A product formula for the eigenfunctions of a quartic oscillator

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A quartic oscillator

We consider solutions to the eigenvalue problem consisting of the differential equation

$$H(a,\lambda)\psi \equiv -\frac{d^2\psi}{dx^2} + \left(ax^2 + \frac{\lambda}{2}x^4\right)\psi = E\psi, \quad x \in \mathbb{R}, \tag{1}$$

and boundary conditions

$$\lim_{x\to\pm\infty} \psi(x) = 0. \tag{2}$$

Throughout we assume that $a \in \mathbb{R}$ and $\lambda > 0$. As is well known, this eigenvalue problem has a discrete spectrum consisting of real eigenvalues $E_0 < E_1 < \cdots < E_k < \cdots$ with $E_k \to \infty$ as $k \to \infty$, and all eigenspaces are one-dimensional.
For any (complex) value of E, the differential equation (1) has a unique solution $\psi = \psi(a,\lambda,E;x)$ with asymptotic behaviour

$$\psi \sim x^{-1}\exp\left(-\frac{(\lambda/2)^{1/2}}{3}x^3 - \frac{a}{2(\lambda/2)^{1/2}}x\right), \quad x \to +\infty,$$

The eigenvalues $E = E_k$ of (1)–(2) are precisely those values of E for which ψ also decays as $x \to -\infty$, which happens if and only if ψ is either even or odd. We let

$$\psi_k(a,\lambda;x) \equiv \psi(a,\lambda,E_k;x)$$

denote the corresponding eigenfunctions. Since ψ_k has k real zeros, ψ_k is even (odd) if and only if k is even (odd).

A product formula

Recall the standard solution $w(x) = \mathrm{Ai}(x)$ of Airy's equation

$$\frac{d^2w}{dx^2} = xw, \tag{3}$$

given explicitly by the integral representation

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} \exp(\zeta^3/3 - \zeta x)d\zeta.$$

(Here, the contour of integration consists of two rays emerging from the origin at angles $\pm\pi/3$.)

Theorem: *For $a \in \mathbb{R}$, $\lambda > 0$ and $k = 0,1,\ldots$, we have a product formula*

$$\psi_k(x)\psi_k(y) = \int_{\mathbb{R}} \psi_k(z)\mathcal{K}(x,y,z)dz \tag{4}$$

with kernel function

$$\mathcal{K}(a,\lambda;x,y,z) \equiv \lambda^{1/3}\exp\left((\lambda/2)^{1/2}xyz\right) \times \mathrm{Ai}\left(\frac{\lambda^{1/3}}{2}(x^2+y^2+z^2) + \frac{a}{\lambda^{2/3}}\right).$$

Moreover, as long as $a > \lambda^{2/3}a_1$ with $a_1 = -2.3381074105\ldots$ being the first zero of $\mathrm{Ai}(x)$, the kernel function $\mathcal{K}(x,y,z)$ is positive for all $x,y,z \in \mathbb{R}$.

Note that this product formula does not have a non-trivial limit as $\lambda \to 0$. In other words it is non-perturbative in the sense that it yields no result for the harmonic oscillator case $\lambda = 0$.

Interpretations

Such a product formula can be viewed in various ways.

Integral equation: For $m = 0,1,\ldots$, the eigenfunction ψ_{2m} is even and we have $\psi_{2m}(0) \neq 0$. Setting $x = 0$ in the product formula (4), we thus obtain the integral equation

$$\psi_{2m}(0)\psi_{2m}(y) = 2\lambda^{1/3}\int_0^\infty \psi_{2m}(z)\mathrm{Ai}\left(\frac{\lambda^{1/3}}{2}(x^2+y^2+z^2) + \frac{a}{\lambda^{2/3}}\right)dz.$$

(For the odd eigenfunctions ψ_{2m+1}, a nontrivial result is obtained by differentiating the product formula before setting $x = 0$.)

Interpretations (contd.)

Harmonic analysis: The product formula (4) entails a convolution relevant to the eigenfunction transform given (under suitable conditions) by

$$\hat{f}_k = \int_{\mathbb{R}} \psi_k(x)f(x)dx, \quad k = 0,1,\ldots$$

Specifically, letting

$$(f * g)(z) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\mathcal{K}(x,y,z)dxdy,$$

we (formally) have

$$\begin{aligned}\widehat{(f*g)}_k &= \int_{\mathbb{R}} \psi_k(z)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\mathcal{K}(x,y,z)dxdy\right)dz\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \psi_k(z)\mathcal{K}(x,y,z)dz\right)f(x)g(y)dxdy\\ &= \int_{\mathbb{R}} \psi_k(x)f(x)dx\int_{\mathbb{R}} \psi_k(y)g(y)dy\\ &= \hat{f}_k \cdot \hat{g}_k.\end{aligned}$$

We believe it would be interesting to consider the implications of such a convolution on the harmonic analysis of expansions in the eigenfunctions ψ_k.

Sketch of proof

Note that $\psi_k(x)\psi_k(y)$ satisfies the partial differential equation (PDE)

$$\big(H(x) - H(y)\big)\Psi(x,y) = 0.$$

A key ingredient in the proof of the theorem is the observation that also the kernel function $\mathcal{K}(x,y,z)$ satisfies this PDE. More generally, the following proposition can be established by direct computations.

Proposition: *Let $w(x)$ be a solution of Airy's equation* (3)*. Then the function*

$$\mathcal{K}(a,\lambda;x,y,z) \equiv \exp\left((\lambda/2)^{1/2}xyz\right)w\left(\frac{\lambda^{1/3}}{2}(x^2+y^2+z^2) + \frac{a}{\lambda^{2/3}}\right)$$

satisfies the identities

$$H(x)\mathcal{K}(x,y,z) = H(y)\mathcal{K}(x,y,z) = H(z)\mathcal{K}(x,y,z).$$

These identities and the (formal) self-adjointness of $H(z)$ entail

$$\begin{aligned}H(x)\int_{\mathbb{R}} \psi_k(z)\mathcal{K}(x,y,z)dz &= \int_{\mathbb{R}} \psi_k(z)H(z)\mathcal{K}(x,y,z)dz\\ &= \int_{\mathbb{R}} \mathcal{K}(x,y,z)H(z)\psi_k(z)dz\\ &= E_k\int_{\mathbb{R}} \psi_k(z)\mathcal{K}(x,y,z)dz,\end{aligned}$$

i.e. the right-hand side of (4) satisfies the differential equation (1). From the asymptotic of $\mathrm{Ai}(x)$ it is readily inferred that it also satisfies the boundary conditions (2). Since all eigenspaces of (1)–(2) are one-dimensional and $\mathcal{K}(x,y,z)$ is invariant under the interchange $x \leftrightarrow y$, we have

$$\int_{\mathbb{R}} \psi_k(z)\mathcal{K}(x,y,z)dz = c_k\psi_k(x)\psi_k(y)$$

for some constant c_k. Finally, computing the asymptotic of the left-hand side and comparing it with the known asymptotic of ψ_k, one can show that in fact $c_k = 1$.

Reference

A detailed account of the results and corresponding proofs sketched above can be found in the following preprint.

M. H. & E. L. (2013). *A product formula for the eigenfunctions of a quartic oscillator.* arXiv:1312.3493.