Drinfeld basis for twisted Yangians

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Yangians and Twisted Yangians

Yangians we first introduced in Faddeev's Leningrad school concerning quantum inverse scattering method in late 70s and early 80s. Named by V.G.Drinfeld to honour C.N.Yang [Drinfeld 85, 86]

Twisted Yangians appeared in the work of G.I.Olshanski in '91 and were generalized by Delius-MacKay-Short '01, MacKay '02 & '03

Plan of the talk:

- Brief review
- Construction of Yangian in Drinfeld Basis (DI)
- Construction of Twisted Yangian in Drinfeld Basis (DI)

Drinfel'd Basis (DI) [Drinfeld'85,86]

- Yangian $Y(\mathfrak{g})$ is a flat deformation of $U(\mathfrak{g}[u])$. It is a Hopf algebra generated by $J(x)$, $x \in \mathfrak{g}$ satisfying:

$$[x_a, x_b] = f_{ab}{}^c x_c, \quad [x_a, J(x_b)] = J([x_a, x_b]) = f_{ab}{}^c J(x_c)$$
$$[J(x_a), J([x_b, x_c])] + [J(x_b), J([x_c, x_a])] + [J(x_c), J([x_a, x_b])] = \hbar^2 \beta^{ijk}_{abc} \{x_i, x_j, x_k\}$$
$$[[J(x_a), J(x_b)], J([x_c, x_d])] + [[J(x_c), J(x_d)], J([x_a, x_b])] = \hbar^2 \gamma^{ijk}_{abcd} \{x_i, x_j, J(x_k)\}$$

- Coalgebra structure is:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{\hbar}{2}[x \otimes 1, \Omega]$$

\+ Minimal realization
\+ Unique form for any simple $\mathfrak{g}$
\+ Very simple presentation, often used in theoretical physics
\+ Simple evaluation modules for $Y(\mathfrak{sl}_N)$

− Non-trivial higher level generators
− Complicated higher-order relations
− Not well suitable for representation theory

Drinfel'd New presentation (DII) [Drinfeld'88]

- The Yangian $Y(\mathfrak{g})$ is isomorphic to the algebra generated by the elements $x_{ir}^{\pm}$, h_{ir} for $i \in I$, $r \in \mathbf{Z}_{\geq 0}$ subject to the relations

$$[h_{ir}, h_{js}] = 0, \quad [h_{i0}, x_{jr}^{\pm}] = \pm(\alpha_i, \alpha_j)x_{jr}^{\pm}, \quad [x_{ir}^{+}, x_{js}^{-}] = \delta_{ij} h_{i,r+s}$$

$$[h_{i,r+1}, x_{js}^{\pm}] - [h_{ir}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{\hbar(\alpha_i, \alpha_j)}{2}(h_{ir}^{\pm} x_{js}^{\pm} + x_{js}^{\pm} h_{ir}^{\pm})$$

$$[x_{i,r+1}^{\pm}, x_{js}^{\pm}] - [x_{ir}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{\hbar(\alpha_i, \alpha_j)}{2}(x_{ir}^{\pm} x_{js}^{\pm} + x_{js}^{\pm} x_{ir}^{\pm})$$

$$\sum_{\sigma \in S_n} [x_{ir_{\sigma(1)}}^{\pm}, [\cdots, [x_{ir_{\sigma(n)}}^{\pm}, x_{js}^{\pm}] \cdots]] = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad n = 1 - a_{ij}$$

$+$ Well suited for representation theory
$+$ Well defined generators and relations for any simple $\mathfrak{g}$
$+$ Loved by mathematicians

$-$ No explicit form of coproduct $\Delta(x_{i,r}^{\pm}) = \ldots \quad \Delta(h_{i,r}) = \ldots$
$-$ Has complicated form of non-simple root vectors

RTT-presentation (FRT) [Fadeev-Reshetikhin-Takhtajan'89]

- The Yangian $Y(\mathfrak{g})$ is isomorphic to the algebra generated by the elements $t_{ij}^{(r)}$ for $0 \leq i, j \leq N$ and $r \in \mathbf{Z}_{\geq 0}$, satisfying:

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v),$$

where

$$T_1(u) = \sum_{i,j=-n}^{n} E_{ij} \otimes 1 \otimes t_{ij}(u), \qquad T_2(u) = \sum_{i,j=-n}^{n} 1 \otimes E_{ij} \otimes t_{ij}(u),$$

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)}\, u^{-r} \in Y(\mathfrak{g})[[u^{-1}]], \qquad t_{ij}^{(0)} = \delta_{ij}$$

- Coalgebra structure is

$$\Delta(t_{ij}(u)) = \sum_{k=0}^{N} t_{ik}(u) \otimes t_{kj}(u)$$

+ Well suited for representation theory
+ Well defined generators and relations
+ Good to treat central elements

− Coproduct has complicated concrete realization*
− Concrete realization of $Y(\mathfrak{g})$ depends on R-matrix $R(u)$
− Lots of technical difficulties*

Twisted Yangian

- Twisted Yangian $Y(\mathfrak{g},\mathfrak{h})$ is a flat deformation of $U(\mathfrak{g}[u]^\rho)$
- It is a coideal subalgebra of $Y(\mathfrak{g})$ introduced by G.I.Olshanski in '92 for $\mathfrak{sl}_N$ and generalized to any simple $\mathfrak{g}$ by Delius-MacKay-Short in '01 playing a major role in quantum integrable models with open boundaries

$$\Delta(Y(\mathfrak{g},\mathfrak{h})) = Y(\mathfrak{g}) \otimes Y(\mathfrak{g},\mathfrak{h})$$

- There is a huge family of $Y(\mathfrak{g},\mathfrak{h})$ that is in a 1-to-1 correspondence with symmetric pairs $(\mathfrak{g},\mathfrak{h})$ of simple complex Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ satifying

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$$

- RTT-type presentation for symmetric pairs of simple complex Lie algebras:

  $AI : (\mathfrak{sl}_N, \mathfrak{so}_N), AII : (\mathfrak{sl}_N, \mathfrak{sp}_{N/2})$ Olshanski'92, Molev-Nazarov-Olshanski'96

  $AIII : (\mathfrak{sl}_N, \mathfrak{sl}_k \oplus \mathfrak{sl}_l)$ Molev-Ragoucy'02

  $CI : (\mathfrak{sp}_N, \mathfrak{gl}_N), CII : (\mathfrak{sp}_N, \mathfrak{sp}_p \oplus \mathfrak{sp}_q), DIII : (\mathfrak{so}_N, \mathfrak{gl}_{N/2})$

  $BDI : (\mathfrak{so}_N, \mathfrak{so}_p \oplus \mathfrak{so}_q), C0 : (\mathfrak{sp}_N, \mathfrak{sp}_N), BD0 : (\mathfrak{so}_N, \mathfrak{so}_N)$ [Guay-VR'14: to appear]

- Generic twisted Yangians $Y(\mathfrak{g},\mathfrak{h})$ for any $(\mathfrak{g},\mathfrak{h})$ Delius-MacKay-Short'01, MacKay'02 '03

  Arnoudon, Avan, Baseilhac, Crampe, Doikou, Frappat, Khoroshkin, Mudrov, Nepomechie, Sklyanin, Sorba...

RA-presentation (RTT)

- Twisted Yangian $Y(\mathfrak{g},\mathfrak{h})$ is isomorphic to the algebra generated by the elements $s_{ij}^{(r)}$ for $0 \le i,j \le N$ and $r \in \mathbf{Z}_{\ge 0}$, satisfying:

$$R(u-v)\,S_1(u)\,R(u+v)\,S_2(v) = S_2(v)\,R(u+v)\,S_1(u)\,R(u-v)$$

  and some additional symmetry relations $S(U) = f(S(U))$, where

$$S(u) = \sum_{i,j=0}^{N} E_{ij} \otimes s_{ij}(u), \qquad s_{ij}(u) = \sum_{r=0}^{\infty} s_{ij}^{(r)}\,u^{-r}, \qquad s_{ij}^{(0)} = b_{ij}$$

- Coideal structure is

$$\Delta(s_{ij}(u)) = \sum_{k,l=0}^{N} t_{ik}(u)\,\theta(t_{jl}(u)) \otimes s_{kl}(u)$$

  \+ Well suited for representation theory
  \+ Well defined generators and relations
  \+ Good to treat central elements

  − Coproduct has complicated concrete realization
  − Requires additional symmetry relations for each $(\mathfrak{g},\mathfrak{h})$
  − Defining relations depend on the type of symmetric pair, R-matrix $R(u)$ and b_{ij}

Drinfeld Basis (DI)

- Twisted Yangian $Y(\mathfrak{g},\mathfrak{h})$ is isomorphic to the algebra generated by elements $x \in \mathfrak{h}$ and $B(y)$ for $y \in \mathfrak{m}$, satisfying:

$$[x_\alpha, x_\beta] = f_{\alpha\beta}{}^{\gamma} x_\gamma, \quad [x_\alpha, B(y_p)] = B([x_\alpha, y_p]) = g_{\alpha p}{}^{q} B(y_q)$$

$$[B(y_p), B(y_q)] + \frac{1}{\bar{\mathfrak{c}}_{(\alpha)}} w_{pq}{}^{\alpha} w_{\alpha}{}^{rs} [B(y_r), B(y_s)] = \hbar^2 \Lambda_{pq}^{\lambda\mu\nu} \{x_\lambda, x_\mu, x_\nu\}$$

$$[[B(y_p), B(y_q)], B(y_r)] + \frac{2}{\mathfrak{c}_{\mathfrak{g}}} \kappa_{\mathfrak{m}}^{tu} w_{pq}{}^{\alpha} g_{r\alpha}{}^{s} [[B(y_s), B(y_t)], B(y_u)] = \hbar^2 \Upsilon_{pqr}^{\lambda\mu u} \{x_\lambda, x_\mu, B(y_u)\}$$

- Twisted Yangian $Y(\mathfrak{g},\mathfrak{g})$ is isomorphic to the algebra generated by elements $G(x)$, $x \in \mathfrak{g}$ satisfying:

$$[x_a, x_b] = \alpha_{ab}{}^{c} x_c, \qquad [x_a, G(x_b)] = G([x_a, x_b]) = \alpha_{ab}{}^{c} G(x_c)$$

$$[G(x_a), G([x_b, x_c])] + [G(x_b), G([x_c, x_a])] + [G(x_c), G([x_a, x_b])]$$
$$= \hbar^2 \Psi_{abc}^{ijk} \{x_i, x_j, G(x_k)\} + \hbar^4 (\Phi_{abc}^{ijk} \{x_i, x_j, x_k\} + \bar{\Phi}_{abc}^{ijklm} \{x_i, x_j, x_k, x_l, x_m\})$$

+ Minimal realization
+ Unique form for any simple $\mathfrak{g}$
+ Very simple presentation, good for theoretical physics

− Non-trivial higher level generators and relations
− Not well suitable for representation theory

Yangian $Y(\mathfrak{g})$

Preliminaries: Lie algebra and Lie bi-algebra

- Let $\mathfrak{g}$ be a complex simple Lie algebra of $\dim(\mathfrak{g}) = n$ with a basis $\{x_a\}$ and a Lie bracket

$$[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \qquad [x_a, x_b] = \alpha_{ab}{}^{c}\, x_c$$

  Let η_{ab} be the Killing form and η^{ab} its inverse:

$$(x_a, x_b)_{\mathfrak{g}} = \eta_{ab} = \alpha_{ac}{}^{d} \alpha_{bd}{}^{c}, \qquad \alpha_{ab}{}^{d} \eta_{dc} = \alpha_{abc}, \qquad \eta_{ab}\eta^{bc} = \delta_{a}{}^{c}$$

  Let $C_{\mathfrak{g}} = \eta^{ab} x_a x_b$ be the Casimir operator and $\mathfrak{c}_{\mathfrak{g}}$ be its eigenvalue in the adjoint representation, then

$$\mathfrak{c}_{\mathfrak{g}}\, \delta_{c}{}^{d} = \eta^{ab} \alpha_{ac}{}^{e} \alpha_{be}{}^{d} = \alpha_{c}{}^{eb} \alpha_{be}{}^{d}, \qquad \alpha_{a}{}^{bc}[x_c, x_b] = \mathfrak{c}_{\mathfrak{g}}\, x_a$$

- A Lie bi-algebra structure on $\mathfrak{g}$ is a skew-symmetric linear map

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$$

  the cocommutator, such that $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$ and δ is a 1-cocycle

$$\delta([x, y]) = x.\delta(y) - y.\delta(x)$$

  where dot denotes the adjoint action on $\mathfrak{g}^* \otimes \mathfrak{g}^*$.

Preliminaries: Half-loop Lie algebra

- Let $\mathcal{L}^+$ be a half-loop Lie algebra generated by elements $\{x_a^{(k)}\}$ with $k \in \mathbb{Z}_{\geq 0}$. It is a graded algebra with $\deg(x_a^{(k)}) = k$ and the defining relations

$$[x_a^{(k)}, x_b^{(l)}] = \alpha_{ab}{}^c \, x_c^{(k+l)}$$

  This algebra can be identified with the set of polynomial maps $f : \mathbb{C} \to \mathfrak{g}$ using the Lie algebra isomorphism $\mathcal{L}^+ \cong \mathfrak{g}[u] = \mathfrak{g} \otimes \mathbb{C}[u]$ with $x_a^{(k)} \cong x_a \otimes u^k$.

- $\mathcal{L}^+$ is isomorphic to an algebra generated by x_a, $J(x_b)$ satisfying $(\mu, \nu \in \mathbb{C})$

$$[x_a, x_b] = \alpha_{ab}{}^c x_c, \quad J(\mu x_a + \nu x_b) = \mu J(x_a) + \nu J(x_b), \quad [x_a, J(x_b)] = \alpha_{ab}{}^c J(x_c)$$
$$[J(x_a), J([x_b, x_c])] + [J(x_b), J([x_c, x_a])] + [J(x_c), J([x_a, x_b])] = 0$$
$$[[J(x_a), J(x_b)], J([x_c, x_d])] + [[J(x_c), J(x_d)], J([x_a, x_b])] = 0$$

- The isomorphism with the standard loop basis is given by the map

$$x_a \mapsto x_a^{(0)}, \qquad J(x_a) \mapsto x_a^{(1)}$$

- Next step: we want to construct a Lie bi-algebra structure $\delta : \mathfrak{L}^+ \to \mathfrak{L}^+ \otimes \mathfrak{L}^+$.

Preliminaries: Manin triple

- A Manin triple is a triple of Lie bi-algebras $(\mathfrak{p}, \mathfrak{p}^+, \mathfrak{p}^-)$ such that
  - $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Lie subalgebras of $\mathfrak{p}$
  - $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ as a vector space
  - $(\ ,\ )_{\mathfrak{p}}$ is isotopic for $\mathfrak{p}^{\pm}$ (i.e. $(\mathfrak{p}^{\pm}, \mathfrak{p}^{\pm})_{\mathfrak{p}} = 0$)
  - $(\mathfrak{p}^+)^* \cong \mathfrak{p}^-$
- For any $\mathfrak{g}$ the is a 1-to-1 correspondence between Lie bi-algebra structures on $\mathfrak{g}$ and the Manin triple $(\mathfrak{p}, \mathfrak{p}^+, \mathfrak{p}^-)$ such that $\mathfrak{p}^+ = \mathfrak{g}$.
- Let $\mathcal{L} = \mathfrak{g}[[u^{\pm 1}]]$ and $\mathcal{L}^- = \mathfrak{g}[[u^{-1}]]$. Then $(\mathcal{L}, \mathcal{L}^+, \mathcal{L}^-)$ is a Manin triple.
- The cocomutator δ on $\mathcal{L}^+$ is deduced from the duality relation

$$(\delta(x), y \otimes z)_{\mathcal{L}} = (x, [y, z])_{\mathcal{L}} \quad \text{where} \quad (x, y)_{\mathcal{L}} = (x, y)_{\mathfrak{g}}\, \delta_{deg(x)+deg(y)+1,0}$$

For $x_a \in \mathcal{L}^+$, $deg(x_a) = 0$ we find

$$(\delta(x_a), y \otimes z)_{\mathcal{L}} = 0 \qquad \Longrightarrow \qquad \delta(x_a) = 0.$$

since $deg(y \otimes z) < 1$ for any $y, z \in \mathcal{L}^-$, and for $J(x_a) \in \mathcal{L}^+$, $deg(J(x_a)) = 1$ we have

$$(\delta(J(x_a)), \alpha_b{}^{cd} x_d^{(-1)} \otimes x_c^{(-1)})_{\mathcal{L}} = \mathfrak{c}_{\mathfrak{g}}\, \eta_{ab}$$

$$\Longrightarrow$$

$$\delta(J(x_a)) = \alpha_a{}^{lk} x_k \otimes x_l = [x_a \otimes 1, \Omega_{\mathfrak{g}}], \qquad \Omega_{\mathfrak{g}} = \eta^{ab} x_a \otimes x_b$$

Preliminaries: Quantization [Drinfeld'85 '86]

- A Hopf algebra is a sextuple $(A, \mu, \imath, \Delta, \varepsilon, S)$ where

$$\begin{array}{rlrl} \textit{product} & \mu : A \otimes A \to A & \textit{unit} & \imath : \mathbb{C} \to A \\ \textit{coproduct} & \Delta : A \to A \otimes A & \textit{couint} & \varepsilon : A \to \mathbb{C} \\ \textit{antipode} & S : A \to A & & \end{array}$$

  such that $(A, \mu, \imath)$ is an algebra and (A, Δ, ε) is a coalgebra.

- Let $(\mathcal{L}^+, \delta)$ be a Lie bi-algebra. We say that a quantized universal enveloping algebra $(\mathcal{U}_\hbar(\mathcal{L}^+), \Delta_\hbar)$ is a quantization of $(\mathcal{L}^+, \delta)$, or that $(\mathcal{L}^+, \delta)$ is the quasi-classical limit of $(\mathcal{U}_\hbar(\mathcal{L}^+), \Delta_\hbar)$, if it is a free $\mathbb{C}[[\hbar]]$ module and
  - $\mathcal{U}_\hbar(\mathcal{L}^+) \,/\, \hbar\,\mathcal{U}_\hbar(\mathcal{L}^+)$ is isomorphic to $\mathcal{U}(\mathcal{L}^+)$ as a Hopf algebra
  - for any $x \in \mathcal{L}^+$ and any $X \in \mathcal{U}_\hbar(\mathcal{L}^+)$ equal to $x\,(\mathit{mod}\,\hbar)$ one has

$$\Big(\Delta_\hbar(X) - \sigma \circ \Delta_\hbar(X)\Big)/\hbar \sim \delta(x) \quad (\mathit{mod}\,\hbar)$$

    with σ the permutation map $\sigma(a \otimes b) = b \otimes a$.

- The simplest solution of the quantization conditions for $x_a, \mathcal{J}(x_a) \in \mathcal{U}_\hbar(\mathcal{L}^+)$ satisfying co-associativity property $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ is:

$$\Delta_h(x_a) = x_a \otimes 1 + 1 \otimes x_a$$

$$\Delta_h(\mathcal{J}(x_a)) = \mathcal{J}(x_a) \otimes 1 + 1 \otimes \mathcal{J}(x_a) + \frac{\hbar}{2}\,[x_a \otimes 1, \Omega_\mathfrak{g}]$$

Yangian [Drinfeld'85 '86]

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. There is, up to isomorphism, a unique homogeneous quantization $\mathcal{Y}(\mathfrak{g}) := \mathcal{U}_\hbar(\mathfrak{g}[u])$ of $(\mathfrak{g}[u], \delta)$ generated by elements x_a, $\mathcal{J}(x_a)$ satisfying:

$$[x_a, x_b] = \alpha_{ab}{}^c\, x_c, \qquad [x_a, \mathcal{J}(x_b)] = \alpha_{ab}{}^c\, \mathcal{J}(x_c),$$

$$[\mathcal{J}(x_a), \mathcal{J}([x_b, x_c])] + [\mathcal{J}(x_b), \mathcal{J}([x_c, x_a])] + [\mathcal{J}(x_c), \mathcal{J}([x_a, x_b])] = \tfrac{1}{4}\hbar^2 \beta^{ijk}_{abc}\, \{x_i, x_j, x_k\}$$

$$[[\mathcal{J}(x_a), \mathcal{J}(x_b)], \mathcal{J}([x_c, x_d])] + [[\mathcal{J}(x_c), \mathcal{J}(x_d)], \mathcal{J}([x_a, x_b])] = \tfrac{1}{4}\hbar^2 \gamma^{ijk}_{abcd}\, \{x_i, x_j, \mathcal{J}(x_k)\}$$

where

$$\beta^{ijk}_{abc} = \alpha_a{}^{il}\alpha_b{}^{jm}\alpha_c{}^{kn}\alpha_{lmn}, \qquad \gamma^{ijk}_{abcd} = \alpha_{cd}{}^e \beta^{ijk}_{abe} + \alpha_{ab}{}^e \beta^{ijk}_{cde}$$

for all $x_a \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$. The antipode is

$$S(x_a) = -x_a, \qquad S(\mathcal{J}(x_a)) = -\mathcal{J}(x_a) + \frac{1}{4}\,\hbar\, \mathfrak{c}_{\mathfrak{g}} x_a.$$

The counit is given by $\varepsilon_\hbar(x_a) = \varepsilon_\hbar(\mathcal{J}(x_a)) = 0$.

Twisted Yangian $Y(\mathfrak{g}, \mathfrak{h})$

Preliminaries: Symmetric pair decomposition

- Let θ be an involution of $\mathfrak{g}$. Then $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{m}) = -\mathfrak{m}$ satisfying

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

  here $\mathfrak{h}$ is a (semi) simple Lie algebra, such that (at most) $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{k}$.

- The pair $(\mathfrak{g}, \mathfrak{h})$ is called a symmetric pair

- Let $X_\alpha \in \mathfrak{h}$, $Y_p \in \mathfrak{m}$, and set ($f_{\alpha\beta}{}^{\gamma} = 0$ for $\alpha \neq \beta$)

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^{\gamma}\, X_\gamma, \qquad [X_\alpha, Y_p] = g_{\alpha p}{}^{q}\, Y_q, \qquad [Y_p, Y_q] = w_{pq}{}^{\alpha}\, X_\alpha$$

- The Casimir operator $C_\mathfrak{g}$ in this basis decomposes as ($\mathfrak{c}_\mathfrak{g} = \mathfrak{c}_\mathfrak{a} + \mathfrak{c}_\mathfrak{b} + \mathfrak{c}_\mathfrak{m} + \mathfrak{c}_z$)

$$C_\mathfrak{g} = C_X + C_Y = \kappa^{\alpha\beta} X_\alpha X_\beta + (\kappa_\mathfrak{m})^{pq} Y_p Y_q$$

$$C_X = C + C' + C_z = (\kappa_\mathfrak{a})^{ij} X_i X_j + (\kappa_\mathfrak{b})^{i'j'} X_{i'} X_{j'} + (\kappa_\mathfrak{k})^{zz} X_z X_z$$

- The following relations hold

$$f_\alpha{}^{\beta\nu}\,[X_\nu, X_\beta] = \mathfrak{c}_{(\alpha)} X_\alpha, \qquad g_q{}^{p\alpha}\,[X_\alpha, Y_p] = \frac{\mathfrak{c}_\mathfrak{g}}{2}\, Y_q, \qquad w_\gamma{}^{qp}\,[Y_p, Y_q] = \bar{\mathfrak{c}}_{(\gamma)} X_\gamma$$

$$f_\alpha{}^{\mu\nu} f_{\nu\mu}{}^{\beta} = \mathfrak{c}_{(\alpha)}\, \delta_\alpha^{\ \beta}, \qquad g_p{}^{r\alpha} g_{\alpha r}{}^{q} = \frac{\mathfrak{c}_\mathfrak{g}}{2}\, \delta_p^{\ q}, \qquad w_\alpha{}^{qp}\, w_{pq}{}^{\beta} = \bar{\mathfrak{c}}_{(\alpha)}\, \delta_\alpha^{\ \beta}$$

Preliminaries: Twisted half-loop Lie algebra I

- Let us extend the involution θ of $\mathfrak{g}$ to the whole of $\mathcal{L}^+ \simeq \mathfrak{g}[u]$ as follows

$$\theta(x_a^{(k)}) = (-1)^k \theta_a^{\ b} x_b^{(k)}$$

- We write $\mathcal{L}^+$ in terms of the elements $\{X_\alpha^{(k)}, Y_q^{(k)}\}$ satisfying

$$[X_\alpha^{(k)}, X_\beta^{(l)}] = f_{\alpha\beta}^{\ \ \gamma} X_\gamma^{(k+l)} \quad [X_\alpha^{(k)}, Y_p^{(l)}] = g_{\alpha p}^{\ \ q} Y_q^{(k+l)} \quad [Y_p^{(k)}, Y_q^{(l)}] = w_{pq}^{\ \ \alpha} X_\alpha^{(k+l)}$$

- The twisted half-loop Lie algebra $\mathcal{H}^+ \cong \mathfrak{g}[u]^\theta$ is a subalgebra of $\mathcal{L}^+$ invariant under θ, namely $\mathcal{H}^+ = \{x \in \mathcal{L}^+ \,|\, \theta(x) = x\}$. We have:

$$\mathcal{L}^+ = \mathcal{H}^+ \oplus \mathcal{M}^+, \qquad \mathcal{H}^+ = \{X_\alpha^{(2k)}, Y_q^{(2k+1)}\}, \qquad \mathcal{M}^+ = \{X_\alpha^{(2k+1)}, Y_q^{(2k)}\}$$

Preliminaries: Twisted half-loop Lie algebra II

- Let $\mathrm{rank}(\mathfrak{g}) \geq 2$. Then $\mathcal{H}^+ \simeq \mathfrak{g}[u]^\theta$ is isomorphic to an algebra generated by elements $\{X_\alpha, B(Y_p)\}$ satisfying

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^{\gamma} X_\gamma, \qquad [X_\alpha, B(Y_p)] = g_{\alpha p}{}^{q} B(Y_q)$$

$$[B(Y_p), B(Y_q)] + \frac{1}{\bar{\mathfrak{c}}_{(\alpha)}} w_{pq}{}^{\alpha} w_{\alpha}{}^{rs} [B(Y_r), B(Y_s)] = 0$$

$$[[B(Y_p), B(Y_q)], B(Y_r)] + \frac{2}{\mathfrak{c}_{\mathfrak{g}}} \kappa^{tu}_{\mathfrak{m}} w_{pq}{}^{\alpha} g_{r\alpha}{}^{s} [[B(Y_s), B(Y_t)], B(Y_u)] = 0$$

The isomorphism with the standard twisted half-loop basis is given by the map

$$X_\alpha \mapsto X_\alpha^{(0)}, \qquad B(Y_p) \mapsto Y_p^{(1)}$$

- Let $\mathrm{rank}(\mathfrak{g}) \geq 2$ and $\mathfrak{m} = \{0\}$. Then $\mathcal{H}^+ \simeq \mathfrak{g}[u^2]$ is isomorphic to an algebra generated by elements $\{x_i, G(x_j)\}$ satisfying

$$[x_i, x_j] = \alpha_{ij}{}^{k} x_k, \qquad [x_i, G(x_j)] = \alpha_{ij}{}^{k} G(x_k)$$

$$[G(x_i), G([x_j, x_k])] + [G(x_j), G([x_k, x_i])] + [G(x_k), G([x_i, x_j])] = 0$$

The isomorphism with the standard half-loop basis is given by the map

$$x_i \mapsto x_i^{(0)}, \qquad G(x_i) \mapsto x_i^{(2)}$$

Preliminaries: Lie bi-ideal and twisted Manin triple I [Belliard-Crampe'12]

- The anti-invariant Manin triple twist ϕ of $(\mathcal{L}, \mathcal{L}^+, \mathcal{L}^-)$ is an automorphism of $\mathcal{L}$ satisfying:
  - ϕ is an involution;
  - $\phi(\mathcal{L}^\pm) = \mathcal{L}^\pm$;
  - $(\phi(x), y)_\mathcal{L} = -(x, \phi(y))_\mathcal{L}$ for all $x, y \in \mathcal{L}^+$.

- The twist ϕ gives symmetric pair decomposition of the Manin triple $(\mathcal{L}, \mathcal{L}^+, \mathcal{L}^-)$

  $$\mathcal{L} = \mathcal{H} \oplus \mathcal{M}, \quad \mathcal{L}^\pm = \mathcal{H}^\pm \oplus \mathcal{M}^\pm \quad \text{with} \quad \phi(\mathcal{H}^\pm) = \mathcal{H}^\pm, \quad \phi(\mathcal{M}^\pm) = -\mathcal{M}^\pm$$

  From the anti-invariance of ϕ for $(\ ,\ )_\mathcal{L}$ it follows

  $$(\mathcal{H}^-, \mathcal{H}^+)_\mathcal{L} = (\mathcal{M}^-, \mathcal{M}^+)_\mathcal{L} = 0 \quad \text{and} \quad (\mathcal{H}^\pm)^* \cong \mathcal{M}^\mp$$

- The linear map $\tau : \mathcal{H}^+ \to \mathcal{M}^+ \otimes \mathcal{H}^+$ is a left Lie bi-ideal structure for the couple $(\mathcal{H}^+, \mathcal{M}^+)$ if it is the dual of the following action of $\mathcal{H}^-$ on $\mathcal{M}^-$,

  $$\begin{aligned} \tau^* \ : \ \mathcal{H}^- \otimes \mathcal{M}^- \ &\to \ \mathcal{M}_-, \\ x \otimes y \ &\mapsto \ [x, y]_{\mathcal{L}_-} \ , \end{aligned} \qquad (1)$$

  for all $x \in \mathcal{H}^-$ and $y \in \mathcal{M}^-$.

Preliminaries: Lie bi-ideal and twisted Manin triple II [Belliard-Crampe'12]

- The Lie bi-ideal structure of $(\mathcal{L}^+, \mathcal{H}^+)$, $\tau : \mathcal{H}^+ \to \mathcal{M}^+ \otimes \mathcal{H}^+$ is given by

$$\theta \neq id \;:\; \tau(X_\alpha) = 0, \qquad \tau(B(Y_p)) = [Y_p \otimes 1, \Omega_\mathfrak{h}], \qquad \Omega_\mathfrak{h} = \kappa^{\alpha\beta} X_\alpha \otimes X_\beta$$
$$\theta = id \;:\; \tau(x_a) = 0, \qquad \tau(G(x_a)) = [J(x_a) \otimes 1, \Omega_\mathfrak{g}]$$

- Let $\theta \neq id$. For $X_\alpha^{(0)} = X_\alpha$ we have $(X_\alpha, [y,z])_\mathcal{L} = 0$ for all $y \in \mathcal{H}^-$, $z \in \mathcal{M}^-$ giving

$$\tau(X_\alpha) = 0.$$

For $Y_p^{(1)} = B(Y_p)$ we have

$$(B(Y_p), Y_q^{(-2)})_\mathcal{L} = (\kappa_\mathfrak{m})_{pq}, \qquad Y_q^{(-2)} = 2\,\mathfrak{c}_\mathfrak{g}^{-1} g_q{}^{\alpha p}\,[Y_p^{(-1)}, X_\alpha^{(-1)}]$$

Then

$$(\tau(B(Y_p)), g_q{}^{r\alpha}\, Y_r^{(-1)} \otimes X_\alpha^{(-1)})_\mathcal{L} = \frac{\mathfrak{c}_\mathfrak{g}}{2}(\kappa_\mathfrak{m})_{pq}.$$

Consider an ansatz $\tau(B(Y_p)) = v_p{}^{\beta s}\, Y_s \otimes X_\beta$. Then we must have

$$v_p{}^{\alpha s} g_{qs\alpha} = \frac{\mathfrak{c}_\mathfrak{g}}{2}(\kappa_\mathfrak{m})_{pq} \quad \text{giving} \quad \tau(B(Y_p)) = g_p{}^{\alpha s}\, Y_s \otimes X_\alpha = [Y_p \otimes 1, \Omega_\mathfrak{h}].$$

- The Lie bi-ideal structure for the $\theta = id$ case follows from the pairing $(G(x_a), x_b^{(-3)})_\mathcal{L} = (\kappa_\mathfrak{g})_{ab}$ and using similar arguments as above

Preliminaries: Co-ideal subalgebra

- Let $\mathcal{A} = (A, \mu, \eta, \Delta, \varepsilon)$ be a bi-algebra. Then $\mathcal{B} = (B, m, i, \mathit{\Delta}, \epsilon)$ is a left coideal subalgebra of $\mathcal{A}$ if:

  1. the triple (B, m, i), where m is the multiplication and i is the unit, is an algebra;
  2. B is a subalgebra of A, i.e. there exists an injective homomorphism $\varphi : B \to A$;
  3. coaction $\mathit{\Delta}$ is a coideal map $\mathit{\Delta} : B \to A \otimes B$
  4. the following identities hold

$$(\Delta \otimes id) \circ \mathit{\Delta} = (id \otimes \mathit{\Delta}) \circ \mathit{\Delta}$$

$$(id \otimes \varphi) \circ \mathit{\Delta} = \Delta \circ \varphi$$

  5. $\epsilon : B \to \mathbb{C}$ is the counit.

- The identities above are called *coideal-coassoctivity*, it is an analogue of coassociativity for coideal algebras, and *coideal-compatibility*

Preliminaries: Quantization

- Let $(\mathcal{L}^+, \delta)$ be a Lie bi-algebra and $(\mathcal{H}^+, \tau)$ be a left Lie bi-ideal of $(\mathcal{L}^+, \delta)$. We say that a left coideal subalgebra $(\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+), \mathit{\Delta}_\hbar)$ is a quantization of $(\mathcal{H}^+, \tau)$, or that $(\mathcal{H}^+, \tau)$ is the quasi-classical limit of $(\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+), \mathit{\Delta}_\hbar)$, if it is a free $\mathbb{C}[[\hbar]]$ module and:

  1. $(\mathcal{U}_\hbar(\mathcal{L}^+), \Delta_\hbar)$ is a quantization of $(\mathcal{L}^+, \delta)$
  2. $\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+)/\,\hbar\,\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+)$ is isomorphic to $\mathcal{U}(\mathcal{H}^+)$ as a Lie algebra
  3. $(\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+), \mathit{\Delta}_\hbar)$ is a left coideal subalgebra of $(\mathcal{U}_\hbar(\mathcal{L}^+), \Delta_\hbar)$
  4. for any $x \in \mathcal{H}^+$ and any $X \in \mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+)$ equal to $x \,(mod\, \hbar)$ one has

  $$\Big(\mathit{\Delta}_\hbar(X) - (\varphi(X) \otimes 1 + 1 \otimes X)\Big)/\hbar \sim \tau(x) \quad (mod\, \hbar)$$

  with φ the natural embedding $\mathcal{U}_\hbar(\mathcal{L}^+, \mathcal{H}^+) \hookrightarrow \mathcal{U}_\hbar(\mathcal{L}^+)$

Preliminaries: Coideal map

- Let $\theta \neq id$ ($\mathfrak{m} \neq \{0\}$). The simplest solution of the quantization conditions satisfying properties of a co-ideal subalgebra are

$$\begin{aligned}
\varDelta_\hbar(X_\alpha) &= X_\alpha \otimes 1 + 1 \otimes X_\alpha \\
\varDelta_\hbar(\mathcal{B}(Y_p)) &= \varphi(\mathcal{B}(Y_p)) \otimes 1 + 1 \otimes \mathcal{B}(Y_p) + \hbar\,[Y_p \otimes 1, \Omega_X] \\
\varphi(\mathcal{B}(Y_p)) &= \mathcal{J}(Y_p) + \frac{1}{4}\hbar\,[Y_p, C_X]
\end{aligned}$$

  The grading is $\deg(X_\alpha) = 0$, $\deg(\mathcal{B}(Y_p)) = 1$ and $\deg(\hbar) = 1$.

- *The embedding $\varphi(\mathcal{B}(Y_p))$ is usually reffered to as the MacKay twisted Yangian formula.*

- Let $\theta = id$ ($\mathfrak{m} = \{0\}$). In this case we find

$$\begin{aligned}
\varDelta_\hbar(x_a) &= x_a \otimes 1 + 1 \otimes x_a\,, \\
\varDelta_\hbar(\mathcal{G}(x_a)) &= \varphi(\mathcal{G}(x_a)) \otimes 1 + 1 \otimes \mathcal{G}(x_a) + \hbar\,[\mathcal{J}(x_a) \otimes 1, \Omega_\mathfrak{g}] \\
&\quad + \frac{1}{4}\hbar^2([[x_a \otimes 1, \Omega_\mathfrak{g}], \Omega_\mathfrak{g}] + \mathfrak{c}_\mathfrak{g}^{-1}\alpha_a{}^{bc}[[x_c \otimes 1, \Omega_\mathfrak{g}], [x_b \otimes 1, \Omega_\mathfrak{g}]]), \\
\varphi(\mathcal{G}(x_a)) &= \mathfrak{c}_\mathfrak{g}^{-1}\alpha_a^{bc}[\mathcal{J}(x_c), \mathcal{J}(x_b)] + \frac{1}{4}\hbar\,[\mathcal{J}(x_a), C_\mathfrak{g}]
\end{aligned}$$

  The grading is $\deg(x_a) = 0$, $\deg(\hbar) = 1$ and $\deg(\mathcal{G}(x_a)) = 2$.

Twisted Yangian $Y(\mathfrak{g},\mathfrak{h})$ [Belliard-VR'14]

- There is, up to isomorphism, a unique homogeneous quantization $\mathcal{Y}(\mathfrak{g},\mathfrak{h}) := \mathcal{U}_\hbar(\mathcal{L}^+,\mathcal{H}^+)$ of $(\mathcal{L}^+,\mathcal{H}^+,\tau)$. It is generated by X_α, $\mathcal{B}(Y_p)$ satisfying:

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^{\gamma} X_\gamma, \quad [X_\alpha, \mathcal{B}(Y_p)] = g_{\alpha p}{}^{q} \mathcal{B}(Y_q),$$

$$[\mathcal{B}(Y_p), \mathcal{B}(Y_q)] + \frac{1}{\bar{\mathfrak{c}}_{(\alpha)}}\, w_{pq}{}^{\alpha} w_{\alpha}{}^{rs}\, [\mathcal{B}(Y_r), \mathcal{B}(Y_s)] = \hbar^2 \Lambda_{pq}^{\lambda\mu\nu} \{X_\lambda, X_\mu, X_\nu\},$$

$$[[\mathcal{B}(Y_p), \mathcal{B}(Y_q)], \mathcal{B}(Y_r)] + \frac{2}{\mathfrak{c}_{\mathfrak{g}}} \kappa_{\mathfrak{m}}^{tu}\, w_{pq}{}^{\alpha} g_{r\alpha}{}^{s}\, [[\mathcal{B}(Y_s), \mathcal{B}(Y_t)], \mathcal{B}(Y_u)] = \hbar^2 \Upsilon_{pqr}^{\lambda\mu u} \{X_\lambda, X_\mu, \mathcal{B}(Y_u)\}$$

where

$$\Lambda_{pq}^{\lambda\mu\nu} = \frac{1}{3}\left(g^{\mu t}{}_{p}\, g^{\lambda u}{}_{q} + \sum_{\alpha} (\bar{\mathfrak{c}}_{(\alpha)})^{-1} w_{pq}{}^{\alpha} w_{\alpha}{}^{rs} g^{\mu t}{}_{r}\, g^{\lambda u}{}_{s} \right) w_{tu}{}^{\nu},$$

$$\Upsilon_{pqr}^{\lambda\mu u} = \frac{1}{4} \sum_{\alpha} \left(w_{st}{}^{\alpha} g_{p}{}^{\lambda s} g_{q}{}^{\mu t} g_{\alpha r}{}^{u} + \sum_{\beta} w_{pq}{}^{\alpha} f_{\alpha}{}^{\lambda\beta} g_{r}{}^{\mu s} g_{\beta s}{}^{u} \right)$$
$$+ \frac{1}{2\,\mathfrak{c}_{\mathfrak{g}}} \sum_{\alpha,\gamma} \kappa_{\mathfrak{m}}^{vx}\, w_{pq}{}^{\gamma} g_{r\gamma}{}^{y} \left(w_{st}{}^{\alpha} g_{y}{}^{\lambda s} g_{v}{}^{\mu t} g_{\alpha x}{}^{u} + \sum_{\beta} w_{yv}{}^{\alpha} f_{\alpha}{}^{\lambda\beta} g_{x}{}^{\mu s} g_{\beta s}{}^{u} \right).$$

The counit is $\epsilon_\hbar(X_\alpha) = \epsilon_\hbar(\mathcal{B}(Y_p)) = 0$ for all non-central X_α, and $\epsilon_\hbar(X_z) = c$ with $c \in \mathbb{C}$ for X_z central in $\mathfrak{h}$.

Twisted Yangian $Y(\mathfrak{g},\mathfrak{g})$ [Belliard-VR'14]

- There is, up to isomorphism, a unique homogeneous quantization $\mathcal{Y}(\mathfrak{g},\mathfrak{g}) := \mathcal{U}_\hbar(\mathcal{L}^+,\mathcal{H}^+)$ of $(\mathcal{L}^+,\mathcal{H}^+,\tau)$. It is generated by x_i, $\mathcal{G}(x_i)$ satisfying:

$$[x_a, x_b] = \alpha_{ab}{}^c x_c, \qquad [x_a, \mathcal{G}(x_b)] = \alpha_{ab}{}^c \mathcal{G}(x_c)$$

$$\begin{aligned}[\mathcal{G}(x_a),\mathcal{G}([x_b,x_c])] + [\mathcal{G}(x_b),\mathcal{G}([x_c,x_a])] + [\mathcal{G}(x_c),\mathcal{G}([x_a,x_b])] \\ = \hbar^2\,\Psi^{ijk}_{abc}\{x_i,x_j,\mathcal{G}(x_k)\} + \hbar^4\,(\Phi^{ijk}_{abc}\{x_i,x_j,x_k\} + \bar{\Phi}^{ijklm}_{abc}\{x_i,x_j,x_k,x_l,x_m\})\end{aligned}$$

  The co-unit is $\epsilon_\hbar(x_i) = \epsilon_\hbar(\mathcal{G}(x_i)) = 0$.

- Coefficients Ψ^{ijk}_{abc}, Φ^{ijk}_{abc}, $\bar{\Phi}^{ijklm}_{abc}$ have a very large generic form, which can be simplified for $\mathfrak{g}$ or low rank. For example, for $\mathfrak{g} = \mathfrak{sl}_3$ they are

$$\Psi^{ijk}_{abc} = \frac{1}{3}\,\beta^{ijk}_{(abc)} + \alpha_{(ab}{}^d \alpha_{c)l}{}^k \phi_d{}^{lij} - \alpha_{dl}{}^k \alpha_{(ab}{}^d \phi_{c)}{}^{lij}$$

$$\Phi^{ijk}_{abc} = -\frac{1}{6}\,\beta^{ijk}_{abc} \qquad \bar{\Phi}^{ijkln}_{abc} = \frac{1}{36}\,\alpha_{(a}{}^{ir} \alpha_b{}^{js} \beta^{klm}_{c)rs}$$

$$\phi_a{}^{bcd} = \frac{1}{24\,\mathfrak{c}_\mathfrak{g}} \sum_\pi (\alpha_a{}^{jk} \alpha_j{}^{\pi(d)r} \alpha_k{}^{\pi(b)s} \alpha_{sr}{}^{\pi(c)}), \quad \beta^{ijk}_{abc} = \alpha_a{}^{il} \alpha_b{}^{jm} \alpha_c{}^{kn} \alpha_{lmn}$$

Example I: $\mathcal{Y}(\mathfrak{sl}_3, \mathfrak{gl}_2)$

Twisted Yangian $\mathcal{Y}(\mathfrak{sl}_3, \mathfrak{gl}_2)$ is generated by

$$\mathsf{h}, \mathsf{e}, \mathsf{f}, \mathsf{k} \quad \text{and} \quad \mathsf{E}_2, \mathsf{F}_2, \mathsf{E}_3, \mathsf{F}_3$$

satisfying level-0 relations (of the $\mathfrak{gl}_2$ Lie algebra)

$$[\mathsf{e}, \mathsf{f}] = \mathsf{h}, \quad [\mathsf{h}, \mathsf{e}] = 2\mathsf{e}, \quad [\mathsf{h}, \mathsf{f}] = -2\mathsf{f}, \quad [\mathsf{e}, \mathsf{k}] = [\mathsf{f}, \mathsf{k}] = [\mathsf{h}, \mathsf{k}] = 0,$$

level-1 Lie relations

$$\begin{aligned}
&[\mathsf{e}, \mathsf{E}_2] = \mathsf{E}_3, && [\mathsf{f}, \mathsf{F}_2] = \mathsf{F}_3, && [\mathsf{e}, \mathsf{F}_2] = [\mathsf{f}, \mathsf{E}_2] = 0, \\
&[\mathsf{e}, \mathsf{F}_3] = \mathsf{F}_2, && [\mathsf{f}, \mathsf{E}_3] = \mathsf{E}_2, && [\mathsf{e}, \mathsf{E}_3] = [\mathsf{f}, \mathsf{F}_3] = 0, \\
&[\mathsf{h}, \mathsf{E}_2] = -\mathsf{E}_2, && [\mathsf{h}, \mathsf{F}_2] = \mathsf{F}_2, && [\mathsf{k}, \mathsf{E}_i] = 3\mathsf{E}_i, \\
&[\mathsf{h}, \mathsf{E}_3] = \mathsf{E}_3, && [\mathsf{h}, \mathsf{F}_3] = -\mathsf{F}_3, && [\mathsf{k}, \mathsf{F}_i] = -3\mathsf{F}_i,
\end{aligned}$$

level-2 horrific relations

$$[\mathsf{E}_2, \mathsf{E}_3] = 0, \qquad [\mathsf{F}_2, \mathsf{F}_3] = 0,$$

level-3 horrific relations

$$[\mathsf{E}_2, [\mathsf{E}_2, \mathsf{F}_3]] = -2\hbar^2\{\mathsf{E}_2, \mathsf{f}, \mathsf{k}\}, \qquad [\mathsf{F}_2, [\mathsf{E}_3, \mathsf{F}_2]] = -2\hbar^2\{\mathsf{F}_2, \mathsf{f}, \mathsf{k}\}.$$

Example II: $\mathcal{Y}(\mathfrak{sl}_3, \mathfrak{so}_3)$

Twisted Yangian $\mathcal{Y}(\mathfrak{sl}_3, \mathfrak{so}_3)$ is generated by elements

$$\mathsf{h}, \mathsf{e}, \mathsf{f} \quad \text{and} \quad \mathsf{H}, \mathsf{E}, \mathsf{F}, \mathsf{E}_2, \mathsf{F}_2$$

satisfying level-0 relations (of the $\mathfrak{so}_3$ Lie algebra)

$$[\mathsf{e}, \mathsf{f}] = \mathsf{h}, \quad [\mathsf{h}, \mathsf{e}] = \mathsf{e}, \quad [\mathsf{h}, \mathsf{f}] = -\mathsf{f},$$

level-1 Lie relations

$$\begin{aligned}
&[\mathsf{e}, \mathsf{F}] = [\mathsf{E}, \mathsf{f}] = \mathsf{H}, \quad [\mathsf{h}, \mathsf{E}] = \mathsf{E}, \quad [\mathsf{h}, \mathsf{F}] = -\mathsf{F}, \\
&[\mathsf{e}, \mathsf{E}] = 2\mathsf{E}_2, \quad [\mathsf{f}, \mathsf{F}] = 2\mathsf{F}_2, \quad [\mathsf{e}, \mathsf{E}_2] = [\mathsf{f}, \mathsf{F}_2] = 0, \\
&[\mathsf{e}, \mathsf{F}_2] = \mathsf{F}, \quad [\mathsf{f}, \mathsf{E}_2] = \mathsf{E}, \quad [\mathsf{h}, \mathsf{F}_2] = -2\mathsf{F}_2, \quad [\mathsf{h}, \mathsf{E}_2] = 2\mathsf{E}_2, \\
&[\mathsf{H}, \mathsf{e}] = 3\mathsf{E}, \quad [\mathsf{H}, \mathsf{f}] = -3\mathsf{F}, \quad [\mathsf{H}, \mathsf{h}] = 0,
\end{aligned}$$

level-2 horrific relation

$$[\mathsf{E}, \mathsf{F}] + [\mathsf{E}_2, \mathsf{F}_2] = \tfrac{1}{4}\hbar^2\big(\{\mathsf{h}, \mathsf{h}, \mathsf{h}\} - 3\{\mathsf{e}, \mathsf{f}, \mathsf{h}\}\big),$$

level-3 horrific relation

$$[[\mathsf{E}, \mathsf{F}], \mathsf{H}] = \tfrac{3}{2}\hbar^2\big(\{\mathsf{E}_2, \mathsf{f}, \mathsf{f}\} + \{\mathsf{F}_2, \mathsf{e}, \mathsf{e}\}\big) + \tfrac{15}{4}\hbar^2\big(\{\mathsf{E}, \mathsf{f}, \mathsf{h}\} - \{\mathsf{F}, \mathsf{e}, \mathsf{h}\}\big).$$

Thank $Y(\mathfrak{o})u$

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