial^{(p+1)} + \partial^{(p+1)} \mathrm{d}_{\mathbf{Q}}^{(p)} = 0.$$

As the associated (direct sum) total complex $(\mathrm{Tot}^{\oplus} \mathrm{P}_*, \mathrm{d}^{\mathrm{Tot}})$ we obtain

$$\mathrm{Tot}^{\oplus} \mathrm{P}_n = \bigoplus_k \mathrm{P}_{k,n-k}, \quad \mathrm{d}^{\mathrm{Tot}} = \mathrm{d}^{\mathrm{h}} + \mathrm{d}^{\mathrm{v}}.$$

Notice that

$$\mathrm{Tot}^{\oplus} \mathrm{P}_n = \mathbf{K}(R; I^s)_n, \quad \mathrm{d}^{\mathrm{Tot}} = \mathrm{d}^{(s)}$$

Hence

$$\mathrm{H}_*(\mathrm{Tot}^{\oplus} \mathrm{P}_*, \mathrm{d}^{\mathrm{Tot}}) = R/I^s.$$

Applying the functor $R/I \otimes (\ )$ we obtain another double complex $(\overline{\mathrm{P}}_{*,*}, \mathrm{d}^{\mathrm{h}}, \mathrm{d}^{\mathrm{v}})$ where

$$\overline{\mathrm{P}}_{p,q} = R/I \otimes \mathrm{P}_{p,q}.$$

The associated total complex $(\mathrm{Tot}^{\oplus} \overline{\mathrm{P}}_*, \mathrm{d}^{\mathrm{Tot}})$ has

$$\mathrm{Tot}^{\oplus} \overline{\mathrm{P}}_n = R/I \otimes \mathbf{K}(R; I^s)_n, \quad \mathrm{d}^{\mathrm{Tot}} = 1 \otimes \mathrm{d}^{(s)}$$

and homology

$$\mathrm{H}_*(\mathrm{Tot}^{\oplus} \overline{\mathrm{P}}_*, \mathrm{d}^{\mathrm{Tot}}) = \mathrm{Tor}_*^R(R/I, R/I^s).$$

Filtering by columns we obtain a spectral sequence with

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\mathrm{H}_q(\overline{\mathrm{P}}_{*,*}, \mathrm{d}^{\mathrm{v}}), \mathrm{d}^{\mathrm{h}}) \Longrightarrow \mathrm{Tor}_{p+q}^R(R/I, R/I^s). \tag{2.1}$$

Here

$$\mathrm{H}_*(\overline{\mathrm{P}}_{p,*}, \mathrm{d}^{\mathrm{v}}) = \mathrm{H}_*(R/I \otimes \mathbf{Q}_*^{(p)}, 1 \otimes \mathrm{d}_{\mathbf{Q}}^{(p)}) = \mathrm{Tor}_*^R(R/I, I^p/I^{p+1})$$

and $\mathrm{H}_*(\mathrm{H}_q(\overline{\mathrm{P}}_{*,*},\mathrm{d}^{\mathrm{v}}),\mathrm{d}^{\mathrm{h}})$ is the homology of the complex

$$0 \to \operatorname{Tor}^R_q(R/I,R/I) \xrightarrow{\partial^{(1)}_*} \operatorname{Tor}^R_q(R/I,I/I^2) \longrightarrow \cdots \longrightarrow \operatorname{Tor}^R_q(R/I,I^2/I^3) \xrightarrow{\partial^{(s-1)}_*} \operatorname{Tor}^R_q(R/I,I^{s-1}/I^s) \to 0.$$

Lemma 2.1. *For $s \geqslant 2$, the complex of graded R/I-modules*

$$0 \to \operatorname{Tor}^R_*(R/I,R/I) \xrightarrow{\partial^{(1)}_*} \operatorname{Tor}^R_*(R/I,I/I^2) \longrightarrow \cdots \longrightarrow \operatorname{Tor}^R_*(R/I,I^2/I^3) \xrightarrow{\partial^{(s-1)}_*} \operatorname{Tor}^R_*(R/I,I^{s-1}/I^s) \to 0$$

is exact, hence the spectral sequence of (2.1) *collapses at* E^2 *to give*

$$\operatorname{Tor}^R_n(R/I,R/I^s) = \begin{cases} R/I & \textit{if } n=0, \\ \operatorname{coker}\partial^{(s-1)}_*\colon \operatorname{Tor}^R_n(R/I,I^{s-2}/I^{s-1}) \longrightarrow \operatorname{Tor}^R_n(R/I,I^{s-1}/I^s) & \textit{if } n\neq 0. \end{cases}$$

With its natural R/I-algebra structure, $\operatorname{Tor}^R_*(R/I,R/I^s)$ has trivial products.

Proof. Our proof uses the observation that this complex is equivalent to part of the Koszul complex $\Lambda_{R/I[\widetilde{u}_i:i]}(\widetilde{e}_i:i)$ which provides a free resolution of $R/I = R/I[\widetilde{u}_i:i]/(\widetilde{u}_i:i)$ as an $R/I[\widetilde{u}_i:i]$-module. Up to a sign, the differential $\widetilde{\mathrm{d}}$ agrees with that of the complex in Lemma 2.1. The result follows by exactness of the Koszul complex since the generators $\widetilde{u}_1,\widetilde{u}_2,\ldots$ form a regular sequence in $R/I[\widetilde{u}_i:i]$. We now proceed to give the details.

For a commutative unital ring $\Bbbk$, make $\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)$ a bigraded $\Bbbk$-algebra for which

$$\operatorname{bideg}\widetilde{e}_i = (1,0), \quad \operatorname{bideg}\widetilde{u}_i = (1,-1).$$

For each grading $p \geqslant 0$ of $\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)$,

$$\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^p = \bigoplus_{q\geqslant 0}\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^{p+q,-q}$$

and the differential

$$\mathrm{d}^p\colon \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^p \longrightarrow \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^{p+1}$$

decomposes as a sum of components

$$\mathrm{d}^{p+q,-q}\colon \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^{p+q,-q} \longrightarrow \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)^{p+q,-q-1},$$

since

$$\mathrm{d}^p(\widetilde{e}_{i_1}\cdots\widetilde{e}_{i_p}\widetilde{u}_{j_1}\cdots\widetilde{u}_{j_q}) = \sum_{k=1}^{p}(-1)^{k-1}\widetilde{e}_{i_1}\cdots\widetilde{e}_{i_{k-1}}\widetilde{e}_{i_{k+1}}\cdots\widetilde{e}_{i_p}\widetilde{u}_{i_k}\widetilde{u}_{j_1}\cdots\widetilde{u}_{j_q}.$$

Exactness of d on $\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)$ is equivalent to the fact that for all pairs p,q,

$$\ker\mathrm{d}^{p+q,-q} = \operatorname{im}\ker\mathrm{d}^{p+q,-q+1}.$$

Hence for all q we have

$$\bigoplus_{p\geqslant 0}\ker\mathrm{d}^{p+q,-q} = \bigoplus_{p\geqslant 0}\operatorname{im}\mathrm{d}^{p+q,-q+1},$$

which is equivalent to the exactness of

$$\Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)\underset{\Bbbk}{\otimes}\Bbbk[\widetilde{u}_i:i]_{q-1} \xrightarrow{\mathrm{d}} \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)\underset{\Bbbk}{\otimes}\Bbbk[\widetilde{u}_i:i]_{q} \xrightarrow{\mathrm{d}} \Lambda_{\Bbbk[\widetilde{u}_i:i]}(\widetilde{e}_i:i)\underset{\Bbbk}{\otimes}\Bbbk[\widetilde{u}_i:i]_{q+1},$$

where $\Bbbk[\widetilde{u}_i:i]_n \subseteq \Bbbk[\widetilde{u}_i:i]$ denotes the homogeneous polynomials of degree n.

The statement about products is now immediate since the spectral sequence is clearly multiplicative. Actually the full force of this is not really needed since

$$\operatorname{Tor}_*^R(R/I, R/I^s) \cong R/I \oplus \operatorname{coker} \partial_*^{(s-1)}$$

and products of elements in the bottom filtration $\operatorname{coker} \partial_*^{(s-1)}$ are zero in $\mathrm{E}^\infty = \mathrm{E}^2$. $\square$

We can strengthen our hold on $\operatorname{Tor}_*^R(R/I, R/I^s)$ using the ideas in the last proof.

Proposition 2.2. *For $s \geqslant 1$, $\operatorname{Tor}_*^R(R/I, R/I^s)$ is a free R/I-module.*

Proof. The case $s = 1$ is of course a consequence of Corollary 0.2.

Using the notation of the proof of Lemma 2.1, notice that in terms of the $\Bbbk$-basis of elements $\widetilde{e}_{i_1} \cdots \widetilde{e}_{i_p} \widetilde{u}_{j_1} \cdots \widetilde{u}_{j_q}$, each $\mathrm{d}^{p+q,-q}$ is actually given by a $\mathbb{Z}$-linear combination. Therefore we can reduce to the case where $\Bbbk = \mathbb{Z}$, and then tensor up over $\mathbb{Z}$ with an arbitrary $\Bbbk$.

For each pair $p, q \geqslant 0$, $\Lambda_{\mathbb{Z}[\widetilde{u}_i:i]}(\widetilde{e}_i : i)^{p+q,-q}$ breaks up into a direct sum of $\mathbb{Z}$-submodules $M^{p+q,-q}(S)$ where S is a set of exactly $p + q$ elements of the indexing set for the u_i's and $M^{p+q,-q}(S)$ is spanned by the finitely many elements $\widetilde{e}_{i_1} \cdots \widetilde{e}_{i_p} \widetilde{u}_{j_1} \cdots \widetilde{u}_{j_q}$ with

$$S = \{i_1, \dots, i_p, j_1, \dots j_q\}, \quad i_1 < i_2 < \cdots < i_p.$$

Notice that on restriction we have

$$\mathrm{d}_{M(S)}^{p+q,-q} = \mathrm{d}^{p+q,-q} \colon M^{p+q,-q}(S) \longrightarrow M^{p+q,-q-1}(S).$$

By exactness, $\operatorname{im} \mathrm{d}_{M(S)}^{p+q,-q+1} = \ker \mathrm{d}_{M(S)}^{p+q,-q}$. Since $M^{p+q,-q}(S)$ is a finitely generated free module, $\ker \mathrm{d}_{M(S)}^{p+q,-q}$ is indivisible in $M^{p+q,-q}(S)$ and so is a summand. Hence $\operatorname{im} \mathrm{d}_{M(S)}^{p+q,-q+1}$ is always a summand of $M^{p+q,-q}(S)$. Taking the sum over all S and then over all p we find that for each q,

$$\operatorname{im} \mathrm{d} \colon \Lambda_{\mathbb{Z}[\widetilde{u}_i:i]}(\widetilde{e}_i : i) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\widetilde{u}_i : i]_{q-1} \longrightarrow \Lambda_{\mathbb{Z}[\widetilde{u}_i:i]}(\widetilde{e}_i : i) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\widetilde{u}_i : i]_q$$

is a summand in

$$\Lambda_{\mathbb{Z}[\widetilde{u}_i:i]}(\widetilde{e}_i : i) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\widetilde{u}_i : i]_q.$$

$\square$

3. Appendix: Resolutions of extensions

In this Appendix we recall some standard facts about extensions of R-modules, see [8], and also give an interpretation in terms of the derived category of complexes of R-modules. Our aim is to put the construction of the complex $(\mathbf{K}(R; I^s)_*, \mathrm{d}^{(s)})$ into a broader context for the benefit of those unfamiliar with such ideas. In fact, we found this complex by iterating the splicing construction for the resolution of an extension given below; in our case this works well to give a very concrete and manageable resolution.

Suppose that

$$\mathcal{E} \colon \quad 0 \to L \longrightarrow M \longrightarrow N \to 0 \tag{3.1}$$

is a short exact sequence of R-modules and

$$P_* \xrightarrow{\varepsilon} N \to 0$$

is a projective resolution of N. Then there are homomorphisms $\varepsilon_0\colon P_0 \longrightarrow M$ and $\varepsilon_1\colon P_1 \longrightarrow L$ which fit into a commutative diagram

$$\begin{tikzcd} 0 & N \arrow[l] \arrow[d, equal] & P_0 \arrow[l] \arrow[d, "\varepsilon_0"'] & P_1 \arrow[l] \arrow[d, "\varepsilon_1"'] & P_2 \arrow[l] & \cdots \arrow[l] \\ 0 & N \arrow[l] & M \arrow[l] & L \arrow[l] & 0 \arrow[l] \end{tikzcd} \tag{3.2}$$

Then ε_1 is a cocycle in $\mathrm{Hom}_R(P_1, L)$ which represents an element $\Theta(\mathcal{E}) \in \mathrm{Ext}^1_R(N, L)$ classifying the extension $\mathcal{E}$.

Now let $Q_* \xrightarrow{\eta} L \to 0$ be a projective resolution of L with differential d_Q and $Q[-1]_*$ its suspension. Then the differential $d_Q[-1]$ in $Q[-1]_*$ is given by

$$d_Q[-1]x = -d_Q x.$$

It is well known that in the derived category $\mathcal{D}^\flat(R)$ of bounded below complexes of R-modules,

$$\mathrm{Ext}^1_R(N, L) \cong \mathrm{Hom}_{\mathcal{D}^\flat(R)}(P_*, Q[-1]_*). \tag{3.3}$$

Given the diagram (3.2), there is an extension to a diagram

$$\begin{tikzcd} 0 & N \arrow[l] \arrow[d, equal] & P_0 \arrow[l] \arrow[d, "\varepsilon_0"'] & P_1 \arrow[l] \arrow[d, "\varepsilon'_1"'] & P_2 \arrow[l] \arrow[d, "\varepsilon'_2"'] & P_3 \arrow[l] \arrow[d, "\varepsilon'_3"'] & \cdots \arrow[l] \\ 0 & N \arrow[l] & M \arrow[l] & Q_0 \arrow[l] & Q_1 \arrow[l] & Q_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd} \tag{3.4}$$

and hence the element

$$\begin{tikzcd} 0 \arrow[d, equal] & P_0 \arrow[l] \arrow[d, "0"'] & P_1 \arrow[l] \arrow[d, "\varepsilon'_1"'] & P_2 \arrow[l] \arrow[d, "\varepsilon'_2"'] & P_3 \arrow[l] \arrow[d, "\varepsilon'_3"'] & \cdots \arrow[l] \\ 0 & 0 \arrow[l] & Q_0 \arrow[l] & Q_1 \arrow[l] & Q_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

which represents an element of $\mathrm{Hom}_{\mathcal{D}^\flat(R)}(P_*, Q[-1]_*)$. Conversely, a diagram with exact rows such as (3.4) clearly gives rise to an extension of the form (3.2). Perhaps a more illuminating way to view this morphism in $\mathcal{D}^\flat(R)$ is in terms of the diagram

$$\begin{tikzcd} 0 \arrow[d, equal] & P_0 \arrow[l] \arrow[d, "0"'] & P_1 \arrow[l] \arrow[d, "\varepsilon_1"'] & P_2 \arrow[l] \arrow[d] & P_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ 0 & 0 \arrow[l] & L[-1] \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\ 0 \arrow[u, equal] & 0 \arrow[l] \arrow[u] & Q[-1]_0 \arrow[l] \arrow[u, "\varepsilon"] & Q[-1]_1 \arrow[l] \arrow[u] & Q[-1]_2 \arrow[l] \arrow[u] & \cdots \arrow[l] \end{tikzcd}$$

where the augmentation $\varepsilon\colon Q[-1]_* \longrightarrow L[-1]$ is a homology equivalence, hence an isomorphism in $\mathcal{D}^\flat(R)$, so the composite

$$P_* \xrightarrow{\varepsilon_1} L \xrightarrow{\varepsilon^{-1}} Q_*$$

gives an element of $\mathrm{Hom}_{\mathcal{D}^\flat(R)}(P_*, Q[-1]_*)$. Of course all of these classes agree with $\Theta(\mathcal{E})$. Notice that $\Theta(\mathcal{E})$ is determined by the homomorphism $\varepsilon_1\colon P_1 \longrightarrow Q[-1]_1 = Q_0$ lifting the map $P_1 \longrightarrow L$.

We also recall a well known related result, see [8].

Proposition 3.1. *For a ring R, let*

$$0 \leftarrow A \longleftarrow B \longleftarrow C \leftarrow 0$$

be short exact and $P_* \longrightarrow A \to 0$ *and* $Q_* \longrightarrow C \to 0$ *projective resolutions. Then there is a projective resolution of the form* $(P \oplus Q)_* \longrightarrow B \to 0$ *and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longleftarrow & P_* & \longleftarrow & (P \oplus Q)_* & \longleftarrow & Q_* & \longleftarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longleftarrow & A & \longleftarrow & B & \longleftarrow & C & \longleftarrow & 0 \end{array}$$

Proof. The extension is classified by an element of $\mathrm{Hom}_{\mathcal{D}^\flat(R)}(P_*, Q[-1]_*)$ corresponding to a chain map $\partial_*\colon P_* \longrightarrow Q[-1]_*$. Viewed as a sequence of maps $\partial_n\colon P_n \longrightarrow Q[-1]_{n-1}$, ∂_* must satisfy

$$\mathrm{d}_Q\, \partial_n + \partial_{n-1}\, \mathrm{d}_P = 0 \quad (n \geqslant 1). \tag{3.5}$$

The formula

$$\mathrm{d}(x, y) = (\mathrm{d}\, x, \partial_n x + \mathrm{d}_Q\, y) \quad (x \in P_n,\ y \in Q_n)$$

defines the differential in $(P \oplus Q)_*$. □

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