Nonlinear supersymmetry in the quantum Calogero model

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Some history

- 1923 Burchnall & Chaundy:
  odd-order ordinary differential operators commuting with 1d Hamiltonian
- 1978 Krichever:
  their existence is tied to algebro-geometric, or 'finite-gap', nature of Hamiltonian
- 1989 Dunkl:
  commuting operators combining partial differentials and Coxeter reflections
- 1990 Chalykh & Veselov:
  "commutative rings of partial differential operators and Lie algebras"
  1st examples of 2d finite-gap Hamiltonians, construction of intertwiners for $g=2$
- 1991 Heckman:
  uses Dunkl operators to construct intertwiners for any multiplicity $g-1$
- 1990s Berest, Chalykh, Etingof, M. Feigin, Ginzburg, Styrkas, Veselov:
  extension to higher dimension $n-1$ and 'multiplicity' $g-1$, in particular:
  construction of Baker-Akhiezer functions, explicit formulæ for add'l charges,
  including via Darboux dressing with intertwiners (only for $n=3$, $g=2$)

Integrals for generic coupling

A_{n-1} Calogero Hamiltonian: $\quad H = \frac{1}{2}\sum_i p_i^2 + \sum_{i<j} \frac{g(g-1)}{(x^i-x^j)^2} \qquad i,j = 1,2,\ldots,n$

$[x^i, p_j] = \mathrm{i}\,\delta^i{}_j$ and define $\quad P = \sum_{i=1}^n p_i \quad , \quad X = \frac{1}{n}\sum_{i=1}^n x^i$

Dunkl operators: $\quad \pi_i = p_i + \sum_{j(\neq i)} \frac{\mathrm{i}\,g}{x^i-x^j} s_{ij} \quad , \quad s_{ij} =$ permutation operators

Liouville charges: $\quad I_k = \mathrm{res}\Big(\sum_i \pi_i^k\Big) \quad , \quad k = 1,2,\ldots,n \quad \Rightarrow \quad [I_k, I_\ell] = 0$

$$I_0 = n\mathbb{1} \quad , \quad I_1 = P \quad , \quad I_2 = 2H \quad , \quad I_3 = \sum_i p_i^3 + 3\sum_{i<j} \frac{g(g-1)}{(x^i-x^j)^2}(p_i+p_j)$$

important observation (hidden in ‘potential-free gauge’): $\quad I_k(g) = I_k(1-g)$

‘dynamical’ conformal symmetry: H is part of $sl(2,\mathbb{R})$ algebra, with other generators

$$D \;=\; \tfrac12 \sum_i (x^i p_i + p_i x^i) \qquad \text{and} \qquad K \;=\; \tfrac12 \sum_i (x^i)^2$$

$$[D,H] \;=\; 2\mathrm{i}H \qquad\qquad [D,K] \;=\; -2\mathrm{i}K \qquad\qquad [K,H] \;=\; \mathrm{i}D$$

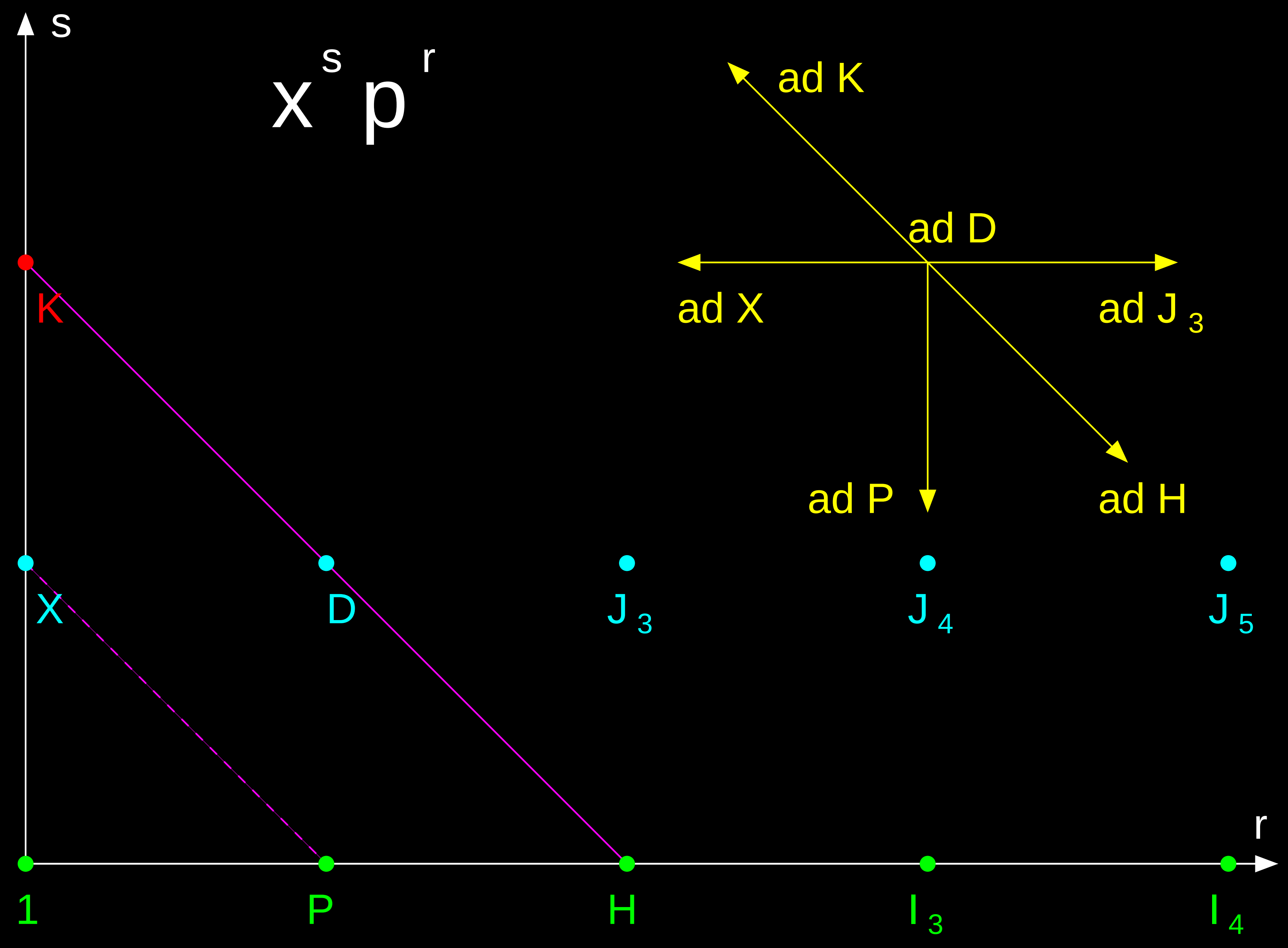
with other Liouville charges: $\quad \tfrac{1}{\mathrm{i}}[D,I_k] \;=\; k I_k \quad , \quad \tfrac{1}{\mathrm{i}}[K,I_\ell] \;=:\; \ell J_\ell$

$$J_1 \;=\; nX \quad , \quad J_2 \;=\; D \quad , \quad J_3 \;=\; \tfrac12 \sum_i \Big(x^i p_i^2 + p_i^2 x^i\Big) + g(g{-}1) \sum_{i<j} \frac{x^i{+}x^j}{(x^i{-}x^j)^2}$$

they are ‘almost’ conserved (linear time dependence) and form a Witt algebra:

$$\tfrac{1}{\mathrm{i}}[D,J_\ell] \;=\; (\ell{-}2)J_\ell \qquad \text{and} \qquad \tfrac{1}{\mathrm{i}}[H,J_\ell] \;=\; -I_\ell$$

$$\tfrac{\mathrm{i}}{k}[I_k,J_\ell] \;=\; I_{k+\ell-2} \;=\; \tfrac{\mathrm{i}}{\ell}[I_\ell,J_k] \qquad \text{and} \qquad \mathrm{i}[J_k,J_\ell] \;=\; (k{-}\ell)J_{k+\ell-2}$$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universal enveloping algebra provides an infinity of quadratic conserved charges:

$$L_{k,\ell} \;=\; \tfrac{1}{2}\{I_k, J_\ell\} - \tfrac{1}{2}\{I_\ell, J_k\} \;=\; -L_{\ell,k} \qquad \Rightarrow \qquad [H, L_{k,\ell}] \;=\; 0$$

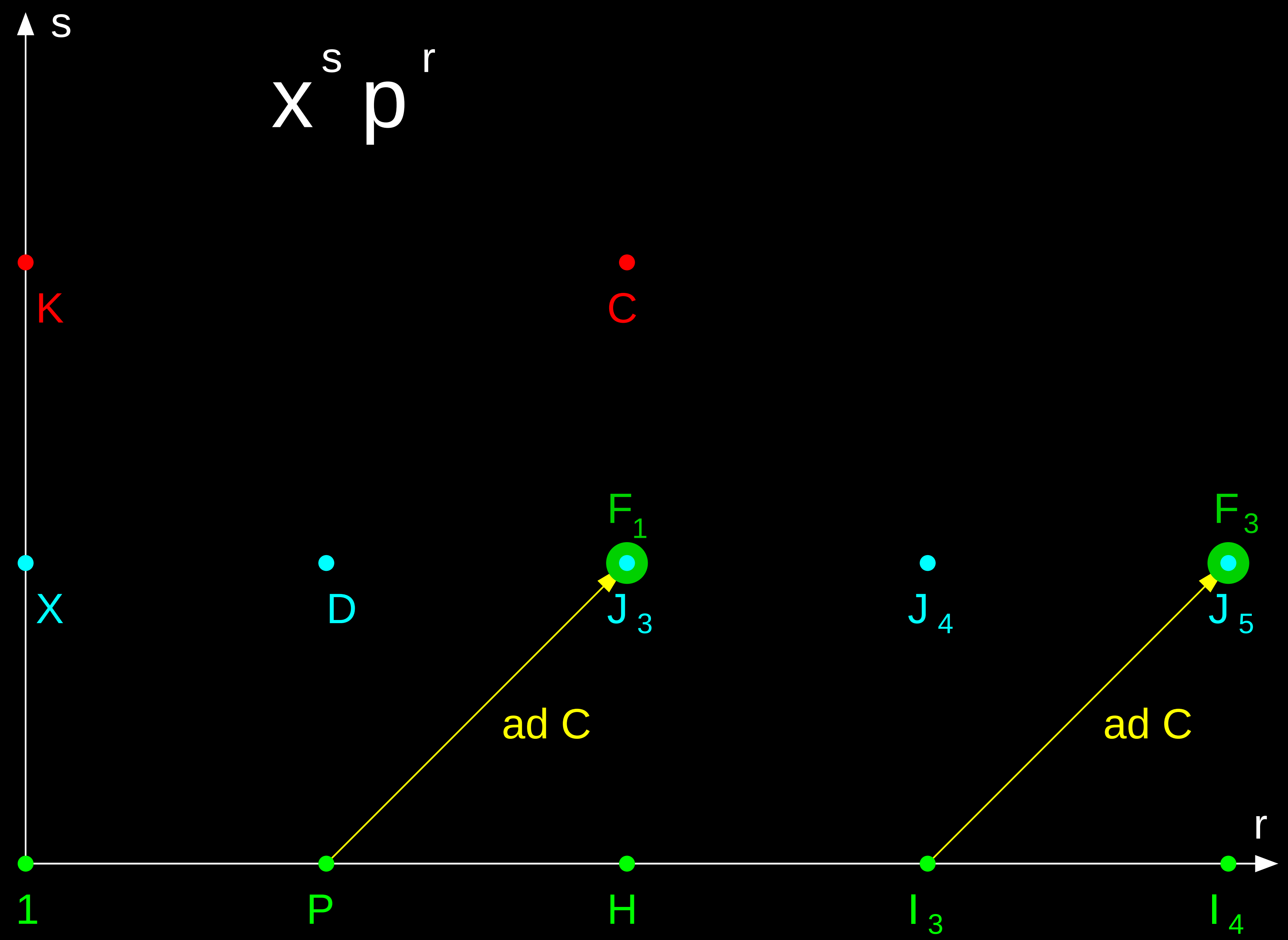
for an algebraically independent set, we take

$$L_{2,\ell} \;=:\; F_\ell \;=\; \{H, J_\ell\} - \tfrac{1}{2}\{I_\ell, D\} \;=\; \tfrac{1}{\mathrm{i}\ell}[C, I_\ell] \qquad \Rightarrow \qquad [H, F_\ell] \;=\; 0$$

special cases: $\quad F_2 \;\equiv\; 0 \quad$ and $\quad F_1 \;=\; \{H, nX\} - \tfrac{1}{2}\{P, D\}$

the F_ℓ are also generated by the adjoint action of the $sl(2,\mathbb{R})$ Casimir

$$C \;=\; KH + HK - \tfrac{1}{2}D^2 \qquad \Rightarrow \qquad [C, H] = [C, D] = [C, K] \;=\; 0$$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the F_ℓ are not in involution but obey a quadratic algebra:

$$\mathrm{i}[I_k, I_\ell] = 0 \quad , \qquad \tfrac{\mathrm{i}}{k}[I_k, F_\ell] = I_{k+\ell-2}I_2 - I_k I_\ell = \tfrac{\mathrm{i}}{\ell}[I_\ell, F_k]$$

$$\mathrm{i}[F_k, F_\ell] = (k-\ell)\tfrac{1}{2}\{F_{k+\ell-2}, I_2\} - (k-2)\tfrac{1}{2}\{F_k, I_\ell\} + (\ell-2)\tfrac{1}{2}\{F_\ell, I_k\}$$

special cases: $\mathrm{i}[I_1, F_\ell] = I_{\ell-1}I_2 - I_\ell I_1$ but $[I_2, F_\ell] \equiv 2[H, F_\ell] = 0$

however, independent set is $\{I_1, I_2, \ldots, I_n; F_1, F_3, \ldots, F_n\}$
higher I_k and F_k are order-k polynomials in lower ones
algebra becomes polynomial of order $2n-1$

was already mostly known to
Barucchi & Regge 1977 / Wojciechowski 1983 / Kuznetsov 1996

Integrals for integer coupling

Heckman (1991) constructed intertwiners $g \leftrightarrow g{+}1$ via Dunkl operators:

$$M(g)\,I_k(g) \;=\; I_k(g{+}1)\,M(g) \qquad \text{for} \qquad M(g) \;=\; \text{res}\Big(\prod_{i<j}(\pi_i{-}\pi_j)(g)\Big)$$

$$M(g)^*\,I_k(g{+}1) \;=\; I_k(g)\,M(g)^* \qquad \text{for} \qquad M(g)^* \;=\; \text{res}\Big(\prod_{i<j}(\pi_i{-}\pi_j)(-g)\Big)$$

$$M(1{-}g)\,I_k(g) \;=\; I_k(g{-}1)\,M(1{-}g) \qquad \Leftarrow \qquad M(g)^* \;=\; M(-g)$$

immediate consequence:

$$\big[M(g)^*M(g)\;,\;I_k(g)\big] \;=\; 0 \qquad \text{and} \qquad \big[M(g)\,M(g)^*\;,\;I_k(g{+}1)\big] \;=\; 0$$

new conserved charge? no, because it is a polynomial in the Liouville charges:

$$M(g)^*M(g) \;=\; M(-g)\,M(-g)^* \;=\; \mathcal{R}\big(I(g)\big) \;=:\; \mathcal{R}(g)$$

coefficients of $\mathcal{R}(I)$ do not depend on g $\quad\rightarrow\quad$ evaluate for $g{=}0$:

$$\mathcal{R}(0) \;=\; M(0)^* M(0) \;=\; \prod_{i<j} (p_i - p_j)^2$$

$$= \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}^2 \;=\; \begin{vmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \dots & p_n^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & p_1 & \dots & p_1^{n-1} \\ 1 & p_2 & \dots & p_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & p_n & \dots & p_n^{n-1} \end{vmatrix}$$

$$= \left| \left(\sum_k p_k^{i+j-2} \right)_{ij} \right| \;=\; \det\Big(I_{i+j-2}(0) \Big)_{ij}$$

hence:

$$\mathcal{R}(g) \;=\; \det\Big(I_{i+j-2}(g) \Big)_{ij}$$

$$n=2: \quad \mathcal{R} \;=\; I_1^2 - 2 I_2$$

$$n=3: \quad \mathcal{R} \;=\; I_1^2 I_4 - 2 I_1 I_2 I_3 + I_2^3 - 3 I_2 I_4 + 3 I_3^2$$

$$= \tfrac{1}{6}\Big(I_1^6 - 9 I_1^4 I_2 + 8 I_1^3 I_3 + 21 I_1^2 I_2^2 - 36 I_1 I_2 I_3 - 3 I_2^3 + 18 I_3^2 \Big)$$

so far, $g \in \mathbb{R}$ generic; but $g \in \mathbb{N}$ admits intertwiner with free theory ($g=1$):

$$L(g) \;=\; M(g-1)M(g-2)\cdots M(2)M(1) \qquad \Rightarrow \qquad L(g)I_k(1) \;=\; I_k(g)L(g)$$

$$L(g)^* \;=\; M(-1)M(-2)\cdots M(2-g)M(1-g)$$ is conjugate intertwiner

$$\Rightarrow \qquad L(g)\,L(g)^* \;=\; \big(\mathcal{R}(g)\big)^{g-1} \qquad \text{and} \qquad L(g)^*L(g) \;=\; \big(\mathcal{R}(g+1)\big)^{g-1}$$

Darboux dressing of some free $G(1)$ with $[G(1), I_k(1)] = 0$ for some k:

$$G(g) \;=\; L(g)\,G(1)\,L(g)^* \qquad \Rightarrow \qquad \big[G(g)\;,\;I_k(g)\big] \;=\; 0$$

consistent with involution of Liouville charges:

$$L(g)\,I_k(1)\,L(g)^* \;=\; \big(\mathcal{R}(g)\big)^{g-1} I_k(g)$$

large choice of 'naked' $G(1)$: any polynomial in $\{p_i\}$ with constant coefficients
identical particles $\rightarrow$ observables totally (anti)symmetric under s_{ij}
totally symmetric $\rightarrow$ Liouville integrals; totally antisymmetric $\rightarrow$ simplest is

$$G(1) = M(0) = \prod_{i<j} (p_i - p_j)$$

Darboux dressing:

$$\begin{aligned} Q(g) &= L(g)\, M(0)\, L(g)^* \\ &= M(g-1) M(g-2) \cdots M(1) M(0) M(-1) \cdots M(2-g) M(1-g) \end{aligned}$$

builds a chain relating $I_k(g) = I_k(1-g)$ back to $I_k(g)$:

$$Q(g)\, I_k(1-g) = I_k(g)\, Q(g) \qquad \Rightarrow \qquad \big[Q(g)\, ,\ I_k(g)\big] = 0$$

a conserved charge of weight $\frac{1}{2}n(n-1)(2g-1)$ algebraically independent of $\{I_k, F_\ell\}$

seeming other option $g \in \mathbb{N}+\frac{1}{2}$ fails:

$$M(g-1) \cdots M(\tfrac{3}{2}) M(\tfrac{1}{2}) M(-\tfrac{1}{2}) M(-\tfrac{3}{2}) \cdots M(1-g) = \big(\mathcal{R}(g)\big)^{g-\frac{1}{2}}$$

check the square of the new charge:

$$\begin{aligned}
Q(g)^2 &= M(g-1)\cdots M(3-g)M(2-g)\underbrace{M(1-g)M(g-1)}M(g-2)\cdots M(1-g)\\
&= M(g-1)\cdots M(3-g)M(2-g)\mathcal{R}(g-1)M(g-2)M(g-3)\cdots M(1-g)\\
&= M(g-1)\cdots M(3-g)\underbrace{M(2-g)M(g-2)}\mathcal{R}(g-2)M(g-3)\cdots M(1-g)\\
&= M(g-1)\cdots M(3-g)\big(\mathcal{R}(g-2)\big)^2 M(g-3)\cdots M(1-g)\\
&\ \vdots\\
&= M(g-1)M(1-g)\big(\mathcal{R}(1-g)\big)^{2g-2} = \big(\mathcal{R}(1-g)\big)^{2g-1} = \big(\mathcal{R}(g)\big)^{2g-1}
\end{aligned}$$

again a polynomial in the Liouville integrals, so formally $Q = \mathcal{R}^{g-\frac{1}{2}}$ for $g \in \mathbb{N}$

it remains to compute the action of ad$Q$ on the extra charges F_ℓ ...

... that is: $\mathrm{i}[Q\,,\,F_\ell] \;=\; H\,[\mathrm{i}Q\,,\,J_\ell] + [\mathrm{i}Q\,,\,J_\ell]\,H - \tfrac{1}{2}n(n{-}1)(2g{-}1)\,Q\,I_\ell$

sidestep: compute $[Q, J_\ell]$

from the observation $\big[I_k, [\mathrm{i}Q, J_\ell]\big] \;=\; \big[\mathrm{i}Q, [I_k, J_\ell]\big] \;=\; k\,\big[Q, I_{k+\ell-2}\big] \;=\; 0$

it follows that $\mathrm{i}[Q, J_\ell] \;=\; (2g{-}1)\,Q\,\mathcal{G}_\ell(I)$

with $\mathcal{G}_\ell$ being a g-independent polynomial in the I_k of weight $\ell{-}2$

compute it at $g{=}0$ (with a few tricks...) to arrive at

$$\mathcal{G}_\ell(I) \;=\; \tfrac{1}{2}\sum_{j=0}^{\ell-2} I_{\ell-2-j}I_j - \tfrac{1}{2}(\ell{-}1)I_{\ell-2}$$

the first few polynomials:

$\mathcal{G}_1 = 0 \qquad \mathcal{G}_2 = \tfrac{1}{2}n(n{-}1) \qquad \mathcal{G}_3 = (n{-}1)P \qquad \mathcal{G}_4 = (n{-}\tfrac{3}{2})2H + \tfrac{1}{2}P^2$

$\mathcal{G}_5 = (n{-}2)I_3 + 2HP \qquad \mathcal{G}_6 = (n{-}\tfrac{5}{2})I_3 + I_3P + \tfrac{1}{2}4H^2$

$$\Rightarrow \qquad \mathrm{i}[Q\,,\,F_\ell] \;=\; (2g{-}1)\,Q\,\Big(2\mathcal{G}_\ell H - \tfrac12 n(n{-}1)I_\ell\Big) \;=:\; (2g{-}1)\,Q\,\mathcal{C}_\ell(I)$$

with $\mathcal{C}_\ell$ being a g-independent polynomial in the I_k of weight ℓ

the first few polynomials:

$$\mathcal{C}_1 = -\tfrac12 n(n{-}1)P \qquad \mathcal{C}_2 = 0 \qquad \mathcal{C}_3 = (n{-}1)\Big(2HP - \tfrac{n}{2}I_3\Big)$$

$$\mathcal{C}_4 = (4n{-}6)H^2 + HP^2 - \tfrac12 n(n{-}1)I_4$$

the complete nonlinear ($\mathbb{Z}_2$ graded) algebra of $2n$ conserved charges:

$$[I_k, I_\ell] = 0 \qquad \mathrm{i}[I_k, F_\ell] = \mathcal{A}_{k,\ell}(I) \qquad \mathrm{i}[F_k, F_l] = \mathcal{B}_{k,\ell}(I,F)$$

$$[Q, I_\ell] = 0 \qquad \mathrm{i}[Q, F_\ell] = (2g{-}1)\,Q\,\mathcal{C}_\ell(I) \qquad \{Q, Q\} = 2\Big(\mathcal{R}(I)\Big)^{2g-1}$$

with right-hand sides:

$$\mathcal{A}_{k,\ell}(I) \;=\; k\,\Big(I_{k+\ell-2}I_2 - I_kI_\ell\Big)$$

$$\mathcal{B}_{k,\ell}(I,F) \;=\; (k{-}\ell)\tfrac12\{F_{k+\ell-2}, I_2\} - (k{-}2)\tfrac12\{F_k, I_\ell\} + (\ell{-}2)\tfrac12\{F_\ell, I_k\}$$

$$\mathcal{C}_\ell(I) \;=\; \tfrac12\textstyle\sum_{j=0}^{\ell-2} I_{\ell-2-j}I_jI_2 - \tfrac12(\ell{-}1)I_{\ell-2}I_2 - \tfrac12 n(n{-}1)I_\ell$$

Two particles

$$x \equiv x^{12} := x^1 - x^2 \quad , \qquad 2p \equiv p_{12} := p_1 - p_2 \quad , \qquad \pi_{12} := \pi_1 - \pi_2 = 2\Big(p + \frac{\mathrm{i}g}{x} s_{12}\Big)$$

$$P = p_1 + p_2$$

$$H = \tfrac{1}{4}P^2 + \tfrac{1}{4}\mathrm{res}\big(\pi_{12}^2\big) = \tfrac{1}{4}P^2 + \Big(p - \frac{\mathrm{i}g}{x}\Big)\Big(p + \frac{\mathrm{i}g}{x}\Big) = \tfrac{1}{4}P^2 + p^2 + \frac{g(g-1)}{x^2}$$

$$F_1 = (x^2 p_1 - x^1 p_2)(p_1 - p_2) + \tfrac{\mathrm{i}}{2}(p_1 + p_2) + 2g(g-1)\frac{x^1 + x^2}{(x^1 - x^2)^2}$$

$$C = \tfrac{1}{2}(x^2 p_1 - x^1 p_2)^2 - \tfrac{1}{2} + g(g-1)\frac{(x^1)^2 + (x^2)^2}{(x^1 - x^2)^2}$$

remove center of mass:

$$I_k(g)\Big|_{P=X=0} =: \widetilde{I}_k(g) \qquad \Rightarrow \qquad \widetilde{H}(g) = H - \tfrac{1}{4}P^2 = p^2 + \frac{g(g-1)}{x^2}$$

for integer coupling:

$$M(g) = 2\,\mathrm{res}\Big(p + \tfrac{\mathrm{i}g}{x} s_{12}\Big) = 2\Big(p + \frac{\mathrm{i}g}{x}\Big) = 2x^{g+1}\big(\tfrac{1}{x}\,p\big)x^{-g} = 2x^g p\, x^{-g}$$

$$M(-g)M(g) = -4\widetilde{H}(g) \quad \text{and} \quad (p+\tfrac{\mathrm{i}g}{x})(p^2+\tfrac{g(g-1)}{x^2}) = (p^2+\tfrac{g(g+1)}{x^2})(p+\tfrac{\mathrm{i}g}{x})$$

$$Q(g) = M(g-1)\cdots M(1-g) \qquad \Rightarrow$$

$$Q(1) = 2\,p$$

$$Q(2) = 8\Big(p^3 + \tfrac{3}{x^2}p + \tfrac{3\mathrm{i}}{x^3}\Big)$$

$$Q(3) = 32\Big(p^5 + \tfrac{15}{x^2}p^3 + \tfrac{45\mathrm{i}}{x^3}p^2 - \tfrac{45}{x^4}p\Big)$$

$$Q(4) = 128\Big(p^7 + \tfrac{42}{x^2}p^5 + \tfrac{210\mathrm{i}}{x^3}p^4 - \tfrac{315}{x^4}p^3 + \tfrac{630\mathrm{i}}{x^5}p^2 - \tfrac{2835}{x^6}p - \tfrac{2835\mathrm{i}}{x^7}\Big)$$

nontrivial commutators:

$$\mathrm{i}[P, F_1] = 4\widetilde{H} \quad , \quad \mathrm{i}[Q, F_1] = -(2g-1)\,Q\,P \quad , \quad Q^2 = (-4\widetilde{H})^{2g-1}$$

Three particles

$$x^{ij} := x^i - x^j \quad , \qquad p_{ij} := p_i - p_j \quad , \qquad \pi_{ij} = p_{ij} + \frac{2\mathrm{i}g}{x^{ij}} s_{ij} - \frac{\mathrm{i}g}{x^{jk}} s_{jk} - \frac{\mathrm{i}g}{x^{ki}} s_{ki}$$

$$\begin{aligned}
P &= p_1 + p_2 + p_3 \\
H &= \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + g(g-1)\Big(\frac{1}{(x^{12})^2} + \frac{1}{(x^{23})^2} + \frac{1}{(x^{31})^2}\Big) \\
I_3 &= \big(p_1^3 + p_2^3 + p_3^3\big) + 3g(g-1)\Big(\frac{p_1+p_2}{(x^{12})^2} + \frac{p_2+p_3}{(x^{23})^2} + \frac{p_3+p_1}{(x^{31})^2}\Big) \\
F_1 &= 2(x^1 + x^2 + x^3)H - (x^1 p_1 + x^2 p_2 + x^3 p_3 - \mathrm{i})P \\
F_3 &= 2J_3 H - (x^1 p_1 + x^2 p_2 + x^3 p_3 - 2\mathrm{i})I_3 \\
J_3 &= p_1 x^1 p_1 + p_2 x^2 p_2 + p_3 x^3 p_3 + g(g-1)\Big(\frac{x^1+x^2}{(x^{12})^2} + \frac{x^2+x^3}{(x^{23})^2} + \frac{x^3+x^1}{(x^{31})^2}\Big)
\end{aligned}$$

$$C = \frac{1}{2}\sum_{i<j}(x^j p_i - x^i p_j)^2 - \frac{3}{8} + g(g-1)\sum_i (x^i)^2 \sum_{j<k} \frac{1}{(x^{jk})^2}$$

$$\Delta = x^{12}x^{23}x^{31}$$

$$
\begin{aligned}
M(g) &= \mathrm{res}\Big(\pi_{12}(g)\pi_{23}(g)\pi_{31}(g)\Big) \\
&= \Delta^{g}\Big(p_{12}\,p_{23}\,p_{31} - \tfrac{\mathrm{i}g}{x^{12}}p_{12}^2 - \tfrac{\mathrm{i}g}{x^{23}}p_{23}^2 - \tfrac{\mathrm{i}g}{x^{31}}p_{31}^2 \\
&\qquad + \tfrac{2g}{(x^{12})^2}p_{12} + \tfrac{2g}{(x^{23})^2}p_{23} + \tfrac{2g}{(x^{31})^2}p_{31}\Big)\,\Delta^{-g} \\
&= p_{12}\,p_{23}\,p_{31} + \tfrac{2\mathrm{i}g}{x^{12}}p_{23}\,p_{31} + \tfrac{2\mathrm{i}g}{x^{23}}p_{31}\,p_{12} + \tfrac{2\mathrm{i}g}{x^{31}}p_{12}\,p_{23} \\
&\quad - \tfrac{4g^2}{x^{12}x^{23}}p_{31} - \tfrac{4g^2}{x^{23}x^{31}}p_{12} - \tfrac{4g^2}{x^{31}x^{12}}p_{23} + \tfrac{g(g-1)}{(x^{12})^2}p_{12} + \tfrac{g(g-1)}{(x^{23})^2}p_{23} + \tfrac{g(g-1)}{(x^{31})^2}p_{31} \\
&\quad - \tfrac{6\mathrm{i}\,g^2(g+1)}{x^{12}x^{23}x^{31}} + 2\mathrm{i}g(g-1)(g+2)\Big(\tfrac{1}{(x^{12})^3} + \tfrac{1}{(x^{23})^3} + \tfrac{1}{(x^{31})^3}\Big)
\end{aligned}
$$

$$M^*M = 3I_3^2 - 12I_3HP + \tfrac{4}{3}I_3P^3 - 4H^3 + 14H^2P^2 - 3HP^4 + \tfrac{1}{6}P^6$$

$$
\begin{aligned}
Q(2) \;=\; & \tfrac{1}{6}\, p_{12}^3\, p_{23}^3\, p_{31}^3 \\
& + \frac{3}{(x^{12})^2}\Big(p_{12}^3\, p_{23}^2\, p_{31}^2 + 2\, p_{12}\, p_{23}^3\, p_{31}^3\Big) \\
& + \frac{12\mathrm{i}}{(x^{12})^3}\Big(p_{12}^2\, p_{23}^3\, p_{31} + p_{23}^3\, p_{31}^3 + 4\, p_{12}^2\, p_{23}^2\, p_{31}^2\Big) \\
& - \left(\frac{12}{(x^{12})^4} - \frac{24}{(x^{12})^2(x^{31})^2}\right) p_{12}^3\, p_{23}^2 \\
& + \left(\frac{264}{(x^{23})^4} - \frac{180}{(x^{12})^4} - \frac{168}{(x^{12})^2(x^{23})^2}\right) p_{12}\, p_{23}^2\, p_{31}^2 \\
& + \mathrm{i}\left(\frac{1440}{(x^{12})^5} - \frac{720}{(x^{12})^3(x^{31})^2} - \frac{720}{(x^{12})^2(x^{31})^3}\right) p_{12}^3\, p_{23} \\
& + \mathrm{i}\left(\frac{1080}{(x^{12})^5} - \frac{360}{(x^{31})^5} - \frac{360}{(x^{12})^3(x^{31})^2} - \frac{1080}{(x^{12})^2(x^{31})^3}\right) p_{12}^2\, p_{23}^2 \\
& + \left(\frac{4200}{(x^{12})^6} + \frac{3360}{(x^{23})^6} - \frac{1920}{(x^{12})^3(x^{23})^3} + \frac{1200}{(x^{23})^3(x^{31})^3} + \frac{2880}{(x^{12})^2(x^{31})^4}\right) p_{12}^3 \\
& - \frac{4320}{(x^{12})^2(x^{31})^4}\, p_{12}^2\, p_{23} - \frac{5760}{(x^{12})^2(x^{31})^4}\, p_{12}\, p_{23}\, p_{31} \\
& + \mathrm{i}\left(\frac{25200}{(x^{12})^7} - \frac{10080}{(x^{23})^7} - \frac{7200}{(x^{12})^5(x^{23})^2} - \frac{5760}{(x^{12})^4(x^{23})^3} + \frac{10080}{(x^{12})^3(x^{23})^4} - \frac{1440}{(x^{12})^2(x^{23})^5}\right) p_{12}^2 \\
& - \left(\frac{90720}{(x^{12})^8} + \frac{198720}{(x^{12})^7(x^{23})} - \frac{129600}{(x^{12})^6(x^{23})^2} + \frac{34560}{(x^{12})^5(x^{23})^3} - \frac{17280}{(x^{12})^3(x^{23})^5}\right) p_{12} \\
& - \frac{181440\mathrm{i}}{(x^{12})^9} - \frac{60480\mathrm{i}}{(x^{12})^7 x^{23} x^{31}} \;+\; \text{all permutations in } (123)
\end{aligned}
$$

nontrivial commutators:

$$\mathrm{i}[P, F_1] = 6H - P^2$$

$$\mathrm{i}[I_3, F_1] = 12H^2 - 3I_3P$$

$$\mathrm{i}[P, F_3] = 4H^2 - I_3P$$

$$\mathrm{i}[I_3, F_3] = -3I_3^2 + 8I_3HP + 12H^3 - 12H^2P^2 + HP^4$$

$$\mathrm{i}[F_1, F_3] = \tfrac{1}{2}(F_1I_3 + I_3F_1 + F_3P + PF_3)$$

$$\mathrm{i}[Q, F_1] = -3(2g-1)\,Q\,P$$

$$\mathrm{i}[Q, F_3] = -3(2g-1)\,Q\Big(I_3 - \tfrac{4}{3}HP\Big)$$

$$Q^2 = \Big(3I_3^2 - 12I_3HP + \tfrac{4}{3}I_3P^3 - 4H^3 + 14H^2P^2 - 3HP^4 + \tfrac{1}{6}P^6\Big)^{2g-1}$$

Four particles

Dunkl operators:

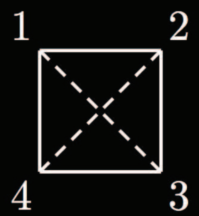
$$\pi_{12} = p_{12} + \frac{2\mathrm{i}g}{x^{12}} s_{12} + \frac{\mathrm{i}g}{x^{13}} s_{13} - \frac{\mathrm{i}g}{x^{23}} s_{23} + \frac{\mathrm{i}g}{x^{14}} s_{14} - \frac{\mathrm{i}g}{x^{24}} s_{24}$$ and permutations

conserved for any g value: $P\,,\; H\,,\; I_3\,,\; I_4\,,\; F_1\,,\; F_3\,,\; F_4$

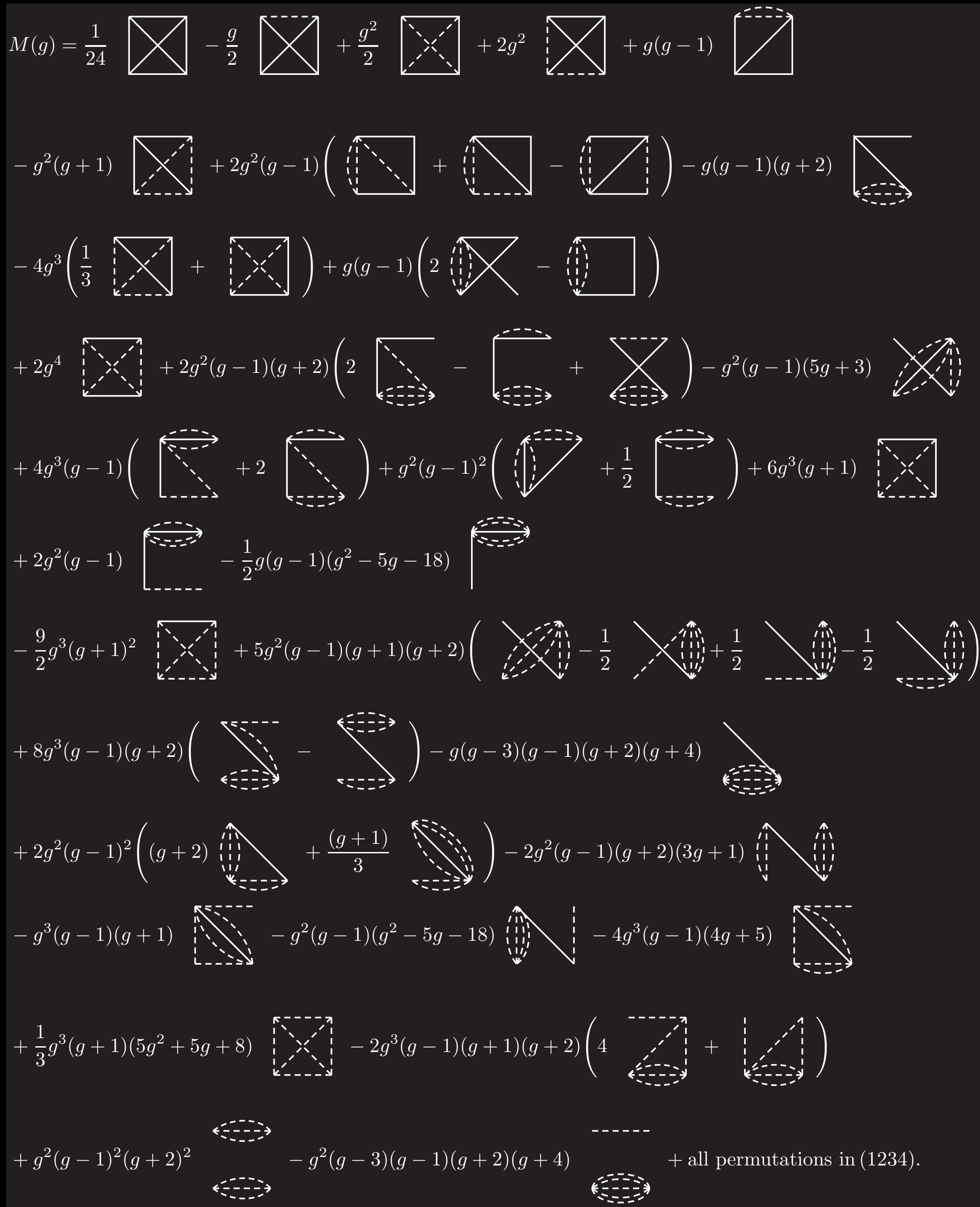
conformal Casimir: $$C = \frac{1}{2}\sum_{i<j}(x^j p_i - x^i p_j)^2 - \frac{3}{8} + g(g-1)\sum_i (x^i)^2 \sum_{j<k}\frac{1}{(x^{jk})^2}$$

basic intertwiner:

$$M(g) = \mathrm{res}\Big(\pi_{12}(g)\pi_{31}(g)\pi_{14}(g)\pi_{23}(g)\pi_{24}(g)\pi_{34}(g)\Big)$$



$$= \frac{1}{x^{31}x^{24}} p_{12}\, p_{23}\, p_{34}\, p_{14}$$



$$
\begin{aligned}
M^*M \;=\; \frac{1}{576}\Big(& P^{12} - 48P^{10}H + 840P^8H^2 - 6368P^6H^3 + 19344P^4H^4 \\
& - 21888P^2H^5 + 4608H^6 + 40P^9I_3 - 1296P^7HI_3 \\
& + 12576P^5H^2I_3 - 33344P^3H^3I_3 + 24576PH^4I_3 \\
& + 544P^6I_3^2 - 9024P^4HI_3^2 + 16896P^2H^2I_3^2 - 4352H^3I_3^2 \\
& + 2496P^3I_3^3 - 1152PHI_3^3 - 192I_3^4 - 36P^8I_4 \\
& + 1152P^6HI_4 - 10800P^4H^2I_4 + 24768P^2H^3I_4 \\
& - 11520H^4I_4 - 1008P^5I_3I_4 + 16416P^3HI_3I_4 \\
& - 25344PH^2I_3I_4 - 7200P^2I_3^2I_4 + 2304HI_3^2I_4 + 468P^4I_4^2 \\
& - 7488P^2HI_4^2 + 9216H^2I_4^2 + 6912PI_3I_4^2 - 2304I_4^3\Big)
\end{aligned}
$$

nontrivial commutators:

$$\mathrm{i}[P,F_1]=8H-P^2\,,\quad \mathrm{i}[I_3,F_1]=12H^2-3I_3P\,,\quad \mathrm{i}[P,F_3]=4H^2-I_3P$$
$$\mathrm{i}[I_3,F_3]=6I_4H-3I_3^2\,,\quad \mathrm{i}[P,F_4]=2I_3H-I_4P\,,\quad \mathrm{i}[I_4,F_1]=8I_3H-4I_4P$$
$$\mathrm{i}[I_3,F_4]=-3I_4I_3+\tfrac{15}{2}I_4HP+10I_3H^2-5I_3HP^2-15H^3P+5H^2P^3-\tfrac{1}{4}HP^5$$
$$\mathrm{i}[I_4,F_3]=-4I_4I_3+10I_4HP+\tfrac{40}{3}I_3H^2-\tfrac{20}{3}I_3HP^2-20H^3P+\tfrac{20}{3}H^2P^3-\tfrac{1}{3}HP^5$$
$$\mathrm{i}[I_4,F_4]=-4I_4^2+12I_4H^2+6I_4HP^2+\tfrac{8}{3}I_3^2H-\tfrac{16}{3}I_3HP^3-8H^4-12H^3P^2+6H^2P^4-\tfrac{1}{3}HP^6$$

$$\mathrm{i}[F_1,F_3]=\tfrac{1}{2}(F_1I_3+I_3F_1+F_3P+PF_3)$$
$$\mathrm{i}[F_1,F_4]=\tfrac{1}{2}(F_1I_4+I_4F_1+2F_4P+2PF_4-12F_3H)$$
$$\mathrm{i}[F_3,F_4]=F_4I_3+I_3F_4-2F_4HP-\tfrac{1}{2}F_3I_4-\tfrac{1}{2}I_4F_3-F_3H(2H-P^2)-\tfrac{1}{2}I_4HF_1+H^3F_1$$
$$+\tfrac{1}{3}I_3H(F_1P+PF_1)-\tfrac{1}{3}H^2(F_1P^2+PF_1P+P^2F_1)+\tfrac{1}{60}H(F_1P^4+\ldots+P^4F_1)$$

$$\mathrm{i}[Q,F_1]=-6(2g-1)\,Q\,P$$
$$\mathrm{i}[Q,F_3]=-6(2g-1)\,Q\,\big(I_3-HP\big)$$
$$\mathrm{i}[Q,F_4]=-6(2g-1)\,Q\,\big(I_4-\tfrac{5}{3}H^2-\tfrac{1}{6}HP^2\big)$$

$$Q^2=(M^*M)^{2g-1}$$

Summary

- for generic coupling g, rank-n Calogero models are superintegrable
- n conserved Liouville charges $\{I_1{=}P, I_2{=}2H, I_3, \ldots, I_n\}$ in involution
- ad$C$ produces $n{-}1$ additional charges $\{F_1, F_3, F_4, \ldots, F_n\}$ not in involution
- they obey a nonlinear commutator algebra (polynomial of order $2n{-}1$)
- Dunkl operators allow one to construct intertwiners $g \leftrightarrow g{+}1$
- for integer coupling g, chain of intertwiners link with free theory ($g{=}1$)
- Vandermonde $\prod_{i<j}(p_i{-}p_j)$ gets mapped to new 'odd' conserved charge Q
- Q is of weight $\frac{1}{2}n(n{-}1)(2g{-}1)$ and squares to a polynomial in the I_k
- Q extends commutator algebra to a Z_2 graded one $\rightarrow$ nonlinear 'supersymmetry'
- explicitly worked out the cases $n{=}2, 3, 4$ (partially for the first time)

THANK YOU !