Characters of the W_3 algebra

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The W_3 algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \delta_{m+n,0} \frac{c}{12} m \left(m^2 - 1\right)$$

$$[L_m, W_n] = (2m-n) W_{m+n}$$

$$[W_m, W_n] = (m-n) \left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right] L_{m+n} + \beta (m-n) \Lambda_{m+n} + \delta_{m+n,0} \frac{c}{360} m \left(m^2-1\right)\left(m^2-4\right)$$

where

$$\Lambda_n = \sum_{p=-\infty}^{\lfloor (n-1)/2 \rfloor} L_p L_{n-p} + \sum_{p=\lceil n/2 \rceil}^{\infty} L_{n-p} L_p + \gamma(n) L_n,$$

$$\beta = \frac{16}{22+5c}, \qquad \text{and} \qquad \gamma(n) = \begin{cases} -\frac{1}{20}(n^2-4) & n \text{ even} \\ -\frac{1}{20}(n^2-9) & n \text{ odd} \end{cases}$$

Null states and modules

- Highest-weight state $|h, w\rangle$, descendant states $L_{-1}|h, w\rangle$, $W_{-1}|h, w\rangle$, etc.
- Verma module V:

Level 0	$\|h, w\rangle$
Level 1	$L_{-1}\|h, w\rangle, W_{-1}\|h, w\rangle$
Level 2	$L_{-2}\|h, w\rangle, W_{-2}\|h, w\rangle$, $L_{-1}^2\|h, w\rangle, W_{-1}^2\|h, w\rangle, L_{-1}W_{-1}\|h, w\rangle$

- A descendant state $|N\rangle$ is null if $L_n|N\rangle = 0$ for all $n > 0$.
- Irreducible module: $L = V/\{|N\rangle\}$

$$\mathcal{O}_{m,n}^{L} = \sum_{r,s=-\infty}^{\infty} \sum_{\omega \in \mathbb{W}(a_2)} (-1)^{l(\omega)} \mathcal{O}_{\omega(m,n)+pr\alpha_1+ps\alpha_2}^{V}$$

Characters

Virasoro partition function on a torus ($q = \exp 2\pi i\tau$):

$$Z = \mathrm{Tr}_{\mathcal{H}}\left(q^{L_0 - \frac{c}{24}}\,\bar{q}^{\bar{L}_0 - \frac{c}{24}}\right) = \sum_{h,\bar{h}} N_{h,\bar{h}}\ \chi_h^{L_{Vir}}\ \chi_{\bar{h}}^{L_{Vir}}$$

where the Virasoro character is

$$\chi^{M_{Vir}} = \mathrm{Tr}_{M_{Vir}}\left(q^{L_0 - \frac{c}{24}}\right).$$

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W_3 partition function:

$$Z = \mathrm{Tr}_{\mathcal{H}}\left(e^{2\pi i z W_0} q^{L_0 - \frac{c}{24}} \cdot \mathrm{barred}\right) = \sum_{h,w;\,\bar{h},\bar{w}} N_{h,w;\,\bar{h},\bar{w}}\ \chi_{h,w}^{L}\ \chi_{\bar{h},\bar{w}}^{L}$$

with W_3 character

$$\begin{aligned}\chi^{M} &= \mathrm{Tr}_{M}\left(e^{2\pi i z W_0} q^{L_0 - \frac{c}{24}}\right) \\ &= \mathrm{Tr}_{M}\left(q^{L_0 - \frac{c}{24}}\right) + 2\pi i z\,\mathrm{Tr}_{M}\left(W_0 q^{L_0 - \frac{c}{24}}\right) + \frac{(2\pi i z)^2}{2}\,\mathrm{Tr}_{M}\left(W_0^2 q^{L_0 - \frac{c}{24}}\right) + \dots\end{aligned}$$

Outline

- Introduction
  - The W_3 algebra
  - Null states and modules
  - Characters
- Character calculations
  - 'Brute force'
  - Null states
  - Exact results
- Modular transformation

'Brute force' results

Example: contribution of the state $L_{-2}|h,w\rangle$ to $\mathrm{Tr}(W_0)$

$$W_0 L_{-2}|h,w\rangle = (L_{-2}W_0 + 4W_{-2})|h,w\rangle = wL_{-2}|h,w\rangle + 4W_{-2}|h,w\rangle$$

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Results:

$$\mathrm{Tr}_V\left(W_0 q^{L_0-\frac{c}{24}}\right) = q^{-\frac{c}{24}}\left(wq^h + 2wq^{h+1} + 5wq^{h+2} + 10wq^{h+3} + 20wq^{h+4} + \dots\right)$$

$$\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = q^{-\frac{c}{24}}\left(w^2q^h + \left(2w^2 + \frac{4}{22+5c}(32h-c+2)\right)q^{h+1} + \dots\right)$$

$$\mathrm{Tr}_V\left(W_0^3 q^{L_0-\frac{c}{24}}\right) = q^{-\frac{c}{24}}\left(w^3q^h + \left(2w^3 + \frac{12w}{22+5c}(32h-c+2)\right)q^{h+1} + \dots\right)$$

Results from null states

Null state condition: $\mathrm{Tr}_L\left(N_0 q^{L_0-\frac{c}{24}}\right) = 0$

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Example:

$$\begin{aligned}\mathrm{Tr}_L\left(L_{-p}L_p q^{L_0-\frac{c}{24}}\right) &= q^p\mathrm{Tr}_L\left(L_pL_{-p}q^{L_0-\frac{c}{24}}\right)\\ &= q^p\mathrm{Tr}_L\left(L_{-p}L_p q^{L_0-\frac{c}{24}}\right)+q^p\mathrm{Tr}_L\left(\left[L_p,L_{-p}\right]q^{L_0-\frac{c}{24}}\right)\\ &= \frac{q^p}{1-q^p}\mathrm{Tr}_L\left(\left[2pL_0+\frac{c}{12}p\left(p^2-1\right)\right]q^{L_0-\frac{c}{24}}\right)\end{aligned}$$

i.e. if $N_0=\sum_{\mathrm{p}\geq 1}L_{-p}L_p$, then, using $D:=q\frac{d}{dq}+\frac{c}{24}=L_0$,

$$0=2\sum_{\mathrm{p}\geq 1}\frac{p\,q^p}{1-q^p}\mathrm{Tr}_L\left(L_0q^{L_0-\frac{c}{24}}\right)+\frac{c}{12}\sum_{\mathrm{p}\geq 1}\frac{\left(p^3-p\right)q^p}{1-q^p}\mathrm{Tr}_L\left(q^{L_0-\frac{c}{24}}\right)$$

$$\Rightarrow\quad 0=\left[2\sum_{\mathrm{p}\geq 1}\frac{p\,q^p}{1-q^p}q\frac{d}{dq}+\frac{c}{12}\sum_{\mathrm{p}\geq 1}\frac{p^3q^p}{1-q^p}\right]\mathrm{Tr}_L\left(q^{L_0-\frac{c}{24}}\right)$$

Results from null states

Three-state Potts model: $c = 4/5$, and

$$(h, w) = (0, 0)\,, \left(\frac{1}{15}, \pm\frac{1}{9}\sqrt{\frac{2}{195}}\right), \left(\frac{2}{3}, \pm\frac{2}{9}\sqrt{\frac{26}{15}}\right), \left(\frac{2}{5}, 0\right).$$

At level seven, this model has a null state with zero mode

$$N_0^{(7)} = -\frac{27}{121} L_0^2 W_0 + \frac{9}{55} L_0 W_0 - \frac{6}{605} W_0 + \dots$$

Substituting this into the null state condition gives

$$0 = \left[q^2 \frac{d^2}{dq^2} + \left(1 - \frac{2}{3}E_2\right) q \frac{d}{dq} + \left(\frac{1}{12}E_2^2 - \frac{14}{225}E_4\right)\right] \mathrm{Tr}_L\left(W_0 q^{L_0 - \frac{c}{24}}\right)$$

This has solutions

$$\mathrm{Tr}_L\left(W_0 q^{L_0 - \frac{c}{24}}\right) = \pm\frac{1}{9}\sqrt{\frac{2}{195}}\, q^{\frac{1}{15} - \frac{c}{24}} \left(1 + 46q + 74q^2 + 192q^3 - 121q^4 + \dots\right)$$

$$\mathrm{Tr}_L\left(W_0 q^{L_0 - \frac{c}{24}}\right) = \pm\frac{2}{9}\sqrt{\frac{26}{15}}\, q^{\frac{2}{3} - \frac{c}{24}} \frac{1}{26} \left(26 + 143q + 142q^2 + 214q^3 - 22q^4 + \dots\right)$$

Results from null states

Similarly, there is a level six null state with zero mode

$$N_0^{(6)} = W_0^2 - \frac{95}{117} L_0^3 + \frac{5}{13} L_0^2 - \frac{14}{585} L_0 + \dots$$

that leads to $\mathrm{Tr}_L\left(W_0^2 q^{L_0 - \frac{c}{24}}\right) = \mathcal{D}\left\{\mathrm{Tr}_L\left(q^{L_0-\frac{c}{24}}\right)\right\}$, which we can solve:

$$\mathrm{Tr}_L\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = q^{-\frac{c}{24}}\left(12q^3 + \frac{4352}{65}q^4 + \frac{3064}{13}q^5 + \frac{50864}{65}q^6 + \dots\right)$$

$$\mathrm{Tr}_L\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = q^{\frac{1}{15}-\frac{c}{24}}\left(\frac{2}{15795} + \frac{4232}{15795}q + \frac{22868}{3159}q^2 + \frac{227216}{5265}q^3 + \dots\right)$$

$$\mathrm{Tr}_L\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = q^{\frac{2}{3}-\frac{c}{24}}\left(\frac{104}{1215} + \frac{3146}{1215}q + \frac{365224}{15795}q^2 + \frac{279764}{3159}q^3 + \dots\right)$$

$$\mathrm{Tr}_L\left(W_0^2 q^{L_0-\frac{c}{24}}\right) = q^{\frac{2}{5}-\frac{c}{24}}\left(\frac{28}{13}q + \frac{256}{13}q^2 + \frac{6872}{65}q^3 + \frac{20992}{65}q^4 + \frac{61872}{65}q^5 + \dots\right)$$

Exact results for Verma modules

- For $\mathrm{Tr}_V\left(W_0 q^{L_0-\frac{c}{24}}\right)$, we need to consider

$$W_0\mathcal{P}\left|h,w\right\rangle = [W_0,\mathcal{P}]\left|h,w\right\rangle + w\mathcal{P}\left|h,w\right\rangle$$

The W_3 algebra - a reminder

$$[W, ABCD\ldots] = [W, A]\, BCD\ldots + A\, [W, B]\, CD\ldots + AB\, [W, C]\, D\ldots + \ldots$$

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \delta_{m+n,0} \frac{c}{12} m \left(m^2 - 1\right) \\
[L_m, W_n] &= (2m-n)\, W_{m+n} \\
[W_m, W_n] &= (m-n) \left[\frac{1}{15} (m+n+3)(m+n+2) - \frac{1}{6} (m+2)(n+2) \right] L_{m+n} \\
&\quad + \beta\, (m-n)\, \Lambda_{m+n} + \delta_{m+n,0} \frac{c}{360} m \left(m^2 - 1\right)\left(m^2 - 4\right)
\end{aligned}$$

$$\Lambda_n = \sum_{p=-\infty}^{\lfloor (n-1)/2 \rfloor} L_p L_{n-p} + \sum_{p=\lceil n/2 \rceil}^{\infty} L_{n-p} L_p + \gamma(n)\, L_n$$

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  and so we find

$$\mathrm{Tr}_V\left(W_0 q^{L_0-\frac{c}{24}}\right) = w\,q^{h-\frac{c}{24}}\prod_{p\geq 1}\frac{1}{\left(1-q^p\right)^2} = \frac{w\,q^{h-\frac{c}{24}}}{\phi\left(q\right)^2}$$
$$= q^{h-\frac{c}{24}}\left(w + 2wq + 5wq^2 + 10wq^3 + 20wq^4 + \ldots\right)$$

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- For $\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right)$, things are slightly more complicated:

$$W_0^2\mathcal{P}\left|h,w\right\rangle = w^2\mathcal{P}\left|h,w\right\rangle + 2w[W_0\,,\mathcal{P}]\,\left|h,w\right\rangle + [W_0\,,[W_0\,,\mathcal{P}]]\,\left|h,w\right\rangle$$

Exact results for Verma modules

$$\mathrm{Tr}_V\left(W_0^2 q^{L_0-\frac{c}{24}}\right)$$

$$= \frac{q^{h-\frac{c}{24}}}{\phi(q)^2}\left[\begin{array}{c} w^2 + \frac{4}{15}\sum_{\mathrm{p}\geq 1}\frac{p^2(p^2-4)q^p}{(1-q^p)^2} \\ + 4\beta\sum_{\mathrm{p}\geq 1}\frac{p^2 q^p}{(1-q^p)^2}\left[2h+\gamma(p)-2\frac{pq^{2p}}{1-q^{2p}}+4\sum_{k=1}^{p}\frac{kq^k}{1-q^k}\right] \\ + 8\beta\sum_{\mathrm{p}\geq 1}\frac{pq^p}{1-q^p}\sum_{s>p/2}^{p-1}\frac{q^s}{1-q^s}\left[\frac{p(2s-p)}{1-q^p}+\frac{s(3s-2p)}{1-q^s}\right] \end{array}\right]$$

Summary

- A series expansion for $\mathrm{Tr}_V\left(W_0 q^{L_0-\frac{c}{24}}\right)$ was found by brute force.

  $\mathrm{Tr}_L\left(W_0 q^{L_0-\frac{c}{24}}\right)$ was found as a series expansion for the Potts model.

  $\mathrm{Tr}_V\left(W_0 q^{L_0-\frac{c}{24}}\right)$ was found exactly for any model.

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- Higher powers $\mathrm{Tr}_V\left(W_0^n q^{L_0-\frac{c}{24}}\right)$ were found as series expansions for any model.

Thank you!