Bethe Ansatz equations and the classical $A_{n-1}^{(1)}$ Toda field theories

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Plan

- Main motivation
- Classical integrable PDEs : $A_{n-1}^{(1)}$ Toda field equations
- Modified $A_{n-1}^{(1)}$ Toda equations
- Particular example of $A_2^{(1)}$ case
- Asymptotic solutions of modified $A_2^{(1)}$ Toda field equations
- Solutions to the associated linear problem for $A_2^{(1)}$
- Functional relations
- Connection to quantum integrability
- Generalisation to $A_{n-1}^{(1)}$ cases

Integrable PDEs and quantum Integrable Models

Relation of quantum integrable models to classical integrable PDEs has been observed

- Barouch, McCoy, Tracy and Wu : spin-spin correlation function in the scaling limit of the 2d Ising model in terms of solutions to Painlevé III.
- Zamolodchikov, Fendley & Saleur : sine-Gordon partition function in terms of other solutions to Painlevé III.
- More recently: Lukyanov & Zamolodchikov showed a way to connect the classical sinh-Gordon model to the quantum massive sine(h)-Gordon model.

Generalising in such way the so-called ODE/IM correspondence to a PDE/IM correspondence which encompasses massive quantum field theories.

The ODE/IM Correspondence

- The ODE/IM Correspondence (Dorey, Tateo, Dunning, Bazhanov, Lukyanov, Zamolodchikov, Suzuki) is a link between particular linear ODEs defined in the complex plane and the conformal field theory limit of certain quantum integrable models in two dimensions (six-vertex model)
- This link is based mainly on certain functional relations that appear on both sides of the correspondence
- On the ODE side: functional relations are satisfied by spectral determinants related to certain eigenvalue problems for the ODEs
- On the quantum integrable model side: Baxter's TQ relation, T and Q operators of Bazhanov, Lukyanov, Zamolodchikov for quantum field theory satisfy functional relations

The ODE/IM Correspondence

Until recently:

- The correspondence concerned the mapping of certain ODEs to massless quantum field theories
- Lukyanov & Zamolodchikov showed how to include massive quantum field theories.
- They had as a starting point the classical sinh-Gordon equation
- Here a correspondence between classical $A_{n-1}^{(1)}$ Toda field theories and A_{n-1} Bethe Ansatz systems will be presented
- We will consider the particular example of $A_2^{(1)}$ Toda equations

$A_{n-1}^{(1)}$ Toda field equations

The two-dimensional $A_{n-1}^{(1)}$ Toda field theories are described by the Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{n}(\partial_t \eta_i)^2 - (\partial_x \eta_i)^2 - \sum_{i=1}^{n} \exp(2\eta_{i+1} - 2\eta_i)$$

with $\eta_i \equiv \eta_i(x,t)$, periodic boundary conditions $\eta_{n+1} = \eta_1$ and $\sum_{i=1}^{n} \eta_i = 0$. Using coordinates $w = x + t$ and $\bar{w} = x - t$, which are considered to be complex, the corresponding equations of motion are

$$2\,\partial_{\bar{w}}\partial_w \eta_i = \exp(2\eta_i - 2\eta_{i-1}) - \exp(2\eta_{i+1} - 2\eta_i) \quad \text{with} \quad i = 1, \ldots, n\,.$$

Modified $A_{n-1}^{(1)}$ Toda field equations

Transformation A

In order to make the connection with quantum integrability we consider a modified version of the $A_{n-1}^{(1)}$ Toda equations. Transformation that relates

$$A_{n-1}^{(1)} \text{ Toda equations} \longleftrightarrow \text{modified } A_{n-1}^{(1)} \text{ Toda equations:}$$

- Change of variables

$$\mathrm{d}w = p(z)^{1/n}\mathrm{d}z\,, \quad \mathrm{d}\bar{w} = p(\bar{z})^{1/n}\mathrm{d}\bar{z}\,,$$

  where the function $p(t)$ is of the form

$$p(t) = t^{nM} - s^{nM}\,, \quad M, s \in \mathbb{R}_+\,.$$

- Transformation of the fields

$$\eta_i(z,\bar{z}) \to \eta_i(z,\bar{z}) + \frac{n-(2i-1)}{4n}\ln(p(z)p(\bar{z}))\,.$$

Modified $A_{n-1}^{(1)}$ Toda field equations

This transformation brings the $A_{n-1}^{(1)}$ Toda equations to the *modified* $A_{n-1}^{(1)}$ *Toda equations*

$$2\partial_{\bar{z}}\partial_z \eta_i = \mathrm{e}^{2\eta_i - 2\eta_{i-1}} - \mathrm{e}^{2\eta_{i+1} - 2\eta_i} \quad \text{for} \quad i = 2, \ldots, n-1\,,$$

$$2\partial_{\bar{z}}\partial_z \eta_1 = p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_n} - \mathrm{e}^{2\eta_2 - 2\eta_1}\,,$$

$$2\partial_{\bar{z}}\partial_z \eta_n = \mathrm{e}^{2\eta_{n-1} - 2\eta_n} - p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_n}\,.$$

with $\eta_i \equiv \eta_i(z, \bar{z})$.

It is convenient for later to introduce $z = \rho\,\mathrm{e}^{i\phi}$, $\bar{z} = \rho\,\mathrm{e}^{-i\phi}$ with $\rho, \phi \in \mathbb{R}$.

Example: $A_2^{(1)}$ Toda field equations

When $n = 3$:

- We have the $A_2^{(1)}$ Toda field equations for the fields η_1, η_3

$$2\partial_{\bar{w}}\partial_w \eta_1 = \mathrm{e}^{2\eta_1 - 2\eta_3} - \mathrm{e}^{-4\eta_1 - 2\eta_3}\,,$$

$$2\partial_{\bar{w}}\partial_w \eta_3 = \mathrm{e}^{4\eta_3 + 2\eta_1} - \mathrm{e}^{2\eta_1 - 2\eta_3}\,.$$

- The corresponding modified version of the $A_2^{(1)}$ field equations is

$$2\partial_{\bar{z}}\partial_z \eta_1 = p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_3} - \mathrm{e}^{-4\eta_1 - 2\eta_3}\,,$$

$$2\partial_{\bar{z}}\partial_z \eta_3 = \mathrm{e}^{4\eta_3 + 2\eta_1} - p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1 - 2\eta_3}\,.$$

Modified $A_2^{(1)}$ Toda field equations

- We are interested in a particular class of solutions η_1, η_3 to the modified equations which are real-valued and respect certain discrete symmetries of the equations.

- In order to obtain these particular asymptotic solutions to the modified $A_2^{(1)}$ Toda equations we first apply asymptotic analysis in certain asymptotic limits to the original $A_2^{(1)}$ Toda equations.

We observe that the combination $w\bar{w}$ remains invariant under a scaling of the variables, therefore we perform a *symmetry reduction*. We consider the transformation

$$t = \sqrt{2w\bar{w}}\,, \quad \eta_1(w,\bar{w}) = y_1(t)\,, \quad \eta_3(w,\bar{w}) = y_3(t)$$

which brings the $A_2^{(1)}$ Toda equations to the form

$$\frac{d^2}{dt^2}\,y_1 + \frac{1}{t}\frac{d}{dt}\,y_1 + \mathrm{e}^{-4y_1-2y_3} - \mathrm{e}^{2y_1-2y_3} = 0\,,$$

$$\frac{d^2}{dt^2}\,y_3 + \frac{1}{t}\frac{d}{dt}\,y_3 + \mathrm{e}^{2y_1-2y_3} - \mathrm{e}^{4y_3+2y_1} = 0\,.$$

Setting $y_i(t) = \ln g_i(t)$, $i = 1, 3$, brings the system of equations to a Painlevé III-type form.

Painlevé analysis of this system of equations is of particular interest.

$A_2^{(1)}$ Toda field equations

Asymptotic Analysis

Asymptotic analysis to the system of equations provides the following leading order behaviours for $y_i(t)$

- As $t \to 0$

$$y_1(t) \sim (2 - g_2)\ln t + b_1 + \text{power series in t},$$

$$y_3(t) \sim -g_0 \ln t + b_1 + \text{power series in t},$$

  with g_i, b_i constants.

- As $t \to \infty$

$$y_i(t) = O(1).$$

The constants g_i will be related to certain parameters which enter the particular ODE of the ODE/IM correspondence. The asymptotic analysis provides for free certain relations which were imposed to these parameters on the ODE side (in the massless ODE/IM correspondence).

Thus, we obtain the following asymptotic behaviours for the solutions $\eta_i(z,\bar{z})$ to the modified equations

- As $z\bar{z} \to 0$

$$\eta_1(z,\bar{z}) \sim (1-\tfrac{g_2}{2})\ln(z\bar{z})+b_1+\sum_{k=1}^{\infty}\gamma_{ik}\left(z^{3kM}+\bar{z}^{3kM}\right)+\text{power series in } z\bar{z}\,,$$

$$\eta_3(z,\bar{z}) \sim -\tfrac{g_0}{2}\ln(z\bar{z})+b_3+\sum_{k=1}^{\infty}\gamma_{ik}\left(z^{3kM}+\bar{z}^{3kM}\right)+\text{power series in } z\bar{z}\,.$$

- As $z\bar{z} \to \infty$

$$\eta_1(z,\bar{z}) = -\tfrac{M}{2}\ln(z\bar{z})+o(1)\,, \quad \eta_3(z,\bar{z}) = \tfrac{M}{2}\ln(z\bar{z})+o(1)\,.$$

Linear problem $A_{n-1}^{(1)}$

The $A_{n-1}^{(1)}$ Toda field equations are integrable and admit a zero-curvature representation

$$\left(\partial_w + \widehat{U}(w,\bar{w},\lambda)\right)\mathbf{\Phi} = 0\,, \quad \left(\partial_{\bar{w}} + \widehat{V}(w,\bar{w},\lambda)\right)\mathbf{\Phi} = 0\,,$$

where $\widehat{U}$, $\widehat{V}$ are $n \times n$ matrices which depend on a spectral parameter $\lambda \in \mathbb{C}$ and the Toda fields η_i with

$$\widehat{U}(w,\bar{w},\lambda) = \partial_w \eta_i\,\delta_{ij} + \lambda\,C\,, \quad \widehat{V}(w,\bar{w},\lambda) = -\partial_{\bar{w}}\,\eta_i\,\delta_{ij} + \lambda^{-1}C\,,$$

$$(C)_{ij} = \exp(\eta_{j+1} - \eta_j)\,\delta_{i-1,j} \quad j = 1,\ldots,n\,.$$

The compatibility condition of the linear system of equations reads

$$\partial_w \widehat{V} - \partial_{\bar{w}} \widehat{U} + [\widehat{U}, \widehat{V}] = 0$$

(zero-curvature condition) and is equivalent to the $A_{n-1}^{(1)}$ Toda field equations.

Linear problem $A_{n-1}^{(1)}$

The linear problem for $A_{n-1}^{(1)}$ Toda equations is associated to that for the modified $A_{n-1}^{(1)}$ Toda equations by a gauge transformation.

$A_{n-1}^{(1)}$ linear problem $\longleftrightarrow$ modified $A_{n-1}^{(1)}$ linear problem:

$$\left.\begin{array}{l}\big(\partial_w + \widehat{U}(w,\bar{w},\lambda)\big)\mathbf{\Phi} = 0\\ \big(\partial_{\bar{w}} + \widehat{V}(w,\bar{w},\lambda)\big)\mathbf{\Phi} = 0\end{array}\right\} \xrightarrow{transf.\,A} \left.\begin{array}{l}\big(\partial_z + \tilde{U}(z,\bar{z},\lambda)\big)\mathbf{\Phi} = 0\\ \big(\partial_{\bar{z}} + \tilde{V}(z,\bar{z},\lambda)\big)\mathbf{\Phi} = 0\end{array}\right\} \xrightarrow{gauge\ transf.}$$

$$\big(\partial_z + U(z,\bar{z},\lambda)\big)\mathbf{\Psi} = 0\,, \quad \big(\partial_{\bar{z}} + V(z,\bar{z},\lambda)\big)\mathbf{\Psi} = 0\,,$$

with

$$A(z,\bar{z},\lambda) = g^{-1}g_z + g^{-1}\tilde{A}(z,\bar{z},\lambda)g\,, \qquad \mathbf{\Phi} = g\,\mathbf{\Psi}$$

and

$$(g)_{ij} = \left(\frac{p(\bar{z})}{p(z)}\right)^{n-\frac{2i-1}{4n}} \delta_{ij}\,.$$

Linear problem $A_2^{(1)}$

The linear problem associated to the modified $A_2^{(1)}$ Toda equations is

$$\big(\partial_z + U(z,\bar{z},\lambda)\big)\boldsymbol{\Psi} = 0\,, \quad \big(\partial_{\bar{z}} + V(z,\bar{z},\lambda)\big)\boldsymbol{\Psi} = 0\,,$$

with

$$U = \begin{pmatrix} \partial_z \eta_1 & 0 & \lambda\, p(z)\, e^{\eta_1-\eta_3} \\ \lambda e^{-2\eta_1-\eta_3} & -\partial_z\eta_1 - \partial_z\eta_3 & 0 \\ 0 & \lambda e^{2\eta_3+\eta_1} & \partial_z\eta_3 \end{pmatrix}$$

and

$$V = \begin{pmatrix} -\partial_{\bar{z}}\eta_1 & \lambda^{-1} e^{-2\eta_1-\eta_3} & 0 \\ 0 & \partial_{\bar{z}}\eta_1 + \partial_{\bar{z}}\eta_3 & \lambda^{-1} e^{2\eta_3+\eta_1} \\ \lambda^{-1}\, p(\bar{z})\, e^{\eta_1-\eta_3} & 0 & -\partial_{\bar{z}}\eta_3 \end{pmatrix}$$

Observe that the potential $p(z)$ is associated to the extended root of the $A_2^{(1)}$ Lie algebra.

Linear problem $A_2^{(1)}$

Symmetries of the linear problem

It is convenient to introduce $\lambda = \mathrm{e}^{\theta}$ and $z = \rho\,\mathrm{e}^{i\phi}$, $\bar{z} = \rho\,\mathrm{e}^{-i\phi}$ with $\rho, \phi \in \mathbb{R}$.

We define the following transformations:

- $\widehat{\Omega}: \quad \phi \to \phi + \frac{2\pi}{3M}\,, \quad \theta \to \theta - \frac{2\pi i}{3M}$
- $\widehat{S}: \quad A(\theta) \to S\,A(\theta - \frac{2\pi i}{3})\,S^{-1} \quad$ or $\quad A(\lambda) \to S\,A\left(\omega^{-1}\lambda\right)S^{-1}$

  Here $\omega = \exp\left(\frac{2\pi i}{3}\right)$, $(S)_{ij} = \omega^{i}\,\delta_{i,j}$ the 3×3 diagonal matrix and $A(\theta)$ a 3×3 matrix which depends on the spectral parameter.

  $\widehat{S}^3 = id$ so the group generated by the transformation $\widehat{S}$ is isomorphic to $\mathbb{Z}_3$.

  Such groups of transformations are known as reduction groups.

Linear problem $A_2^{(1)}$

Symmetries of the linear problem

For the linear problem associated to $A_2^{(1)}$ Toda field equations:

- The matrices U, V of the linear problem are invariant under the action of these transformations, i.e.

$$\widehat{\Omega}\,(U(\rho,\phi,\theta)) = U(\rho,\phi,\theta)\,, \quad \widehat{\Omega}\,(V(\rho,\phi,\theta)) = V(\rho,\phi,\theta)\,,$$

$$\widehat{S}\,(U(\rho,\phi,\theta)) = U(\rho,\phi,\theta)\,, \quad \widehat{S}\,(V(\rho,\phi,\theta)) = V(\rho,\phi,\theta)\,.$$

- The symmetries of U, V affect the auxiliary solution $\boldsymbol{\Psi}$

Linear problem $A_2^{(1)}$

Solution

Considering a vector $\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3)^{\mathrm{T}}$ a general solution to the linear problem reads

$$\mathbf{\Psi}(z, \bar{z}, \lambda) = \begin{pmatrix} \lambda^{-2}\, e^{3\eta_1+2\eta_3}\, \partial_z \left(e^{-2\eta_1-4\eta_3}\, \partial_z \left(e^{2\eta_3}\, \psi\right)\right) \\ -\lambda^{-1}\, e^{-\eta_1-3\eta_3}\, \partial_z \left(e^{2\eta_3}\, \psi\right) \\ e^{\eta_3}\psi \end{pmatrix} = \begin{pmatrix} e^{-\eta_1}\, \bar{\psi} \\ -\lambda\, e^{3\eta_1+\eta_3}\, \partial_{\bar{z}} \left(e^{-2\eta_1}\, \bar{\psi}\right) \\ \lambda^2\, e^{-2\eta_1-3\eta_3}\, \partial_{\bar{z}} \left(e^{4\eta_1+2\eta_3}\, \partial_{\bar{z}} \left(e^{-2\eta_1}\, \bar{\psi}\right)\right) \end{pmatrix} .$$

The functions ψ, $\bar{\psi}$ satisfy the following third-order ODEs

$$\partial_z^3\psi + u_1(z, \bar{z})\, \partial_z\psi + \left(u_0(z, \bar{z}) + \lambda^3 p(z)\right) \psi = 0\,,$$

$$\partial_{\bar{z}}^3\bar{\psi} + \bar{u}_1(z, \bar{z})\, \partial_{\bar{z}}\bar{\psi} + \left(\bar{u}_0(z, \bar{z}) + \lambda^{-3} p(\bar{z})\right) \bar{\psi} = 0\,,$$

with, e.g.,

$$u_1(z, \bar{z}) = -2\left(2\left(\partial_z\eta_1\right)^2 + 2\partial_z\eta_1\partial_z\eta_3 + 2\left(\partial_z\eta_3\right)^2 + \partial_z^2\eta_1 - \partial_z^2\eta_3\right)\,,$$
$$u_0(z, \bar{z}) = -4\partial_z\eta_3\left(2\partial_z\eta_1\partial_z(\eta_1+\eta_3) + \partial_z^2\eta_1 + 2\partial_z^2\eta_3\right) + 2\partial_z^3\eta_3\,.$$

Linear problem $A_2^{(1)}$

Solution

Interested in solutions to the linear problem:

- The different asymptotic solutions for η_1, η_3 provide with different potentials u_0, u_1 the ODEs for ψ, $\bar{\psi}$.
- Finding specific solutions for ψ, $\bar{\psi}$ will determine a particular solution $\boldsymbol{\Psi}$.

Focus on the third-order ODE for ψ and treat $\bar{z}$ as a parameter:

- In the limit $\rho^2 = z\bar{z} \to 0$ there are three different solutions to the ODE for ψ

$$\chi_0 \sim z^{g_0}, \quad \chi_1 \sim z^{g_1}, \quad \chi_2 \sim z^{g_2}, \quad g_0 + g_1 + g_2 = 3\,.$$

- These provide the following solutions to the linear problem

$$\boldsymbol{\Xi}_0 \sim (0, 0, \mathrm{e}^{g_0(\theta+i\phi)})^{\mathrm{T}}, \quad \boldsymbol{\Xi}_1 \sim (0, \mathrm{e}^{(g_1-1)(\theta+i\phi)}, 0)^{\mathrm{T}},$$

$$\boldsymbol{\Xi}_2 \sim (\mathrm{e}^{(g_2-2)(\theta+i\phi)}, 0, 0)^{\mathrm{T}}\,.$$

Linear problem $A_2^{(1)}$

Solution

- In the limit $\rho^2 = z\bar{z} \to \infty$ the ODE for ψ has a WKB-like solution which decays in the sector $|\phi| < 4\pi/(3M+3)$ and has the form

$$\psi \sim z^{-M} \exp\left(-\lambda \frac{z^{M+1}}{M+1} - \lambda^{-1} \frac{\bar{z}^{M+1}}{M+1}\right)$$

with $M > 1/2$. This asymptotic solution for ψ provides the following solution to the linear problem

$$\boldsymbol{\Psi} \sim \begin{pmatrix} \mathrm{e}^{i\phi M} \\ 1 \\ \mathrm{e}^{-i\phi M} \end{pmatrix} \exp\left(-2 \frac{\rho^{M+1}}{M+1} \cosh(\theta + i\phi(M+1))\right) .$$

Q-functions $A_2^{(1)}$

We can express the solution $\boldsymbol{\Psi}$ in terms of $\boldsymbol{\Xi}_0$, $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ as

$$\boldsymbol{\Psi} = Q_0(\theta)\,\boldsymbol{\Xi}_0 + Q_1(\theta)\,\boldsymbol{\Xi}_1 + Q_2(\theta)\,\boldsymbol{\Xi}_2\,.$$

- The coefficients Q_i can be expressed in terms of solutions to the linear problem as

$$Q_0 = \frac{\det\left(\boldsymbol{\Psi},\boldsymbol{\Xi}_1,\boldsymbol{\Xi}_2\right)}{\det\left(\boldsymbol{\Xi}_0,\boldsymbol{\Xi}_1,\boldsymbol{\Xi}_2\right)},\ Q_1 = \frac{\det\left(\boldsymbol{\Xi}_0,\boldsymbol{\Psi},\boldsymbol{\Xi}_2\right)}{\det\left(\boldsymbol{\Xi}_0,\boldsymbol{\Xi}_1,\boldsymbol{\Xi}_2\right)},$$

$$Q_2 = \frac{\det\left(\boldsymbol{\Xi}_0,\boldsymbol{\Xi}_1,\boldsymbol{\Psi}\right)}{\det\left(\boldsymbol{\Xi}_0,\boldsymbol{\Xi}_1,\boldsymbol{\Xi}_2\right)}.$$

  $(\cdot,\cdot,\cdot)$ denotes the matrix with columns three linearly independent solutions.

- The solutions $\boldsymbol{\Psi}$, $\boldsymbol{\Xi}_i$ are characterised by properties which follow from the symmetries of the linear problem. These properties affect the functions Q_i (periodicity, quantum Wronskian relation).

For example, the relations

$$S\,\mathbf{\Xi}_i\left(\rho,\phi+\tfrac{2\pi}{3M},\theta-\tfrac{2\pi i}{3M}-\tfrac{2\pi i}{3}\right)=\exp(-g_i\tfrac{2\pi i}{3})\,\mathbf{\Xi}_i(\rho,\phi,\theta)$$

and

$$S\,\mathbf{\Psi}\left(\rho,\phi+\tfrac{2\pi}{3M},\theta-\tfrac{2\pi i}{3M}-\tfrac{2\pi i}{3}\right)=\exp\left(\tfrac{4\pi i}{3}\right)\;\mathbf{\Psi}(\rho,\phi,\theta)$$

imply the following property for the Q_i

$$Q_i(\theta)=\exp\left(-\tfrac{2\pi i}{3}(g_i-1)\right)\;Q_j\left(\theta-\tfrac{2\pi i}{3}\tfrac{(M+1)}{M}\right),\quad \text{with}\quad i=0,1,2\,.$$

Functional relations $A_2^{(1)}$

We can show that the Q_i functions satisfy certain functional relations

- Consider the change of variables

$$x = z\,\mathrm{e}^{\frac{\theta}{M+1}},\; E = s^{3M}\mathrm{e}^{\frac{3M\theta}{M+1}},\; \bar{x} = \bar{z}\,\mathrm{e}^{-\frac{\theta}{(M+1)}},\; \bar{E} = s^{3M}\mathrm{e}^{-\frac{3M\theta}{(M+1)}}\,.$$

  Then the ODE for ψ becomes

$$\partial_x^3\psi + u_1(x,\bar{x})\,\partial_x\psi + \left(u_0(x,\bar{x}) + (x^{3M} - E)\right)\psi = 0\,.$$

- The ODE admits the following asymptotic solution

$$\psi \sim x^{-M}\exp\left(-\frac{x^{M+1}}{M+1} - \frac{\bar{x}^{M+1}}{M+1}\right)$$

  as $|x| \to \infty$ in the sector $|\arg x| < 4\pi/3M+3$, treating $\bar{x}$ as a parameter.

Functional relations $A_2^{(1)}$

- Based on the asymptotic solution ψ we define rotated solutions that decay in certain sectors of the complex plane

$$\psi_k(x, \bar{x}, E, \bar{E}) = \omega^k \psi(\omega^{-k} x, \omega^k \bar{x}, \omega^{-3kM} E, \omega^{3kM} \bar{E}),$$

  with $\omega = \exp\left(\frac{2\pi i}{3(M+1)}\right)$.

- The functions ψ_k, ψ_{k+1}, ψ_{k+2} are linearly independent, so we can write

$$\psi_0 = C^{(1)}(E, \bar{E})\,\psi_1 + C^{(2)}(E, \bar{E})\,\psi_2 + C^{(3)}(E, \bar{E})\,\psi_3\,.$$

  The coefficients are called Stokes multipliers and can be expressed in terms of Wronskians of rotated solutions ψ_k.

Functional relations $A_2^{(1)}$

On the other hand:

- Expanding the solution ψ in terms of the basis of solutions to the ODE at the origin we can write

$$\psi = Q_0(E,\bar{E})\chi_0 + Q_1(E,\bar{E})\chi_1 + Q_2(E,\bar{E})\chi_2 \,.$$

- Combining the relations for solutions at the origin and at infinity we can obtain the functional relation

$$\begin{aligned} C^{(1)}(E,\bar{E})\, Q^{(1)}(\omega^{-3M}E,\omega^{3M}\bar{E})\, Q^{(2)}(\omega^{-3M}E,\omega^{3M}\bar{E}) = \\ Q^{(1)}(E,\bar{E})\, Q^{(2)}(\omega^{-3M}E,\omega^{3M}\bar{E})\, \omega^{g_0-1} \\ + Q^{(1)}(\omega^{-6M}E,\omega^{6M}\bar{E})\, Q^{(2)}(E,\bar{E})\, \omega^{g_1-1} \\ + Q^{(1)}(\omega^{-3M}E,\omega^{3M}\bar{E})\, Q^{(2)}(\omega^{-6M}E,\omega^{6M}\bar{E})\, \omega^{2-g_0-g_1} \,, \end{aligned}$$

with $Q^{(1)} = Q_0$ and $Q^{(2)} \sim W[\psi,\psi_1]$.

- Why is this result important for the connection to quantum integrable systems?

  Because the previous ODE appears in the context of the so-called ODE/IM Correspondence.

CFT limit

Considering the limit

$$\bar{z} \to 0\,, \quad z \sim s \to 0\,, \quad \theta \to +\infty$$

the ODE for ψ takes the form

$$\partial_x^3 \psi + \frac{1}{x^2}\,(g_0 g_1 + g_0 g_2 + g_1 g_2 - 2)\,\partial_x \psi - \frac{1}{x^3}\,g_0 g_1 g_2 + (x^{3M} - E)\psi = 0\,,$$

which is the third-order ODE introduced in the context of the ODE/IM Correspondence.

In this limit the coefficients Q_i coincide with those of the massless quantum field theory related to the A_2 Lie algebra.

Generalisation to $A_{n-1}^{(1)}$

- Asymptotic solutions to modified $A_{n-1}^{(1)}$ Toda field equations
- $\widehat{\Omega}$ and $\widehat{S}$ transformations
- Symmetries of the associated $A_{n-1}^{(1)}$ Lax matrices U, V and properties of the auxiliary vector solution $\mathbf{\Psi}$

can be generalised accordingly.

Generalisation to $A_{n-1}^{(1)}$

Considering a vector $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_n)^{\mathrm{T}}$, a general solution to the $A_{n-1}^{(1)}$ linear problem reads

$$\Psi_i(z,\bar{z},\lambda) = \begin{cases} -\lambda^{-1} e^{\eta_i - \eta_{i+1}} \left(\partial_z \Psi_{i+1} + \partial_z \eta_{i+1} \Psi_{i+1}\right) & \text{for} \quad i = n-1, \ldots, 1 \\ e^{\eta_n} \psi & \text{for} \quad i = n \end{cases}$$

$$= \begin{cases} e^{-\eta_1} \bar{\psi} & \text{for} \quad i = 1 \\ -\lambda\, e^{\eta_{i-1} - \eta_i} \left(\partial_{\bar{z}} \Psi_{i-1} - \partial_{\bar{z}} \eta_{i-1} \Psi_{i-1}\right) & \text{for} \quad i = n-1, \ldots, 1\,. \end{cases}$$

The $\psi \equiv \psi(z,\bar{z},\lambda)$ and $\bar{\psi} \equiv \bar{\psi}(z,\bar{z},\lambda)$ satisfy n^{th}-order differential equations

$$\Big((-1)^{n+1} D_n(\eta) + \lambda^n p(z)\Big)\psi = 0\,,$$

$$\Big((-1)^{n+1} \bar{D}_n(\eta) + \lambda^{-n} p(\bar{z})\Big)\bar{\psi} = 0\,,$$

and we have introduced the n^{th}-order operators

$$D_n(\eta) = (\partial_z + 2\,\partial_z \eta_1)(\partial_z + 2\,\partial_z \eta_2) \ldots (\partial_z + 2\,\partial_z \eta_n)\;,$$

$$\bar{D}_n(\eta) = (\partial_{\bar{z}} - 2\,\partial_{\bar{z}} \eta_n) \ldots (\partial_{\bar{z}} - 2\,\partial_{\bar{z}} \eta_2)(\partial_{\bar{z}} - 2\,\partial_{\bar{z}} \eta_1)\;.$$

Outlook/Conclusion

- Classical Integrable PDEs
- Asymptotic solutions
- Linear problem: linear ODEs
- Connection with Quantum Integrability (using the ODE/IM Correspondence)

Starting from a classical integrable PDE we can recover a certain type of ODE which can then be mapped to a massive quantum integrable system, with s playing the role of the mass scale.

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