RATIONAL HOMOLOGY SPHERES AND THE FOUR-BALL GENUS OF KNOTS

BRENDAN OWENS AND SAŠO STRLE

Abstract. Using the Heegaard Floer homology of Ozsváth and Szabó we investigate obstructions to a rational homology sphere bounding a four-manifold with a definite intersection pairing. As an application we obtain new lower bounds for the four-ball genus of Montesinos links.

1. Introduction

Let Y be a rational homology three-sphere and X a smooth negative-definite four-manifold bounded by Y. For any spin^c structure $\mathfrak{t}$ on Y let $d(Y,\mathfrak{t})$ denote the correction term invariant of Ozsváth and Szabó (see [16] for the definition; this invariant is the Heegaard Floer homology analogue of the Frøyshov invariant in Seiberg-Witten theory). It is shown in [16] that for each spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$,

$$c_1(\mathfrak{s})^2 + \mathrm{rk}(H^2(X;\mathbb{Z})) \leq 4d(Y, \mathfrak{s}|_Y); \tag{1}$$

moreover, both sides of (1) are congruent modulo 8. In order to use these conditions one must study the restriction map $\mathfrak{s} \mapsto \mathfrak{s}|_Y$ from $\mathrm{Spin}^c(X)$ to $\mathrm{Spin}^c(Y)$; this map commutes with the conjugation of spin^c structures. Moreover, since $\mathrm{Spin}^c(\,\cdot\,)$ is an affine $H^2(\,\cdot\,;\mathbb{Z})$ space, the restriction map is equivariant with respect to the action of $H^2(X;\mathbb{Z})$, where this group acts on $\mathrm{Spin}^c(Y)$ through the natural group homomorphism $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$. In this paper we describe an algorithm that for a given second Betti number tests each possible four-manifold X (i.e., each possible negative-definite intersection form) to see if it can give rise to an equivariant map for which (1) and the congruence hold for each $\mathfrak{s} \in \mathrm{Spin}^c(X)$.

The algorithm in principle applies to any rational homology sphere for which the invariants $d(Y,\mathfrak{t})$ are known; this is the case for all Seifert fibered ones ([17]; see also [16] for lens spaces). We describe the situation in detail for four-manifolds X with $b_2(X) \leq 2$. Note that computations are the simplest for homology lens spaces, since in this case the number of possible equivariant maps as above is greatly reduced.

Date: November 18, 2008.

We use this algorithm to find obstructions to the four-ball genus of a link being as small as the signature allows it to be. To this end we encode the information about the link and its slice surface in a manifold pair (X, Y) as above. Specifically, for a link L in the three-sphere and its slice surface F in the four-ball, we let Y be the two-fold cover of S^3 branched along L, and X be the two-fold cover of B^4 branched along F; this is analogous to the slice obstruction of Casson-Gordon [4] and Fintushel-Stern [5]. Applying this to Montesinos links, we get some new bounds on the four-ball genus.

Alternatively, one could try to obtain a lower bound on the four-ball genus of a knot K by attaching a two-handle to B^4 along K. If K is alternating, this approach reproduces the classical bound given by the signature of K; this is reminiscent of the behaviour of the invariant $\tau(K)$ of Ozsváth and Szabó [18]. This is a purely 3-dimensional invariant defined using knot Floer homology; it gives the optimal lower bound for torus knots but agrees with the signature bound for alternating knots. By contrast our method yields new bounds for some alternating knots.

2. Four-manifolds bounded by rational homology spheres

In this section we study the relationship between a smooth four-manifold X and its boundary Y. The following is an extension of [3, Lemma 3].

Lemma 2.1. *Let Y be a rational homology sphere; denote by h the order of $H_1(Y;\mathbb{Z})$. Suppose that Y bounds X and denote by s the absolute value of the determinant of the intersection pairing on $H_2(X,\mathbb{Z})/\mathrm{Tors}$. Then $h = st^2$, where st is the order of the image of $H^2(X;\mathbb{Z})$ in $H^2(Y;\mathbb{Z})$, and t is the order of the image of the torsion subgroup of $H^2(X;\mathbb{Z})$.*

Proof. Note that for $b_2(X) > 0$, X has a non-degenerate integral intersection form
$$Q_X\colon H_2(X;\mathbb{Z})/\mathrm{Tors} \otimes H_2(X;\mathbb{Z})/\mathrm{Tors} \longrightarrow \mathbb{Z};$$
we denote the absolute value of the determinant of this pairing by s. If $b_2(X) = 0$, then set $s = 1$. The long exact sequence of the pair (X, Y) yields the following (with integer coefficients):

$$\begin{array}{ccccccccccc}
0 \longrightarrow & H^2(X,Y) & \xrightarrow{j^*} & H^2(X) & \longrightarrow & H^2(Y) & \longrightarrow & H^3(X,Y) & \longrightarrow & H^3(X) & \longrightarrow 0,\\
& \| & & \| & & & & \| & & \| & \\
& \mathbb{Z}^b \oplus T_2 & & \mathbb{Z}^b \oplus T_1 & & & & T_1 & & T_2 &
\end{array}$$

where T_1, T_2 are torsion groups, and $b = b_2(X)$ (we may assume that $b_1(X) = 0$; if not one may surger out b_1 without changing the conclusion of the lemma).

With respect to appropriate bases for (a compatible choice of) free parts of $H^2(X,Y)$ and $H^2(X)$, we have

$$j^* = \begin{pmatrix} Q & 0 \\ * & \tau \end{pmatrix},$$

where Q is the matrix representation of the intersection pairing on $H_2(X;\mathbb{Z})/\mathrm{Tors}$. Note that $\tau\colon T_2 \to T_1$ is a monomorphism; let $t = |T_1|/|T_2|$. It follows that $h = st^2$, as Q can be thought of as a presentation matrix for a group of order s. □

To state the basic relation between X and Y more explicitly, we need to understand the restriction map from spin^c structures on X to those on Y. Let $\mathcal{T}$ be the image of the torsion subgroup of $H^2(X;\mathbb{Z})$ in $\mathcal{H} := H^2(Y;\mathbb{Z})$, and let $\mathcal{S}$ be the quotient of $H^2(X;\mathbb{Z})$ by the sum of its torsion subgroup and the image of $H^2(X,Y;\mathbb{Z})$. After fixing affine identifications of $\mathrm{Spin}^c(\,\cdot\,)$ with $H^2(\,\cdot\,;\mathbb{Z})$, the restriction map from $\mathrm{Spin}^c(X)$ to $\mathrm{Spin}^c(Y)$ induces an affine monomorphism

$$\rho : \mathcal{S} \to \mathcal{H}/\mathcal{T}.$$

For appropriate choices of origins in the spaces of spin^c structures, ρ becomes a group homomorphism and the conjugation of spin^c structures, denoted by $\mathbf{j}$, corresponds to multiplication by -1. Choose an identification $\mathrm{Spin}^c(Y) \cong \mathcal{H}$ so that a spin structure corresponds to $0 \in \mathcal{H}$, and let $0 \in \mathcal{S}$ correspond to the class of a spin^c structure on X whose Chern class belongs to the sum of the torsion subgroup of $H^2(X;\mathbb{Z})$ and the image of $H^2(X,Y;\mathbb{Z})$. If the order of $\mathcal{H}$ is odd then there is a unique $\mathbf{j}$-fixed element in each of $\mathcal{S}$ and $\mathcal{H}/\mathcal{T}$ and ρ is a group homomorphism. In general, any $\mathbf{j}$-fixed element (i.e., any element of order 2) can be used as origin in $\mathcal{S}$; to make ρ a group homomorphism one needs to choose the right $\mathbf{j}$-fixed element in $\mathcal{H}/\mathcal{T}$.

We assume from now on that X is negative definite. We define two (rational-valued) functions on $\mathcal{S}$; one induced by the intersection pairing on X and the other coming from the correction terms on Y. For each $\alpha \in \mathcal{S}$ let $sq(\alpha)$ be the largest square of the Chern class of any spin^c structure on X in the equivalence class α, and let $d_\rho(\alpha)$ be the minimal value of the correction term for Y on the coset $\rho(\alpha)$.

Theorem 2.2. *Suppose that a rational homology sphere Y bounds a negative definite manifold X. Then, with the above notation,*

$$sq(\alpha) + b_2(X) \le 4d_\rho(\alpha)$$

for all $\alpha \in \mathcal{S}$.

Remark 2.3. *Since both sides in the above inequality are $\mathbf{j}$-invariant, one may work over $\mathcal{S}/\mathbf{j}$.*

Proof. This follows from [16, Theorem 9.6] and the fact that changing a spin^c structure on X by a torsion line bundle does not change the square of its Chern class. $\square$

We note that if Y is a homology lens space, we can choose a labelling $\{\mathfrak{t}_j : j = 0, \dots, h-1\}$ of spin^c structures on Y corresponding to an isomorphism $\mathcal{H} \cong \mathbb{Z}/h$. Similarly, we label a set of spin^c structures $\{\mathfrak{s}_i : i = 0, \dots, s-1\}$ on X, where $\mathfrak{s}_i$ has maximal square in its equivalence class $i \in \mathbb{Z}/s \cong \mathcal{S}$; here s denotes the absolute value of the determinant of the intersection pairing on $H_2(X;\mathbb{Z})$. We call such a collection of spin^c structures on X an *optimal set of spin^c structures.* The condition of Theorem 2.2 can then be expressed as follows: for any $i = 0, \dots, s-1$

$$c_1(\mathfrak{s}_i)^2 + b_2(X) \leq 4d(Y, \mathfrak{t}_{\rho(i)+kst}) \text{ for all } k = 0, \dots, t-1.$$

The following lemma is useful in identifying $\mathbf{j}$-fixed cosets in $\mathcal{H}/\mathcal{T}$.

Lemma 2.4. *Let H be a finite abelian group and T a subgroup with $|T|^2$ dividing $|H|$. If H has no element of order 4 or if $|T|$ is odd, then any element of order 2 in H/T is the image of an element of order 2 in H. Conversely, if H has an element of order 4 then there exists a subgroup T with order as above so that H/T contains an element of order 2 that is not the image of an element of order 2 in H.*

Proof. Suppose H has no element of order 4 or $|T|$ is odd. Let $[s]$ be an element of order 2 in H/T. Then $2s = t \in T$. By hypothesis t has odd order, say $2k+1$. Then $s + kt$ has order 2.

Suppose now that $s \in H$ has order 4. Let T be the subgroup generated by $2s$. Then $[s]$ is an element of order 2 in H/T; its preimage in H consists of s and $3s$, each of which has order 4. $\square$

Linking pairing. A rational homology sphere Y has a non-degenerate bilinear pairing λ on $H_1(Y;\mathbb{Z})$ with values in $\mathbb{Q}/\mathbb{Z}$, called the *linking pairing.* Suppose Y is the boundary of a four-manifold X with no torsion in $H_1(X;\mathbb{Z})$ and that Q is the intersection form on $H_2(X;\mathbb{Z})$. Then

$$\lambda \equiv -Q^{-1} \pmod 1 \tag{2}$$

(see [8, Exercise 5.3.13(g)]). If there is torsion in $H_1(X;\mathbb{Z})$ the same formula holds on the image of $H_2(X,Y;\mathbb{Z})$ in $H_1(Y;\mathbb{Z})$. In particular, λ is constant on the cosets of the image in $H_1(Y;\mathbb{Z})$ of the torsion subgroup of $H_2(X,Y;\mathbb{Z})$. This gives another restriction on the intersection pairings that a given rational homology sphere may bound.

Congruence condition on $d(Y,\mathfrak{t})$ [16, Theorem 1.2]. Let Y be a rational homology sphere and $\mathfrak{t} \in \mathrm{Spin}^c(Y)$. Suppose there exists a negative definite four-manifold X and a spin^c structure $\mathfrak{s}$ on X such that $\partial(X,\mathfrak{s}) = (Y,\mathfrak{t})$. We may suppose that $b_1(X) = 0$. Then

$$d(Y,\mathfrak{t}) = d(Y,\mathfrak{s}|_Y) \equiv \frac{c_1(\mathfrak{s})^2 - \sigma(X)}{4} \pmod 2; \tag{3}$$

this follows from the dimension shift formula for the absolute grading [16, Equation (4)] and the fact that the spin^c cobordism $X - B^4$ induces an isomorphism $HF^\infty(S^3) \to HF^\infty(Y,\mathfrak{t})$ [16, Proof of Theorem 9.6]. It is then clear that (3) holds for *any* $(X,\mathfrak{s})$ with $\partial(X,\mathfrak{s}) = (Y,\mathfrak{t})$ since the right hand side (mod 2) of (3) is an invariant of $(Y,\mathfrak{t})$, called the rho invariant.

Suppose now Y is the boundary of a simply connected definite four-manifold. This is the case for all Seifert fibred rational homology spheres as described in the next section. Then (3) holds for any spin^c structure on Y. Moreover, the linking pairing on Y is determined by the correction terms; it can be recovered from the differences $d(Y,\mathfrak{t}) - d(Y,\mathfrak{t}_0)$, where $\mathfrak{t}_0$ is a spin structure and $\mathfrak{t}$ runs over all spin^c structures on Y.

3. Application to links

Let L be an oriented link with μ components in the three-sphere; denote its signature by $\sigma(L)$. The unlinking number (or unknotting number) $u(L)$ is the minimal number of crossing changes in any diagram of L which yield the trivial μ-component link.

The four-ball genus $g^*(L)$ of L is defined to be the minimal genus of a (connected) oriented surface F admitting a smooth embedding into B^4 which maps ∂F to L. An easy argument shows that $g^*(L) \leq u(L)$. A classical result due to Murasugi [12] states that

$$g^*(L) \geq \frac{|\sigma(L)| - \mu + 1}{2}. \tag{4}$$

Suppose that this bound is attained and fix such a connected minimal surface F. Let X be the branched double cover of B^4 along F. Then $b_1(X) = 0$, $b_2(X) = 2g^*(L)+\mu-1$, and the signature of X is given by $\sigma(L)$ ([11]). After possibly changing its orientation, we may assume that X is negative-definite. Moreover, X is a spin manifold. The complement in X of the branch locus F is spin since it is the double cover of a spin manifold. The neighbourhood of F in X is also spin and the spin structures can be chosen to agree on the common boundary. This follows since the class of the branch surface $[F,K]$ is trivial in $H_2(B^4,S^3;\mathbb{Z})$, which implies that the class of the linking circle of F is of infinite order in $H_1(B^4 - F;\mathbb{Z})$, so also in $H_1(X - F;\mathbb{Z})$.

Note that $Y = \partial X$ is the double cover of S^3 branched along L. If Y is a rational homology sphere (which is the case if the determinant $h = |\Delta_L(-1)|$ of L is non-zero; in this case h is the order of $H_1(Y;\mathbb{Z})$), we may apply Theorem 2.2. We will spell this out in more detail in Section 4.

In Section 5 we list some resulting bounds on the four-ball genus of Montesinos links. In the rest of this section we discuss other classical bounds on the four-ball genus; we describe Montesinos links, their double branched covers, and a spanning surface; and we recall the formulas from [16, 17] for the correction terms of Seifert fibered rational homology spheres.

3.1. **Bounds on four-ball genus from Seifert matrices.** Potentially stronger bounds on $g^*(L)$ may be obtained by replacing $|\sigma(L)|$ in (4) by $|\sigma_\omega(L)| + n_\omega(L)$; here $\omega \in S^1 - 1$, $\sigma_\omega(L)$ is the Tristram-Levine signature and $n_\omega(L)$ is the nullity (see e. g. [6]). These invariants may be computed from any Seifert matrix associated to L. In the case of a knot K, a still stronger bound is given by Taylor [21], which we now describe.

Let $M \in \mathbb{Z}^{a\times a}$ be any Seifert matrix for K. Then M defines a pairing λ on $\mathbb{Z}^a$ by $\lambda(x, y) = x^T My$. Denote by $z(M)$ the maximal rank of a self-annihilating subgroup of λ, that is a sublattice N such that $\lambda(x, y) = 0$ for all $x, y \in N$. Taylor defines an invariant $m(K) = a/2 - z(M)$, and he proves the following inequalities for any ω:

$$g^*(K) \geq m(K) \geq \frac{|\sigma_\omega(K)|}{2}. \tag{5}$$

In Section 5 we will provide examples of knots K for which it follows from Theorem 2.2 that $g^*(K) > m(K)$.

3.2. **Montesinos links and Seifert fibered spaces.** For more details on Montesinos links and their classification see [2]. In Definitions 3.1 and 3.2, e is any integer and $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)$ are pairs of coprime integers, with $\alpha_i > 1$.

Definition 3.1. *A Montesinos link $M(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$ is a link which has a projection as shown in Figure 1(a). There are e half-twists on the left side. A box $\boxed{\alpha, \beta}$ represents a* rational tangle of slope *α/β: given a continued fraction expansion*

$$\frac{\alpha}{\beta} = [a_1, a_2, \ldots, a_m] := a_1 - \cfrac{1}{a_2 - \ddots - \cfrac{1}{a_m}},$$

the rational tangle of slope α/β consists of the four string braid $\sigma_2^{a_1}\sigma_1^{a_2}\sigma_2^{a_3}\sigma_1^{a_4} \ldots \sigma_i^{a_m}$, which is then closed on the right as in Figure 1(b) if m is odd or (c) if m is even.

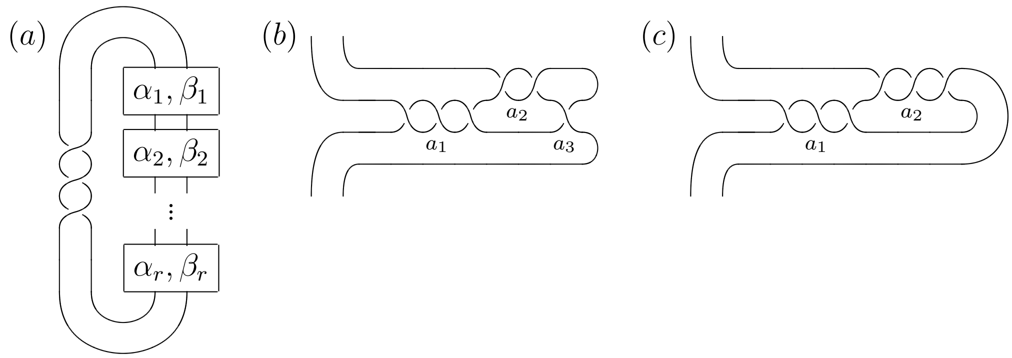

FIGURE 1. **Montesinos links and rational tangles.** Note that $e = 3$ in (a). Also (b) and (c) are both representations of the rational tangle of slope 10/3:

$$10/3 = [3, -2, 1] = [3, -3]$$

(and one can switch between (b) and (c) by simply moving the last crossing).

A two-bridge link $S(p, q)$ (or rational link, or 4-plat) is the reflection of the link formed by closing the rational tangle $\boxed{p, q}$ with two trivial bridges. This is equal to the Montesinos link $M(e; (q, eq + p))$ for any e.

Definition 3.2. *The Seifert fibered space $Y(e; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$ is the oriented boundary of the four-manifold obtained by plumbing disk bundles over the two-sphere according to the weighted graph shown in Figure 2. To each vertex v with multiplicity $m(v)$, associate a disk bundle over S^2 with Euler number $m(v)$. The bundles associated to two vertices are plumbed precisely when the vertices are connected by an edge. (See* [9, 14] *for details on plumbing.) The multiplicities on the graph are obtained from continued fraction expansions*

$$\frac{\alpha_i}{\alpha_i - \beta_i} = [\eta_1^i, \eta_2^i, \ldots, \eta_{s_i}^i].$$

A lens space $L(p, q)$ is a special case of the above; it is the boundary of the plumbed four-manifold associated to a linear graph with weights $-a_1, -a_2, \ldots, -a_m$, where $\frac{p}{q} = [a_1, a_2, \ldots, a_m]$. This is equal to the Seifert fibered space $Y(-e; (q, eq + p))$ for any e.

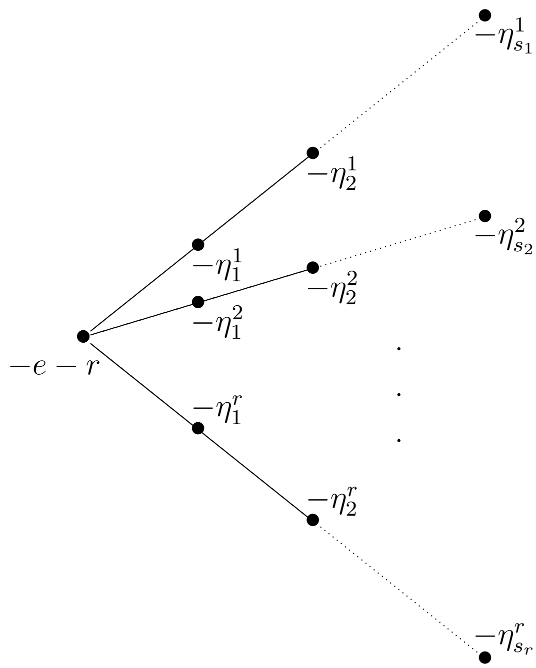

FIGURE 2. **Plumbing description of Seifert fibered space.**

A Seifert fibered space $Y(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ is a rational homology sphere if and only if its *degree* $e+\displaystyle\sum_{i=1}^{r}\frac{\beta_i}{\alpha_i}$ is nonzero.

Proposition 3.3. *The branched double cover of S^3 along the Montesinos link $M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ is the Seifert fibered space $Y(-e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$.*

Note it follows from Proposition 3.3 that the branched double cover of $S(p,q)$ is $L(p,q)$.

Proof. The original proof is in [13]. The result is also proved in [2] but note that on p. 197 an e-twist should correspond to $\alpha_0=1,\beta_0=-e$ (rather than $\beta_0=e$). Since it is particularly important that we correctly identify the branched cover as an oriented manifold we will sketch a proof here.

We start with an alternative description of the Montesinos link $M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$. For each i, let

$$\frac{\alpha_i}{\beta_i}=[a^i_1,a^i_2,\ldots,a^i_{m_i}].$$

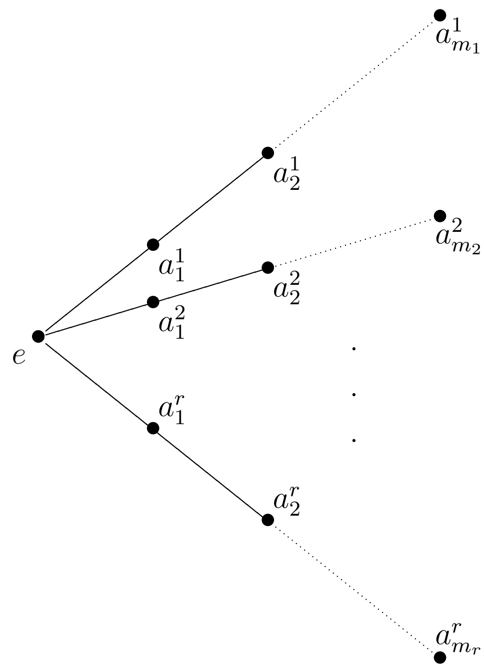

FIGURE 3. **Plumbing description of Montesinos link.** Note that changing the order of the branches may change the link.

Then $M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ is obtained by plumbing twisted bands according to the graph in Figure 3, and then taking the boundary. Here each vertex represents a twisted band, that is a D^1-bundle over S^1, embedded in S^3, with the number of half-twists given by the multiplicity of the vertex. For example, Figure 1(a) is (the boundary of) a band with 3 half-twists, if $r=0$. Bands are plumbed together precisely when the corresponding vertices are adjacent.

We now want to describe the double branched cover of such a plumbed link. Start with the case of a single vertex, with weight a. This gives the two-bridge link $L=S(a,-1)$ formed by closing the four-string braid σ_2^a. Split S^3 along a 2-sphere which separates the link into 4 arcs, so that the braid is contained in one component of S^3-S^2. (If L is pictured as in Figure 1(b), the 2-sphere may be drawn as a vertical line through L on one side of the twists.) This gives (S^3,L) as a union of two balls, each containing two arcs. The branched double covers of these are solid tori, which inherit an orientation from S^3.

The braid operation σ_2 lifts to a right-handed Dehn twist about a longitude of either torus. Use the meridian and this longitude as an ordered basis, oriented to have intersection number $+1$. With respect to this basis, the map induced on homology by the lift of σ_2 has matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Composing the Dehn twists and changing basis to that of the other torus yields the matrix product

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -a & 1 \end{pmatrix}.$$

This is precisely the gluing map for the circle bundle over S^2 with Euler number a.

Now consider a graph with two vertices labeled a_1, a_2 which are joined by an edge. The resulting plumbed link is equivalent to the two-bridge link L formed by closing the four string braid $\sigma_2^{a_1}\sigma_1\sigma_2\sigma_1\sigma_2^{a_2}$. As above split S^3 along a 2-sphere to one side of the braid. The braid σ_1 lifts to a right-handed Dehn twist about the meridian, with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus the double branched cover of L is the union of two solid tori with the gluing map given by the product

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -a_2 & 1 \end{pmatrix},$$

which is the gluing map for the boundary of the manifold formed by plumbing together disk bundles over S^2 with Euler numbers a_1, a_2.

It is now not hard to see that in general if L is the plumbed link associated to a weighted tree T then the double branched cover of L is the Seifert fibered space associated to T. According to Definition 3.2, the Seifert fibered space obtained from the graph in Figure 3 is $Y(-e-r; (\alpha_1, \alpha_1 + \beta_1), \ldots, (\alpha_r, \alpha_r + \beta_r))$. Note that this is orientation-preserving diffeomorphic to $Y(-e; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ as claimed. □

Remark 3.4. *We have used the same orientation convention as Orlik* [14] *and Hirzebruch-Neumann-Koh* [9] *for lens spaces and Seifert fibered spaces. However the opposite convention for lens spaces is used in* [16].

Remark 3.5. *Montesinos links are not in general classified by their double branched cover alone (see* [2, Theorem 12.28]*). However the following equivalence holds:*

$$M(e; \ldots, (\alpha_i, \beta_i), \ldots) = M(e+1; \ldots, (\alpha_i, \alpha_i + \beta_i), \ldots). \tag{6}$$

3.3. **A spanning surface for Montesinos links.** We describe an orientable spanning surface Σ in S^3 for the Montesinos link $L = M(e; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ which is a generalisation of that shown in [2, 12.26] for 2-bridge links (see also

[1]). For knots this will enable us to compute the signature, and also in some cases the Taylor invariant $m(K)$. For links with more than one component both the signature and the four-ball genus depend on a choice of orientation; we will choose the orientation given by $L = \partial\Sigma$ (for either orientation of Σ).

Fixing the surface Σ will require fixing a choice of invariants $(e; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$, and a continued fraction expansion for each pair (α_i, β_i). If α, β are coprime with β odd, the algorithm from [2, 12.16] yields a continued fraction expansion

$$\frac{\alpha}{\beta} = [a_1, a_2, \ldots, a_m]$$

with m odd and $a_2, a_4, \ldots$ even. If α is odd the same algorithm may be adjusted to produce an expansion with $m, a_1, a_3, \ldots$ even.

Colour black or white, in chessboard fashion, the regions of S^2 that form the complement of the projection in Figure 1(a). Start by colouring black the twisted band on the left. There are then two cases to consider.

Case 1: α_i is odd for $i = 1, \ldots, r$. Assume, using (6) if necessary, that

$$1 \leq \beta_i < \alpha_i \quad \text{for all} \quad i = 1, \ldots, r.$$

Then for each i, choose the continued fraction expansion

$$\frac{\alpha_i}{\beta_i} = [a_1^i, a_2^i, \ldots, a_{m_i}^i]$$

with $m_i, a_1^i, a_3^i, \ldots, a_{m_i-1}^i$ even, as above. The white surface is orientable in the resulting diagram.

Case 2: $\{\alpha_i\}$ are not all odd. Using (6) we may assume each β_i is the smallest positive odd integer in its congruence class mod α_i. We also require that $e \equiv r \pmod 2$. If this does not hold, choose the smallest j such that α_j is even and $\dfrac{\beta_j}{\alpha_j} = \min\left\{\dfrac{\beta_i}{\alpha_i} : \alpha_i \text{ is even}\right\}$. Then replace e with $e+1$ and β_j with $\alpha_j + \beta_j$.

Choose continued fraction expansions as above with odd length m_i and with $a_2^i, a_4^i, \ldots, a_{m_i-1}^i$ even. The black surface is orientable in the resulting diagram.

3.4. **The correction term for Seifert fibered spaces.** When Y is the lens space $L(p,q)$ a labelling of $\mathrm{Spin}^c(Y)$ by $\mathbb{Z}/p = \{0, 1, \ldots, p-1\}$ is chosen in [16, §4], and the following recursive formula is given:

$$d(L(p,q), i) = \frac{pq - (2i + 1 - p - q)^2}{4pq} - d(L(q,r), j),$$

where $i \in \mathbb{Z}/p$ and r and j are the reductions modulo q of p and i respectively. (Note Remark 3.4 above concerning orientation conventions.)

The conjugation action on spin^c structures is given by

$$\mathbf{j}(i) = q - i - 1 \pmod{p},$$

so that the $\mathbf{j}$-fixed-point-set is $\mathbb{Z} \cap \left\{ \dfrac{q-1}{2}, \dfrac{p+q-1}{2} \right\}$.

Remark 3.6. *It is shown in* [19] *that the Frøyshov invariant defined using Seiberg-Witten theory satisfies the same recursive formula. Therefore a gauge theoretic version of Theorem 2.2 based on* [7] *gives the same results for lens spaces.*

More generally, if Y is a Seifert fibered rational homology sphere $Y(e; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$, the following formula is given in [17, Corollary 1.5]:

$$d(Y, \mathfrak{t}) = \max \left\{ \frac{c_1(\mathfrak{s})^2 + |G|}{4} : \mathfrak{s} \in \mathrm{Spin}^c(X_G), \mathfrak{s}|_Y = \mathfrak{t} \right\}.$$

Here G is a graph as in Definition 3.2 for which the plumbed manifold X_G is negative definite with $\partial X_G = Y$, and $|G|$ is the number of vertices of G. This formula may be interpreted as saying that equality is obtained in (1) for some $\mathfrak{s} \in \mathrm{Spin}^c(X_G)$. Thus computing the correction terms for Y is equivalent to computing the sq function on $\mathcal{S}(X_G)$; in Section 4 we indicate how to do this for any negative definite four-manifold.

4. Obstruction algorithm

Given a rational homology sphere Y with the order of $\mathcal{H} := H^2(Y; \mathbb{Z})$ equal to h, and an integer $b \geq 0$, we want to know if Y can bound a negative definite four-manifold X with $b_2(X) = b$. In view of the results in Section 2 this can be checked in the following sequence of steps:

(1) consider all factorizations $h = st^2$ with s, $t \geq 1$;
(2) for a fixed factorization, consider all order t subgroups $\mathcal{T}$ of $\mathcal{H}$, and for a fixed $\mathcal{T}$ consider all order s subgroups $\mathcal{S}$ of $\mathcal{H}/\mathcal{T}$;
(3) for a fixed $\mathcal{S}$, consider all negative definite symmetric forms of rank b that present $\mathcal{S}$;
(4) for a fixed form, represented by a matrix Q, determine the function $sq : \mathcal{S} \to \mathbb{Q}$ (see discussion preceding Theorem 2.2);
(5) for all choices of $\mathbf{j}$-fixed origin in $\mathcal{H}/\mathcal{T}$ consider all group monomorphisms $\rho : \mathcal{S} \to \mathcal{H}/\mathcal{T}$, and for a fixed ρ determine the function $d_\rho : \mathcal{S} \to \mathbb{Q}$;

(6) if for a particular set of choices above the conclusion of Theorem 2.2 and the congruence condition (3) holds, then there is no obstruction to Y bounding a negative definite four-manifold X with $b_2(X) = b$.

Note that when $b = 0$ the above procedure simplifies significantly (see below for details). For $b > 0$ there is only a finite number of possible choices in steps (3) and (5); in particular, a complete (but not minimal) set of forms of given rank and determinant, due to Hermite, is described in [10, Theorem 23]. When determining the function sq in step (4) one can restrict to spin^c structures whose Chern classes c (modulo torsion) are characteristic vectors in the hypercube

$$x_i^2 \leq c(x_i) < |x_i^2|, \qquad i = 1, \dots, b\,,$$

where $\{x_i, i = 1, \dots, b\}$ is a basis for $H_2(X, \mathbb{Z})/\mathrm{Tors}$. To see this note that if the inequality is violated for some i, changing c by an even multiple of the Poincaré dual of x_i to make this particular inequality hold, will result in a vector with no smaller square; moreover, the square only stays the same if $c(x_i) = |x_i^2|$ (see [17] for details). A characteristic vector is the Chern class of a $\mathbf{j}$-fixed element if and only if it is in the image of $Q : \mathbb{Z}^b \to \mathbb{Z}^b$.

In the rest of this section we describe in detail the cases $b = 0$, 1 and 2 with emphasis on the application to four-ball genus of knots and links. For this application it suffices to show Y cannot bound a spin manifold X; therefore in step (3) above we need only consider even forms.

4.1. Case $b = 0$.

A necessary condition for a rational homology sphere Y to bound a rational homology ball X is that the order of the first homology of Y is a square (Lemma 2.1). The algorithm described above yields the following.

Proposition 4.1. *Let X be a smooth four-manifold with boundary Y, and suppose that $H_*(X;\mathbb{Q}) \cong H_*(B^4;\mathbb{Q})$ and the order of $\mathcal{H} = H^2(Y;\mathbb{Z})$ is $h = t^2$. Then there is a spin^c structure $\mathfrak{t}_0$ on Y so that*

$$d(Y, \mathfrak{t}_0 + \beta) = 0 \ \text{ for all } \ \beta \in \mathcal{T}\,,$$

where $\mathcal{T}$ denotes the image of $H^2(X;\mathbb{Z})$ in $\mathcal{H}$. The image of $\mathfrak{t}_0$ in $\mathcal{H}/\mathcal{T}$ is $\mathbf{j}$-invariant. If $\mathcal{H}$ contains no element of order 4 or if $|\mathcal{T}|$ is odd or if X is spin, then $\mathfrak{t}_0$ may be chosen to be a spin structure.

In particular, if Y is a homology lens space, then given a labelling $\{\mathfrak{t}_0, \dots, \mathfrak{t}_{h-1}\}$ of spin^c structures on Y, there is a j_0 so that

$$d(Y, \mathfrak{t}_{j_0+kt}) = 0 \ \text{ for all } \ k = 0, \dots, t-1.$$

Proof. Denote by $\mathfrak{t}_0$ a spin^c structure on Y that extends to X. Then the set of spin^c structures on Y that extend to X is $\mathfrak{t}_0 + \mathcal{T}$. Given that all spin^c structures on the rational homology ball X are torsion, Theorem 2.2 implies that $d(Y, \mathfrak{t}_0 + \beta) \geq 0$ for all β. Finally, changing the orientation of X and using the fact that the correction term changes its sign under this operation, gives the other inequality.

If $\mathcal{H}$ contains no element of order 4 or if $|\mathcal{T}|$ is odd, then it follows from Lemma 2.4 that there is a spin structure on Y which maps to the same element of $\mathcal{H}/\mathcal{T}$ as $\mathfrak{t}_0$. □

Corollary 4.2. *Let K be a knot in S^3 with branched double cover Y. If K is slice, then Y satisfies the conclusion of Proposition 4.1 with $\mathfrak{t}_0$ a spin structure.*

Proof. From the discussion in Section 3 we see that if K is slice then Y bounds a spin rational homology ball. □

4.2. **Case $b = 1$.**
When $b_2(X) = 1$, the intersection form Q_X of a negative definite manifold X is represented by $[-s]$, where $h = st^2$ is the order of the second cohomology of $Y = \partial X$. Note that in this case $\mathcal{S} \cong \mathbb{Z}/s$ is cyclic, and so Y can only bound such an X if $\mathcal{H}/\mathcal{T}$ contains a cyclic subgroup of order s (see discussion preceding Theorem 2.2 for notation).

Characteristic vectors in $H^2(X;\mathbb{Z})/\mathrm{Tors}$ are given by numbers $x \in \mathbb{Z}$ with the same parity as s. A set of spin^c structures on X with maximal square in their equivalence class in $\mathcal{S}$ is given by $\mathfrak{s}_i$, $i = 0, \ldots, s-1$, where the image of $c_1(\mathfrak{s}_i)$ modulo torsion is $x_i = 2i - s$, and its square is

$$sq(i) = c_1(\mathfrak{s}_i)^2 = -\frac{(2i-s)^2}{s}\,.$$

Note that $x_0 = -s$ corresponds to a $\mathbf{j}$-fixed element in $\mathcal{S}$; in case s is even, $x_{s/2} = 0$ also gives a $\mathbf{j}$-fixed element.

Let L be a two component link in S^3 with branched double cover Y. If the signature $\sigma(L) = -1$, then according to Murasugi's result L may bound a cylinder in the four-ball. If this is the case, then Y bounds a negative definite spin four-manifold X with $b_2(X) = 1$ (see Section 3). We may use the above algorithm to check if this is possible.

4.3. **Case $b = 2$.**
We now suppose a rational homology sphere Y bounds a negative definite four-manifold X with $b_2(X) = 2$. We denote the order of the second cohomology of Y by h, and fix a factorization $h = st^2$, where s is the determinant of the intersection pairing Q_X of X. Note that in this case $\mathcal{S} \cong \mathbb{Z}^2/Q\mathbb{Z}^2$ has at most two exponents, which puts a homological restriction on Y bounding such a manifold.

The following classification theorem for rank two quadratic forms is a modified version of [10, Theorem 76].

Theorem 4.3. *Any negative definite form with integer coefficients of rank two and determinant $s > 0$ is equivalent to a* reduced form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $0 \geq 2b \geq a \geq c$. (It follows that $|a| \leq 2\sqrt{s/3}$.)

Let Q be a reduced form as above with coefficients a, b, c. Note that this form presents $\mathbb{Z}/e_1 \oplus \mathbb{Z}/e_2$, where $e_2|e_1$ and $e_1e_2 = r$, if and only if $\gcd(a, b, c) = e_2$. Denote by Ω_Q the set of points (x, y) in the plane satisfying the following conditions:

$$\begin{aligned} a \leq x &< |a|, \\ c \leq y &< |c|, \\ a - 2b + c \leq x - y &< |a - 2b + c|. \end{aligned}$$

Call a lattice point $(x, y) \in \mathbb{Z}^2$ *characteristic* if $x \equiv a,\ y \equiv c \pmod 2$.

Proposition 4.4. *Fix a basis for $H_2(X;\mathbb{Z})/\mathrm{Tors}$ so that the matrix representative Q of the intersection pairing Q_X is reduced. Then the characteristic lattice points in Ω_Q are the (images of the) first Chern classes of a set of spin^c structures on X that have maximal square in their class in $\mathcal{S}$. Moreover, any characteristic vector among*

$$(0, 0), \quad (a, b), \quad (b, c), \quad (a - b, b - c)$$

gives rise to a $\mathbf{j}$-fixed element of $\mathcal{S}$.

Proof. Note that two characteristic points correspond to spin^c structures with isomorphic restrictions to Y if and only if they differ by $2m(a, b) + 2n(b, c)$, for some integers m, n. A complete set of characteristic representatives is given by the parallelogram with vertices $\pm(a + b, b + c), \pm(a - b, b - c)$ (taking all characteristic points in the interior and those in one component of the boundary minus $\pm(a-b, b-c)$). Observe that each of these points is equivalent to exactly one characteristic point in Ω_Q. It therefore remains to show that the corresponding spin^c structures have maximal square in their equivalence class. Recall that the cup product pairing on $H^2(X;\mathbb{Z})/\text{Tors}$ is well defined over $\mathbb{Q}$, and its matrix with respect to the Hom-dual basis is Q^{-1}.

As observed after the description of the algorithm, we need only consider points in the rectangle $\{(x,y)|a \le x < |a|, c \le y < |c|\}$. Thus it only remains to choose the point with larger square from any equivalence class having more than one characteristic point in the rectangle. This is done by eliminating points inside the triangles cut out of the rectangle by the lines $x - y = \pm|a-2b+c|$. □

It follows from Proposition 4.4 that the numbers $sq(\alpha)$ (for $\alpha \in \mathcal{S}$) from Theorem 2.2 are given, as an unordered set, by the squares with respect to Q^{-1} of characteristic points $(x,y) \in \Omega_Q$. It remains to order these points with respect to the group structure on $\mathcal{S} \cong \mathbb{Z}^2/Q\mathbb{Z}^2 \cong \mathbb{Z}/e_1 \oplus \mathbb{Z}/e_2$. The point $(x_{0,0}, y_{0,0})$ may be chosen arbitrarily; for convenience we choose it to be $\mathbf{j}$-fixed. Then choose $(x_{1,0}, y_{1,0})$ so that $\delta_1 = \frac{1}{2}(x_{1,0} - x_{0,0}, y_{1,0} - y_{0,0})$ has order e_1. Finally choose $(x_{0,1}, y_{0,1})$ so that $\delta_2 = \frac{1}{2}(x_{0,1} - x_{0,0}, y_{0,1} - y_{0,0})$ has order e_2 and the subgroups of $\mathcal{S}$ generated by δ_1 and δ_2 have trivial intersection. These choices determine the ordering of the remaining points: $(x_{i,j}, y_{i,j})$ is the unique characteristic point in Ω_Q with

$$\frac{1}{2}(x_{i,j} - x_{0,0}, y_{i,j} - y_{0,0}) = i\delta_1 + j\delta_2 + m(a,b) + n(b,c), \quad m,n \in \mathbb{Z}\,,$$

and

$$sq(i,j) = (x_{i,j}\ y_{i,j})Q^{-1}(x_{i,j}\ y_{i,j})^T\,,$$

for $i = 0, \ldots, e_1 - 1$ and $j = 0, \ldots, e_2 - 1$.

Suppose that K is a knot in S^3 with signature -2 and branched double cover Y. From Section 3 we know that if $g^*(K) = 1$, then Y bounds a negative-definite spin four-manifold with $b_2 = 2$, and we may use the algorithm described at the beginning of this section to seek a contradiction. We may similarly get an obstruction to a three component link with signature -2 bounding a genus zero slice surface.

5. Examples

In this section we list examples of knots and links for which our obstruction shows that inequality (4) is strict. We begin with a proof that the unknotting number of the knot 10_{145} is 2. We list two-bridge examples in 5.3 and Montesinos examples in 5.4.

5.1. **Unknotting number of 10_{145}.** The knot 10_{145} in the Rolfsen table is the Montesinos knot $M(1;(3,1),(3,1),(5,2))$. This knot has signature 2 and determinant 3. Its branched double cover is the Seifert fibered space $Y(-1;(3,1),(3,1),(5,2))$. We will show that $-Y$ cannot bound a negative definite 4-manifold with $b_2 = 2$. (In [15] we show that $-Y$ cannot bound a negative-definite form of any rank.) The correction terms are

$$d(-Y) = \left\{-\frac{3}{2}, -\frac{1}{6}, -\frac{1}{6}\right\}.$$

There are two reduced negative definite forms of rank 2 and determinant 3, namely the diagonal form $\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$, and $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$. For the first the region Ω_Q of Proposition 4.4 yields optimal spin^c structures with squares $\left\{-4, -\frac{4}{3}, -\frac{4}{3}\right\}$, and for the second, $\left\{0, -\frac{8}{3}, -\frac{8}{3}\right\}$. In either case there is clearly no map

$$\rho : \mathbb{Z}/3 \to \mathbb{Z}/3$$

which satisfies

$$c_1(\mathfrak{s}_i)^2 + 2 \leq 4d(-Y, \mathfrak{t}_{\rho(i)}).$$

It follows that $g^*(10_{145}) > 1$. From the knot diagram as in Figure 1 it is easy to see that the unknotting number u is at most 2. Since the unknotting number is bounded below by the four-ball genus, we conclude that $g^* = u = 2$. (This was first shown by Tanaka [20].)

Finally we note that the spanning surface described in 3.3 yields the Seifert matrix

$$M = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The vector $x = (1,1,1,0)^T$ satisfies $x^T M x = 0$. It follows that the Taylor invariant $m(10_{145})$ (which is the optimal lower bound for g^* from a Seifert matrix) is 1.

5.2. **A non-cyclic example.** The Montesinos knot $M(1;(5,2),(5,2),(5,2))$ has signature 4 and determinant 25. Its branched double cover $Y = Y(-1;(5,2),(5,2),(5,2))$ has $H^2(Y) \cong \mathbb{Z}/5 \oplus \mathbb{Z}/5$. The correction terms are

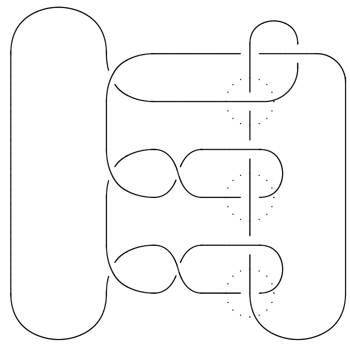

FIGURE 4. **The Montesinos knot M(1;(5,2),(5,2),(5,2)).** Note that changing the circled crossings will give the unknot.

$$d(-Y) = \frac{1}{5}\left\{\begin{array}{ccccc} -5 & 1 & -1 & -1 & 1 \\ -7 & -3 & -7 & 1 & 1 \\ -3 & -1 & -7 & -1 & -3 \\ -3 & -3 & -1 & -7 & -1 \\ -7 & 1 & 1 & -7 & -3 \end{array}\right\},$$

where the array structure indicates the $\mathbb{Z}/5 \oplus \mathbb{Z}/5$ action on $\mathrm{Spin}^c(-Y)$.

Suppose that $-Y$ bounds a negative definite manifold X with $b_2(X) = 4$. Then by Lemma 2.1 the intersection pairing of X is either unimodular or has determinant 25. If unimodular then it is equivalent to $-I$ (see for example [10, Corollary 23]); in this case $\mathcal{S}$ contains just 1 element with maximal square -4. The inequality in Theorem 2.2 now simply becomes $0 \leq d_\rho(\alpha)$; however, the correction term of the $\mathbf{j}$-fixed element is -1.

Now suppose that Q_X has determinant 25. Note that there are 6 nonnegative correction terms in the above array. There are 3 Hermite-reduced negative definite rank 4 forms with determinant 25 which present $\mathbb{Z}/5 \oplus \mathbb{Z}/5$. Each of these gives at least 10 elements $\alpha \in \mathcal{S}$ with $sq(\alpha) + 4 \geq 0$. It follows from Theorem 2.2 that $-Y$ cannot bound these forms.

This implies $g^*(M(1;(5,2),(5,2),(5,2))) > 2$. From the knot diagram in Figure 4 we see that the unknotting number is at most 3; thus $g^* = u = 3$. As with the previous example the Taylor invariant of this knot equals $|\sigma|/2$.

5.3. **Two-bridge examples.** We start with the question of slice two-bridge knots. Recall a knot K is slice if $g^*(K)$=0. It is called *ribbon* if it bounds a smoothly immersed disk in S^3 whose singularities come from identifying spanning arcs in D^2 with interior arcs in D^2. Ribbon implies slice, however, it is unknown whether every slice knot is ribbon.

Table 1 lists two-bridge knots and links $S(p, q)$ with $1 \leq |\sigma| \leq 4$ for which the obstruction algorithm shows that inequality (4) is strict. The table includes all knots with $p < 120$ and all links with $p < 60$.

TABLE 1. **Genus bounds for two-bridge links.** Here σ is the signature, and m is Taylor's lower bound for the 4-ball genus (knots only).

Link	σ	$g^* \geq$	m
$S(12, 7)$	1	1	
$S(28, 15)$	1	1	
$S(32, 19)$	1	1	
$S(42, 25)$	1	1	
$S(44, 23)$	1	1	
$S(52, 31)$	1	1	
$S(52, 33)$	-1	1	
$S(67, 39)$	2	2	1
$S(91, 22)$	2	2	1
$S(91, 53)$	2	2	1
$S(107, 28)$	-2	2	1
$S(115, 28)$	2	2	1
$S(115, 67)$	2	2	1

Any slice 2-bridge knot $S(p, q)$ must have $p = t^2$ from Lemma 2.1. A set of values of t and q for which $S(t^2, q)$ is ribbon is given in [3]. Using the Atiyah-Singer G-signature theorem, Casson and Gordon [3] defined an invariant which detects when a two-bridge knot is not ribbon and showed that the known ribbon two-bridge knots provide the only ribbon examples $S(t^2, q)$ with $t \leq 105$. Fintushel and Stern showed in [5] that the Casson-Gordon invariant is equal to an invariant they defined using Yang-Mills theory, and also showed the invariant detects when a knot is not slice. The obstruction algorithm described in Subsection 4.1 seems to give the same results as Casson-Gordon and Fintushel-Stern; we have verified this for $t \leq 105$.

Finally we note that the knots $S(187, 101)$ and $S(187, 117)$ have the same Alexander polynomials and Taylor invariants. The latter has $g^* = 1$, but our algorithm can be used to show that the former has $g^* = 2$.

5.4. **More Montesinos examples.** Table 2 contains obstructed Montesinos links $M(e;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3)$ with $-2 \le e \le 1$, $\alpha_i \le 5$, and $|\sigma| \le 4$. We have also restricted to links with determinant less than 150.

TABLE 2. **Genus bounds for Montesinos links.** Here μ is the number of components and Y is the branched double cover.

Link	μ	σ	$H_1(Y)$	$g^* \ge$	m
$M(-2;(2,1),(2,1),(5,4))$	2	1	$\mathbb{Z}/76$	1	
$M(-2;(3,1),(3,1),(5,3))$	1	2	$\mathbb{Z}/147$	2	1
$M(-1;(2,1),(2,1),(5,2))$	2	-1	$\mathbb{Z}/48$	1	
$M(-1;(3,1),(3,1),(5,3))$	2	1	$\mathbb{Z}/102$	1	
$M(-1;(3,2),(3,2),(5,1))$	2	1	$\mathbb{Z}/114$	1	
$M(0;(2,1),(2,1),(3,2))$	2	-1	$\mathbb{Z}/20$	1	
$M(0;(3,1),(3,1),(5,4))$	2	1	$\mathbb{Z}/66$	1	
$M(0;(3,2),(3,2),(5,2))$	2	1	$\mathbb{Z}/78$	1	
$M(0;(3,2),(3,2),(5,4))$	2	-1	$\mathbb{Z}/96$	1	
$M(0;(3,2),(4,3),(4,3))$	2	-1	$\mathbb{Z}/104$	1	
$M(0;(3,2),(5,1),(5,1))$	2	1	$\mathbb{Z}/80$	1	
$M(1;(2,1),(2,1),(2,1))$	3	-4	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	2	
$M(1;(3,1),(3,1),(5,2))$	1	2	$\mathbb{Z}/3$	2	1
$M(1;(3,1),(5,2),(5,2))$	2	3	$\mathbb{Z}/10$	2	
$M(1;(3,1),(5,4),(5,4))$	2	-1	$\mathbb{Z}/70$	1	
$M(1;(5,1),(5,1),(5,2))$	1	4	$\mathbb{Z}/5 \oplus \mathbb{Z}/5$	3	2
$M(1;(5,2),(5,2),(5,2))$	1	4	$\mathbb{Z}/5 \oplus \mathbb{Z}/5$	3	2

Remark 5.1. *The reflection of $M(e;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ is $M(r-e;(\alpha_1,\alpha_1-\beta_1),\dots,(\alpha_r,\alpha_r-\beta_r))$. The four-ball genus of a knot is equal to that of its reflection. However, the signature and four-ball genus of links depend on a choice of orientation, and the orientation convention from Subsection 3.3 is not preserved under reflection. For example the 3-component link $M(5;(2,1),(2,1),(2,1))$, oriented as in Subsection 3.3, has signature -2 and is shown by our algorithm to have nonzero four-ball genus. Its reflection $M(-2;(2,1),(2,1),(2,1))$ also has signature -2, but the algorithm yields no information.*

Remark 5.2. *The obstruction algorithm uses the inequality (1)* and *the congruence (3). It is interesting to note that either testing only the inequality or only the congruences yields most of the results. In Table 1, the link $S(52,33)$ is obstructed by the congruence test but not by the inequality, while $S(32,19)$ is obstructed by the inequality but not by the congruence. All other entries in the table are obstructed using either test. Similarly in Table 2,*

$M(-2;(2,1),(2,1),(5,4))$ *is obstructed by the congruence but not by the inequality; there are four knots and links which are obstructed by the inequality but not by the congruence, namely* $M(-2;(3,1),(3,1),(5,3))$*,* $M(0;(3,2),(3,2),(5,4))$*,* $M(0;(3,2),(4,3),(4,3))$ *and* $M(1;(3,1),(3,1),(5,2))$*. All other links in the table are obstructed by either condition.*

Acknowledgements. Most of this work was completed while the authors were supported as Britton postdoctoral fellows at McMaster University, Hamilton, Ontario. Part of this work was completed while the first author was supported as an EDGE postdoc at Imperial College, London. We are grateful to Peter Ozsváth and Daniel Ruberman for helpful suggestions.

References

[1] G. Burde, *Uber das Geschlecht und die Faserbarkeit von Montesinos-knoten*, Abh. Math. Sem. Univ. Hamburg **54** 1984, 199–226.

[2] G. Burde & H. Zieschang, *Knots*, Walter de Gruyter, 1985.

[3] A. J. Casson & C. McA. Gordon, *Cobordism of classical knots*, À la recherche de la topologie perdue, Progr. Math. **62** 1986, 181–199.

[4] A. J. Casson & C. McA. Gordon, *On slice knots in dimension three*, Algebraic and geometric topology, Part 2, pp. 39–53, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., 1978.

[5] R. Fintushel & R. J. Stern, *Rational homology cobordisms of spherical space forms*, Topology **26** 1987, 385–393.

[6] V. Florens & P. M. Gilmer, *On the slice genus of links*, Algebr. Geom. Topol. **3** 2003, 905–920.

[7] K. A. Frøyshov, *The Seiberg-Witten equations and four-manifolds with boundary*, Math. Res. Lett. **3** 1996, 373–390.

[8] R. E. Gompf & A. I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Math. **20**, Amer. Math. Soc., 1999.

[9] F. Hirzebruch, W. D. Neumann & S. S. Koh, *Differentiable manifolds and quadratic forms*, Lecture Notes in Pure and Applied Math. **4**, Marcel Dekker, 1971.

[10] B. W. Jones, *The arithmetic theory of quadratic forms*, Math. Assoc. of America, 1950.

[11] L. H. Kauffman & L. R. Taylor, *Signature of links*, Trans. Amer. Math. Soc. **216** 1976, 351–365.

[12] K. Murasugi, *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117** 1965, 387–422.

[13] J. M. Montesinos, *Seifert manifolds that are ramified two-sheeted cyclic coverings*, Bol. Soc. Mat. Mexicana (2) **18** 1973, 1–32.

[14] P. Orlik, *Seifert manifolds*, Lecture Notes in Math. **291**, 1972.

[15] B. Owens & S. Strle, *A characterisation of the* $\mathbb{Z}^{n-1} \oplus 3\mathbb{Z}$ *lattice and applications to rational homology spheres*, math.GT/0312266, 2003.

[16] P. Ozsváth & Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics **173** 2003, 179–261.

[17] P. Ozsváth & Z. Szabó, *On the Floer homology of plumbed three-manifolds*, Geometry and Topology **7** 2003, 225–254.

[18] P. Ozsváth & Z. Szabó, *Knot Floer homology and the four-ball genus*, Geometry and Topology **7** 2003, 615–639.

[19] S. Strle, *In preparation*
[20] T. Tanaka, *Unknotting numbers of quasipositive knots*, Topology Appl. **88** 1998, 239–246.
[21] L. R. Taylor, *On the genera of knots*, Lecture Notes in Math. **722** 1972, 144–154.