Vertex operators in algebraic topology

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Introduction

This paper is intended for two rather different audiences. First we aim to provide algebraic topologists with a timely introduction to some of the algebraic ideas associated with vertex operator algebras. Second we try to demonstrate to algebraists that many of the constructions involved in some of the most familiar vertex operator algebras have topological (and indeed geometric) significance. We hope that both of these mathematical groups will benefit from recognition of their links in this area. Rather than simply attempting to survey the area, we have reworked some aspects to emphasise integrality and other algebraic features that are less well documented in the literature on vertex operator algebras, but probably well understood by experts.

The notion of a *vertex operator algebra* is due to R. Borcherds and arose in the algebraicization of structures first uncovered in the context of Conformal Field Theory and representations of infinite dimensional Lie algebras and groups. A spectacular example is provide by the *Monster vertex operator algebra,* $V^\natural$, whose automorphism group is the Monster simple group $\mathbb{M}$. As well as the book of Frenkel, Lepowsky and Meurman **[5]**, the paper of Dong **[2]** and the memoir of Frenkel, Y.-Z. Huang, J. Lepowsky **[4]**, provide algebraic details on vertex operator algebras, and we take these as basic references.

The work of **[6]** already gives a hint that there is an 'integral' structure underlying some of the algebraic aspects of Conformal Field Theory. In this paper we will show that there are integral (at least after inverting 2) structures within some of the most basic examples of vertex operator algebras associated to positive definite even lattices. We will also interpret such algebras in terms of the (co)homology of spaces related to the classifying space of K-theory. In future work we will further clarify the topological connections by explaining their origins in the geometry of certain free loop spaces as described in the work of Pressley and Segal **[9]**,**[8]**.

Although our topological interpretation of vertex operator algebras involves *homology,* it could just as easily (and perhaps more naturally) be given in terms of *cohomology.* We could even describe such structures in generalized (co)homology theories, particularly complex oriented theories. There is some evidence that vertex operators may usefully be viewed as giving rise to families of unstable operations in such theories, perhaps leading to algebraic generalizations of vertex operator algebras appropriate to the study of some important examples, and we intend to consider these issues in future work.

For the benefit of topologists we note that vertex operator algebras have appeared in work of H. Tamanoi and others in connection with elliptic genera and elliptic cohomology as well the study of loop spaces and particularly loop groups. Indeed it is possible that a geometric model of elliptic cohomology will involve vertex operator algebras and their modules.

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§1 Vertex operator algebras and their modules

Let $\Bbbk$ be a field of characteristic 0. Let $\mathcal{V} = \mathcal{V}_\bullet$ denote a $\mathbb{Z}$-graded vector space over $\Bbbk$; following **[5]**, we denote the nth grading by $\mathcal{V}_{(n)}$ and for $v \in \mathcal{V}_{(n)}$ we refer to n as the *weight* of v and write $\operatorname{wt} v = n$. Whenever we refer to elements of $\mathcal{V}$, we always assume that they are homogeneous. Suppose that there is a $\Bbbk$-linear map

$$\mathrm{Y}(\ ,z)\colon \mathcal{V} \longrightarrow \operatorname{End}_{\Bbbk}(\mathcal{V})[[z,z^{-1}]],$$

where for any abelian group M,

$$M[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} m_n z^n : m_n \in M \right\}.$$

We write

$$\mathrm{Y}(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1},$$

where $v_n \in \operatorname{End}_{\Bbbk}(\mathcal{V})$, $v_n u = v_n(u)$ and

$$\mathrm{Y}(v,z)u = \sum_{n\in\mathbb{Z}} (v_n u) z^{-n-1}.$$

The pair $(\mathcal{V},\mathrm{Y})$ gives rise to a *vertex operator algebra* if the following axioms are satisfied.

VOA-1 For each $n \in \mathbb{Z}$, $\dim_{\Bbbk} \mathcal{V}_{(n)} < \infty$.
VOA-2 For $n \ll 0$, $\dim_{\Bbbk} \mathcal{V}_{(n)} = 0$.
VOA-3 Given elements $u, v \in \mathcal{V}$, $u_n v = 0$ for $0 \ll n$.
VOA-4 There are two distinguished elements $\mathbf{1}, \boldsymbol{\omega} \in \mathcal{V}$ and a rational number $\operatorname{rank} \mathcal{V}$. We set $\boldsymbol{\omega}_{n+1} = \mathrm{L}_n$.

VOA-5 For any $v \in \mathcal{V}$, we have the identities

$$\begin{aligned}\mathrm{Y}(\mathbf{1}, z) &= \mathrm{Id}_{\mathcal{V}};\\ \mathrm{Y}(v, z)\mathbf{1} &\in \mathcal{V}[[z]];\\ \mathrm{Y}(v, 0)\mathbf{1} &= \lim_{z \longrightarrow 0} \mathrm{Y}(v, z)\mathbf{1} = v.\end{aligned}$$

VOA-6 The following identity amongst operator valued Laurent series in the variables z_0, z_1, z_2 holds:

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) \mathrm{Y}(u, z_1)\mathrm{Y}(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) \mathrm{Y}(v, z_2)\mathrm{Y}(u, z_1) \\ = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \mathrm{Y}(\mathrm{Y}(u, z_0)v, z_2),\end{aligned}$$

where the expansion of the Dirac function δ will be discussed below.

VOA-7 The elements L_n (as operators on $\mathcal{V}$) satisfy

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n} + \frac{(m^3 - m)}{12}(\operatorname{rank} \mathcal{V})\delta_{m+n,0}.$$

VOA-8 For $v \in \mathcal{V}_{(n)}$,

$$\begin{aligned}\mathrm{L}_0 v &= nv = (\operatorname{wt} v)v,\\ \mathrm{L}_n \mathbf{1} &= 0 \quad \text{if } n \geqslant -1,\\ \mathrm{L}_{-2}\mathbf{1} &= \boldsymbol{\omega},\\ \mathrm{L}_0\boldsymbol{\omega} &= 2\boldsymbol{\omega}.\end{aligned}$$

VOA-9 As formal series in z,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Y}(v, z) &= \mathrm{Y}(\mathrm{L}_{-1}v, z),\\ [\mathrm{L}_{-1}, \mathrm{Y}(v, z)] &= \mathrm{Y}(\mathrm{L}_{-1}v, z),\\ [\mathrm{L}_0, \mathrm{Y}(v, z)] &= \mathrm{Y}(\mathrm{L}_0 v, z) + z\mathrm{Y}(\mathrm{L}_{-1}v, z).\end{aligned}$$

These axioms are essentially those of **[3]**,**[5]**. We use the notation

$$(\mathcal{V}_\bullet, \mathrm{Y}, \mathbf{1}, \boldsymbol{\omega}, \operatorname{rank} \mathcal{V})$$

to denote such a vertex operator algebra, often just writing $(\mathcal{V}_\bullet, \mathrm{Y}, \mathbf{1}, \boldsymbol{\omega})$ or even $(\mathcal{V}_\bullet, \mathrm{Y})$. For each element $v \in \mathcal{V}$, the series $\mathrm{Y}(v, z)$ is called the *vertex operator* corresponding to v. The operators L_n are called the *Virasoro operators* and generate an action of the so-called *Virasoro algebra* on the vertex operator algebra. Provided $\operatorname{rank} \mathcal{V} \neq 0$, this action implies that $\dim_{\Bbbk} \mathcal{V}_{(n)} \neq 0$ infinitely often (or equivalently that $\mathcal{V}_\bullet$ is infinite dimensional); this is enough to ensure that a vertex operator algebra is non-trivial, and indeed all examples are complicated to construct.

To expand the Dirac function δ referred to in VOA-6, we define

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n, \tag{1.1}$$

and for three variables z_0, z_1, z_2,

$$\delta\left(\frac{z_1 - z_2}{z_0}\right) = \sum_{n\in\mathbb{Z}} z_0^{-n} z_1^{n}\left(1 - \frac{z_2}{z_1}\right)^{n} \tag{1.2}$$
$$= \sum_{n\in\mathbb{Z}}\sum_{0\leqslant k}(-1)^k\binom{n}{k} z_0^{-n} z_1^{n-k} z_2^{k}.$$

In other words, we expand in terms of the *second* variable in the numerator of the argument.

From **[2]** and **[5]**, we also record the definition of a *module* $(\mathcal{M}, \mathrm{Y}_{\mathcal{M}})$ *over a vertex operator algebra* $(\mathcal{V}, \mathrm{Y}, \mathbf{1}, \boldsymbol{\omega}, \operatorname{rank}\mathcal{V})$. This consists of a $\mathbb{Q}$-graded $\Bbbk$-module $\mathcal{M} = \mathcal{M}_\bullet$ together with a $\Bbbk$-linear map

$$\begin{aligned}\mathrm{Y}_{\mathcal{M}}\colon \mathcal{V} &\longrightarrow \operatorname{End}(\mathcal{M})[[z, z^{-1}]];\\ v &\longmapsto \mathrm{Y}_{\mathcal{M}}(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1},\end{aligned}$$

satisfying the following conditions.

VOM-1 For each $n \in \mathbb{Q}$, $\dim_{\Bbbk}\mathcal{M}_{(n)} < \infty$.

VOM-2 For $n \ll 0$, $\dim_{\Bbbk}\mathcal{M}_{(n)} = 0$.

VOM-3 Given elements $u, v \in \mathcal{V}$, $u_n v = 0$ for $0 \ll n$.

VOM-4 We have the identity

$$\mathrm{Y}_{\mathcal{M}}(\mathbf{1}, z) = \mathrm{Id}_{\mathcal{M}};$$

VOM-5 The following identity amongst Laurent series in z_0, z_1, z_2 holds.

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\mathrm{Y}_{\mathcal{M}}(u, z_1)\mathrm{Y}_{\mathcal{M}}(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\mathrm{Y}_{\mathcal{M}}(v, z_2)\mathrm{Y}_{\mathcal{M}}(u, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\mathrm{Y}_{\mathcal{M}}(\mathrm{Y}_{\mathcal{V}}(u, z_0)v, z_2).$$

VOM-6 The elements $\mathrm{L}_n = \boldsymbol{\omega}_{n+1}$ (as operators on $\mathcal{M}$) satisfy

$$[\mathrm{L}_m, \mathrm{L}_n] = (m-n)\mathrm{L}_{m+n} + \frac{(m^3 - m)}{12}(\operatorname{rank}\mathcal{V})\delta_{m+n,0}.$$

VOM-7 For $w \in \mathcal{V}_{(n)}$,

$$\mathrm{L}_0 w = nw = (\operatorname{wt} w)w.$$

VOM-8 As formal series in z,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Y}(v, z) &= \mathrm{Y}(\mathrm{L}_{-1}v, z),\\ [\mathrm{L}_{-1}, \mathrm{Y}(v, z)] &= \mathrm{Y}(\mathrm{L}_{-1}v, z),\\ [\mathrm{L}_0, \mathrm{Y}(v, z)] &= \mathrm{Y}(\mathrm{L}_0 v, z) + z\mathrm{Y}(\mathrm{L}_{-1}v, z).\end{aligned}$$

Actually, this ‘naive’ notion of module is insufficient for many purposes and there is a more general one of a *twisted module,* which seems to occur in particular in connection with bundles over loop spaces and the elliptic cohomology of non-simply connected spaces.

§2 The homology of the classifying space for K-theory

We take as a general reference for the topology of this section the classic work of Adams **[1]**, which contains full details on many of the points we mention.

We begin with the simplest case, which provides a starting point for several generalizations. We assume throughout that $\Bbbk$ is a commutative, unital ring and $H_*(\ )$ denotes the ordinary homology functor $H_*(\ ;\Bbbk)$.

We recall the space $BU\times\mathbb{Z}$ classifying the K-theory functor

$$KU^0(\ )\cong[\quad;BU\times\mathbb{Z}].$$

In homology we have

$$H_*(BU\times\mathbb{Z})=H_*(BU)[[1],[-1]],$$

where $[n]$ denotes the component of $BU\times\mathbb{Z}=\coprod_{n\in\mathbb{Z}}BU\times\{n\}$, and we identify BU with $BU\times\{0\}$. We follow the notation of Ravenel and Wilson **[7]** by writing $[n]=[1]^n$. This homology ring is a Laurent polynomial ring over the ring $H_*(BU)$ on the generator $[1]$.

We assign the following new grading: an element $x\in H_{2n}(BU)$ is assigned weight n and the element $x[m]$ is assigned weight m^2+n. With this grading, $H_*(BU\times\mathbb{Z})$ is no longer a graded ring. We will write $H_\bullet(BU\times\mathbb{Z})$ to emphasise this regrading.

Now we recall the structure of the bicommutative Hopf algebra $H_*(BU)$. There are standard algebra generators $b_n\in H_{2n}(BU)$ (we set $b_0=1$), coming from complex projective space under the natural embedding in homology. These span a binomial coalgebra, thus their generating function $b(T)=\sum_{n\geqslant 0}b_nT^n$ is a grouplike element of $H_*(BU)[[T]]$. There are also the primitive elements $p_n\in H_{2n}(BU)$, satisfying the Newton recurrence relation

$$\begin{aligned}p_1&=b_1,\\ p_n&=b_1p_{n-1}-b_2p_{n-2}+b_3p_{n-3}-\cdots+(-1)^{n-2}b_{n-1}p_1+(-1)^{n-1}nb_n.\end{aligned}\tag{2.1}$$

When $\Bbbk$ is a $\mathbb{Q}$-algebra, this is equivalent to the generating function identity

$$\ln b(T)=\sum_{n\geqslant 1}\frac{(-1)^{n-1}p_n}{n}T^n,$$

where $\ln(1+Z)=\sum_{n\geqslant 1}(-1)^{n-1}Z^n/n$ is the formal logarithmic series. The primitive submodule of $H_{2n}(BU)$ is generated by p_n, but these elments are not algebra generators of $H_*(BU)$ over a general ring $\Bbbk$, in particular over integers $\mathbb{Z}$. They do however generate over the rationals $\mathbb{Q}$, which accounts for the fact that in **[5]** they are used in giving explicit formulæ for vertex operators.

Dually, in $H^*(BU)$ we have the universal Chern classes $c_n\in H^{2n}(BU)$, which are also polynomials generators and generate a binomial coalgebra by the Cartan formula. The duality is given by the formula

$$\langle c_n, b_1^{r_1}\cdots b_k^{r_k}\rangle = \begin{cases} 1 & \text{if } b_1^{r_1}\cdots b_k^{r_k} = b_1^n, \\ 0 & \text{else.}\end{cases}$$

The primitive submodule in $H^{2n}(BU)$ has generator s_n satisfying the recurrence relation

$$\begin{aligned} s_1 &= c_1, \\ s_n &= c_1 s_{n-1} - c_2 s_{n-2} + c_3 s_{n-3} - \cdots + (-1)^{n-2} c_{n-1} s_1 + (-1)^{n-1} n c_n. \end{aligned} \tag{2.2}$$

We also have the duality formula

$$\langle s_n, b_1^{r_1}\cdots b_k^{r_k}\rangle = \begin{cases} 1 & \text{if } b_1^{r_1}\cdots b_k^{r_k} = b_n, \\ 0 & \text{else.}\end{cases}$$

We will find it convenient to use the *total symmetric functions* h_n, recursively defined by

$$\sum_{0\leqslant k\leqslant n} (-1)^k c_{n-k} h_k = 0,$$

and also satisfying the recurrence relation

$$\begin{aligned} s_1 &= h_1, \\ s_n &= n h_n - (h_1 s_{n-1} + h_2 s_{n-2} + h_3 s_{n-3} + \cdots + h_{n-1} s_1). \end{aligned} \tag{2.3}$$

These formulæ combine to give the following remarkable result.

Theorem 2.1. *There is an isomorphism of (graded) Hopf algebras*

$$\begin{aligned} \Phi\colon H_*(BU) &\longrightarrow H^*(BU); \\ \Phi(b_n) &= h_n \end{aligned}$$

under which $\Phi(p_n) = (-1)^{n-1} s_n$.

Let A be a Hopf algebra over a ring $\Bbbk$ and let $A^* = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$ be its dual. Then there is a $\Bbbk$-algebra homomorphism

$$\begin{aligned} A^* &\longrightarrow \mathrm{End}_{\Bbbk}(A); \\ \alpha &\longmapsto \alpha\cdot\,, \end{aligned}$$

where

$$\alpha\cdot a = \sum \alpha(a')a'',$$

with $\sum a'\otimes a''$ denoting the coproduct on a. This gives a canonical action of A^* on A.

In the case where $A = H_*(BU)$, and $A^* = H^*(BU)$, this action agrees with the *cap product* action of cohomology on homology. Thus, for $u \in H^{2m}(BU)$ and $x \in H_{2n}(BU)$ we have $u\cdot x = u \cap x$.

Now given a $\Bbbk$-algebra homomorphism $\varphi\colon A \longrightarrow A^*$, we have an induced action of A on itself given by

$$a\cdot x = \varphi(a)\cdot x, \quad \text{for } a, x \in A.$$

Taking the case of $A = H_*(BU)$, we obtain the action

$$u \cdot x = \Phi(u) \cap x.$$

Proposition 2.3. *We have the following formulæ.*

1) *For* $a, b, x \in H_*(BU)$, $(ab) \cdot x = a \cdot (b \cdot x)$. *Hence* $\cdot$ *is a left action of* $H_*(BU)$ *on itself.*
2) *For the primitives* p_n $(n > 0)$,
$$p_m \cdot p_n = n\delta_{m,n}.$$
3) *For the standard generators* b_k,
$$b_m \cdot b_n = h_m \cap b_n = \begin{cases} b_0 = 1 & \textit{if } m = n, \\ b_{n-m} & \textit{if } 0 \leqslant m < n, \\ 0 & \textit{otherwise.} \end{cases}$$
In terms of the generating function $b(T)$, *this formula is equivalent to*
$$b(X) \cdot b(Y) = (1 - XY)^{-1} b(Y).$$

In the statement of part (2), the Kronecker symbol $\delta_{m,n}$ is determined by

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

We can now describe a vertex operator algebra associated to $H_\bullet(BU \times \mathbb{Z})$ which we take as the underlying $\Bbbk$-module. We will define a linear map

$$\mathrm{Y}(\ \ , z) \colon H_\bullet(BU \times \mathbb{Z}) \longrightarrow \mathrm{End}(H_\bullet(BU \times \mathbb{Z}))[[z, z^{-1}]]$$

with the relevant properties. The indeterminate z may be interpreted as indexing copies of $BU \times \mathbb{Z}$ corresponding to the spaces classifying the functors $KU^{-2k}(\ )$. Thus we can consider the *graded space*

$$BU \times \mathbb{Z}[z, z^{-1}] = \{BU \times \mathbb{Z}\{z^k\}\}_{k \in \mathbb{Z}},$$

where

$$KU^{-2k}(\ ) \cong [\quad ; BU \times \mathbb{Z}\{z^k\}] \xrightarrow[\cong]{t^k} KU^0(\ ),$$

with the latter being multiplication by the kth power of the Bott element. We can therefore view z as t^{-1} and each vertex operator $\mathrm{Y}(u, z)$ as an element of the graded tensor product

$$H_\bullet(BU \times \mathbb{Z}[z, z^{-1}]) \widehat{\otimes}_{\Bbbk} H^\bullet(BU \times \mathbb{Z}) = H_\bullet(BU \times \mathbb{Z}) \widehat{\otimes}_{\Bbbk} H^\bullet(BU \times \mathbb{Z})[[z, z^{-1}]],$$

which has an action on $H_\bullet(BU \times \mathbb{Z})$ induced by

$$(a \otimes c) \cdot x = a(c \cap x).$$

Using this interpretation, the *normal ordering* convention found in **[5]** becomes the usual commutation rule in the tensor product.

We begin by defining the operators $\mathrm{Y}([a],z)$ for $a\in\mathbb{Z}$. Set

$$\mathrm{Y}([a],z)=[a]b(z)^a\otimes h\,(-1/z)^{2a}\,z^{[a]},$$

where $z^{[a]}$ is given by linearly extending the formula

$$z^{[a]}(x[n])=x[n]z^{2an}\quad\text{for } x\in H_*(BU) \text{ and } n\in\mathbb{Z}.$$

This is essentially the formula found in [**5**, section 7.1]. Next we must explain how to define the more general vertex operators $\mathrm{Y}(x[n],z)$ for arbitrary elements $x\in H_*(BU)$ and $n\in\mathbb{Z}$. Again this is done in [**5**, section 8.5], at least on the assumption that the ground ring $\Bbbk$ is a rational algebra. Our description is valid for any ground ring $\Bbbk$.

By linearity, it suffices to describe the elements $\mathrm{Y}(b_{r_1}\cdots b_{r_k}[a],z)$. This is done using the generating function

$$b(w_1)\cdots b(w_k)=\sum_{r_j\geqslant 0} b_{r_1}\cdots b_{r_k}w_1^{r_1}\cdots w_k^{r_k},$$

and the formula

$$\begin{aligned}(2.4)\quad &\mathrm{Y}(b(w_1)\cdots b(w_k)[a],z)\\ &=b(z+w_1)\cdots b(z+w_k)\otimes h(-1/(z+w_1))\cdots h(-1/(z+w_k))z^{[k]}\mathrm{Y}([a],z)\\ &=[a]b(z+w_1)\cdots b(z+w_k)b(z)^a\otimes h(-1/(z+w_1))\cdots\\ &\qquad\qquad\cdots h(-1/(z+w_k))h(-1/z)^{2a}z^{[a+k]}.\end{aligned}$$

To determine $\mathrm{Y}(b_{r_1}\cdots b_{r_k}[a],z)$, we read off the term in the monomial $w_1^{r_1}\cdots w_k^{r_k}$ appearing in the expansion of $\mathrm{Y}(b(w_1)\cdots b(w_k)[a],z)$ in terms of the variable z, using Equation 2.4. Such 'preferential' treatment of certain variables lies at the heart of the formal variable calculations of [**5**] and is related to normal ordering and time ordering in Quantum Field Theory.

Given the structure we have described we obtain the following result. The proof is essentially given in [**5**], where the case of a $\mathbb{Q}$-algebra $\Bbbk$ is covered; for the general case we need to verify that our integral formulæ agree with their rational results. Set

$$\begin{aligned}\mathcal{V}(\mathbb{Z})_\bullet&=H_\bullet(BU\times\mathbb{Z}),\\ \mathrm{Y}_{\mathbb{Z}}&=\mathrm{Y},\\ \mathbf{1}&=[0],\\ \boldsymbol{\omega}&=\frac{1}{2}b_1^2.\end{aligned}$$

Theorem 2.3. *The quintuple $(\mathcal{V}(\mathbb{Z})_\bullet,\mathrm{Y}_{\mathbb{Z}},\mathbf{1},\boldsymbol{\omega},1)$ is a vertex operator algebra over any $\mathbb{Z}[1/2]$-algebra $\Bbbk$.*

Remark 2.4. The attentive reader will have noticed the appearance of factors of 2 in some of the above formulæ. This is not an accident and corresponds to the fact that we are viewing the integers as the root lattice $\mathbb{A}_1$ of the simple Lie algebra $\mathfrak{sl}_2$. In Section 3 we will make this more explicit. It is perhaps worth remarking that we could use the more 'natural' inner product $\langle a, b\rangle = ab$ to define an algebraic object which is an example of a somewhat weaker notion, that of a *quasi-vertex operator algebra,* as defined in **[4]** and possessing most of the properties of a vertex operator algebra except for those involving the Virasoro operators L_n, which are only required to occur for $n = 0, \pm 1$.

§3 Vertex operator algebras based on positive, even lattices

In this section we generalize the vertex operator algebra construction of Section 2 starting with a positive definite, even lattice $(\mathbb{L}, \langle\ ,\ \rangle)$ in place of the integers $\mathbb{Z}$. Thus, $\mathbb{L}$ is a finite rank free abelian group equipped with a positive definite inner product

$$\langle\ ,\ \rangle : \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{Z}$$

which is *even* in the sense that for $\ell \in \mathbb{L}$,

$$\langle \ell, \ell \rangle \in 2\mathbb{Z}.$$

Following **[5]**, we may construct a vertex operator algebra $(\mathcal{V}(\mathbb{L}), \mathrm{Y}_{\mathbb{L}})$ for which the underlying $\Bbbk$-module is

$$\mathcal{V}(\mathbb{L}) = \Bbbk\{\mathbb{L}\} \underset{\Bbbk}{\otimes} \mathcal{S}(\mathbb{L}).$$

Here $\Bbbk\{\mathbb{L}\}$ is the free $\Bbbk$-module on the elements of $\mathbb{L}$, given the weight grading for which $\operatorname{wt} \ell = \langle \ell, \ell \rangle / 2$. Also the factor $\mathcal{S}(\mathbb{L})$ is a infinite tensor product

$$\mathcal{S}(\mathbb{L}) = \bigotimes_{k \geqslant 1} \Bbbk[b(\ell)_k : \ell \in \mathbb{L}]/(\text{relations})$$

where the symbols $b(\ell)_k$ stand for elements of weight k satisfying relations

$$b(\ell_1 + \ell_2)_k = \sum_{0 \leqslant j \leqslant k} b(\ell_1)_j b(\ell_2)_{k-j}.$$

Thus, if $\mathbb{L}$ has a basis $\ell_1, \ldots, \ell_{\operatorname{rank} \mathbb{L}}$, we have

$$\mathcal{S}(\mathbb{L}) = \Bbbk[b(\ell_1)_k, \ldots, b(\ell_{\operatorname{rank} \mathbb{L}})_k : k \geqslant 1].$$

The recursion of Equation 2.1 can be used to define elements $p(\ell)_k$ for which

$$\begin{aligned}
p(\ell)_1 &= b(\ell)_1, \\
p(\ell)_n &= b(\ell)_1 p(\ell)_{n-1} - b(\ell)_2 p(\ell)_{n-2} + b(\ell)_3 p(\ell)_{n-3} - \cdots \\
&\qquad + (-1)^{n-2} b(\ell)_{n-1} p(\ell)_1 + (-1)^{n-1} n b(\ell)_n.
\end{aligned} \tag{3.1}$$

Our $p(\ell)_k$ is essentially equivalent to the element denoted $\ell(-k)$ in **[5]**. We will use the notation

$$b(T)^{\ell} = \sum_{k \geqslant 0} b(\ell)_k T^k.$$

We define elements $c(\ell)_k, h(\ell)_k$ in the dual of $\mathcal{S}(\mathbb{L})$ by requiring that they satisfy the formulæ

$$c(X)^\ell = \sum_{k\geqslant 0} c(\ell)_k X^k,$$
$$\left\langle c(X)^\ell, b(Y)^{\ell'} \right\rangle = (1-XY)^{\langle \ell,\ell' \rangle},$$

and

$$h(X)^\ell = \sum_{k\geqslant 0} h(\ell)_k X^k,$$
$$\left\langle h(X)^\ell, b(Y)^{\ell'} \right\rangle = (1-XY)^{-\langle \ell,\ell' \rangle}.$$

and form divided power sequences under the coproduct, i.e., for any $x, y \in \mathcal{S}(\mathbb{L})$,

$$\left\langle c(X)^\ell, xy \right\rangle = \left\langle c(X)^\ell, x \right\rangle \left\langle c(X)^\ell, y \right\rangle$$

and

$$\left\langle h(X)^\ell, xy \right\rangle = \left\langle h(X)^\ell, x \right\rangle \left\langle h(X)^\ell, y \right\rangle.$$

Using these together with the approach of Section 2, we have an action of $\mathcal{S}(\mathbb{L})$ on itself by

$$b(X)^\ell \cdot b(Y)^{\ell'} = (1-XY)^{-\langle \ell,\ell' \rangle} b(Y)^{\ell'}. \tag{3.2}$$

We can inflict K-theory with coefficients in $\mathbb{L}$ by forming the even degree K-theory functors

$$KU\mathbb{L}^{-2k}(\ ) = KU^{-2k}(\ ) \underset{\mathbb{Z}}{\otimes} \mathbb{L}.$$

These are all represented by a space

$$BU\mathbb{L} \times \mathbb{L},$$

where $BU\mathbb{L}$ is connected. Of course, given a basis $\{\ell_1, \ldots, \ell_{\operatorname{rank}\mathbb{L}}\}$ for $\mathbb{L}$, we obtain a decomposition (of infinite loop spaces)

$$BU\mathbb{L} \cong BU \times \cdots \times BU,$$

with one factor for each ℓ_j. The homology of this classifying space is of form

$$H_*(BU\mathbb{L} \times \mathbb{L}, \Bbbk) \cong H_*(BU\mathbb{L})[[\ell_j], [\ell_j]^{-1} : 1 \leqslant j \leqslant \operatorname{rank}\mathbb{L}],$$

where

$$H_*(BU\mathbb{L}) = H_*(BU) \circ [\ell_1] \underset{\Bbbk}{\otimes} \cdots \underset{\Bbbk}{\otimes} H_*(BU) \circ [\ell_{\operatorname{rank}\mathbb{L}}].$$

Here the notation $H_*(BU) \circ [\ell]$ denotes the homology of the space

$$BU \times \{[\ell]\} \subset BU\mathbb{L} \times \{[\ell]\}.$$

This notation is suggested by the Hopf ring notation of [7]; we are working with a 'Hopf module' $H_*(BU\mathbb{L}\times\mathbb{L})$ over the Hopf ring $H_*(BU\times\mathbb{Z})$. We will frequently write ℓ in place of $[\ell]$ when there is no chance of confusion.

We now proceed to describe the vertex operators for this example. For $\ell\in\mathbb{L}$, let z^ℓ denote the operator for which

$$z^\ell\cdot(x[\ell']) = x[\ell']z^{\langle\ell,\ell'\rangle},$$

where $\ell'\in\mathbb{L}$ and $x\in H_*(BU\mathbb{L})$. We make the following definitions:

$$\mathrm{Y}([\ell],z) = [\ell]b(z)^\ell\otimes h\left(-1/z\right)^\ell z^\ell,$$

and

$$\begin{aligned}(3.3)\quad & \mathrm{Y}(b(w_1)^{\ell_1}\cdots b(w_k)^{\ell_k}[\ell],z)\\ &= b(z+w_1)^{\ell_1}\cdots b(z+w_k)^{\ell_k}\otimes\\ &\qquad\qquad h(-1/(z+w_1))^{\ell_1}\cdots h(-1/(z+w_k))^{\ell_k}z^{\ell_1+\cdots+\ell_k}\mathrm{Y}([\ell],z)\\ &= [\ell]b(w_1+z)^{\ell_1}\cdots b(w_k+z)^{\ell_k}b(z)^\ell\otimes\\ &\qquad\qquad h(-1/(w_1+z))^{\ell_1}\cdots h(-1/(w_k+z))^{\ell_k}h\left(-1/z\right)^\ell z^{\ell_1+\cdots+\ell_k+\ell}.\end{aligned}$$

Also set

$$\begin{aligned}\mathcal{V}(\mathbb{L})_\bullet &= H_\bullet(BU\mathbb{L}\times\mathbb{L}),\\ \mathrm{Y}_\mathbb{L} &= \mathrm{Y},\\ \mathbf{1} &= [0],\\ \boldsymbol{\omega} &= \sum_{1\leqslant j\leqslant\operatorname{rank}\mathbb{L}}\frac{1}{2\langle\ell_j,\ell_j\rangle}b(\ell_j)_1^2,\end{aligned}$$

where $\{\ell_1,\dots,\ell_{\operatorname{rank}\mathbb{L}}\}$ is a basis for $\mathbb{L}$.

Theorem 3.1. *The quintuple $(\mathcal{V}(\mathbb{L})_\bullet,\mathrm{Y}_\mathbb{L},\mathbf{1},\boldsymbol{\omega},\operatorname{rank}\mathbb{L})$ is a vertex operator algebra over any $\mathbb{Z}[1/2]$-algebra $\Bbbk$.*

Again, the proof is essentially given in [5].

The case discussed in Section 2 amounts to taking the lattice $\mathbb{L}=\mathbb{Z}\sqrt{2}$, rather than $\mathbb{Z}$ itself. This is the root lattice $\mathbb{A}_1$, and root lattices of semi-simple Lie algebras provide many interesting examples.

§4 Some modules over vertex operator algebras

In this section we briefly describe some modules over the vertex operator algebras constructed earlier, including their irreducibles.

Again we assume the data of Section 3. In general, the linear map $\mathbb{L}\longrightarrow\mathrm{Hom}_\mathbb{Z}(\mathbb{L},\mathbb{Z})$ given by

$$\ell\longmapsto\ell^*=\langle\ell,\ \rangle$$

is injective but not surjective; if it *is* an isomorphism then the lattice $\mathbb{L}$ is said to be *unimodular.* We also set

$$\mathbb{L}^0 = \{\ell \in \mathbb{Q} \otimes \mathbb{L} : \forall \ell' \in \mathbb{L},\ \langle \ell, \ell' \rangle \in \mathbb{Z}\} \supseteq \mathbb{L}.$$

The index $|\mathbb{L}^0/\mathbb{L}|$ is of course finite and equal to 1 if and only if $\mathbb{L}$ is unimodular.

We may generalize the definition of the vertex operator algebra

$$(\mathcal{V}(\mathbb{L})_\bullet, \mathrm{Y}_\mathbb{L}, \mathbf{1}, \boldsymbol{\omega}, \operatorname{rank} \mathbb{L})$$

as follows. We replace the set of components $\mathbb{L}$ of $BU\mathbb{L} \times \mathbb{L}$ by a coset $\mathbb{L} + \ell$ of $\mathbb{L}$ in $\mathbb{L}^0$. Thus we take

$$\mathcal{V}(\mathbb{L} + \ell) = \Bbbk\{\mathbb{L} + \ell\} \underset{\Bbbk}{\otimes} \mathcal{S}(\mathbb{L}),$$

where the $\Bbbk$-module is free on the elements of the coset $\mathbb{L}+\ell$. We grade $\mathcal{V}(\mathbb{L}+\ell)$ by decreeing that $\operatorname{wt} \ell = \langle \ell, \ell \rangle / 2$ for $\ell \in \mathbb{L} + \ell$ and extending this to the whole of $\mathcal{V}(\mathbb{L} + \ell)$. Of course, this is now a grading over $\mathbb{Q}$ rather than just $\mathbb{Z}$.

Theorem 4.1. *The graded $\Bbbk$-module admits the structure of a module $(\mathcal{V}(\mathbb{L} + \ell), \mathrm{Y})$ over the vertex operator algebra $(\mathcal{V}(\mathbb{L})_\bullet, \mathrm{Y}_\mathbb{L}, \mathbf{1}, \boldsymbol{\omega}, \operatorname{rank} \mathbb{L})$. In the case of $\mathbb{L} + \ell = \mathbb{L}$, this agrees with the adjoint module (i.e., the natural module structure of a vertex operator algebra over itself).*

These modules may be combined into a single $\mathcal{V}(\mathbb{L})$-module

$$\mathcal{V}(\mathbb{L}^0) = \bigoplus_{\mathbb{L}+\ell \in \mathbb{L}^0/\mathbb{L}} \mathcal{V}(\mathbb{L} + \ell)$$

graded by $\mathbb{L}^0/\mathbb{L}$.

As a particular example, we may consider the situation of Section 2, in which we are taking $\mathbb{L} = \mathbb{Z}\sqrt{2}$. Then $\mathbb{L}^0 = \mathbb{Z}(1/\sqrt{2})$ and

$$\mathbb{L}^0/\mathbb{L} = \{\mathbb{L}, \mathbb{L} + (1/\sqrt{2})\}.$$

The following result of Dong **[2]** explains the significance of these examples.

Theorem 4.2. *Over the vertex operator algebra $(\mathcal{V}(\mathbb{L})_\bullet, \mathrm{Y}_\mathbb{L}, \mathbf{1}, \boldsymbol{\omega}, \operatorname{rank} \mathbb{L})$, each of the modules $(\mathcal{V}(\mathbb{L} + \ell), \mathrm{Y})$ $(\mathbb{L} + \ell \in \mathbb{L}^0/\mathbb{L})$ is irreducible; moreover, every irreducible module is isomorphic to precisely one of these.*

For the root lattice $\mathbb{A}_n$ of $\mathfrak{sl}_{n+1}$, $\mathbb{A}_n^0/\mathbb{A}_n \cong \mathbb{Z}/(n+1)$; hence there are exactly $n+1$ irreducibles. These are indexed by the fundamental weights associated to the $n+1$ nodes in the extended Dynkin diagram for $\mathbb{A}_n$ which has a rotational symmetry of order $n+1$; this symmetry also appears in the vertex operator algebra $\mathcal{V}(\mathbb{A}_n)$ and has the effect of twisting these modules by shifting the indexing element of $\mathbb{Z}/(n+1)$ by 1 modulo $(n+1)$.

In fact, it is possible to incorporate all such modules into a single $\mathbb{Z}/(n+1)$-graded object which possesses the algebraic structure of what has been called an *Abelian intertwining algebra* by Dong and Lepowsky **[3]**. In terms of the underlying geometry of **[8]** this is probably the most natural algebraic object to study.

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