A recursive construction of joint eigenfunctions for the commuting hyperbolic Calogero-Moser Hamiltonians

(Joint work with M. Hallnäs)

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Introduction

Reminders

- The hyperbolic (A_{N-1}) Calogero-Moser systems are integrable systems describing N particles on the line with hyperbolic pair interaction.
- The nonrelativistic quantum version is defined by the Hamiltonian

$$H = -\frac{\hbar^2}{2}\sum_{j=1}^{N}\partial_{x_j}^2 + g(g-\hbar)\sum_{1\le j<l\le N} V(x_j - x_l),$$

with $\hbar > 0$ (Planck's constant), $g \in \mathbb{R}$ (coupling constant), and pair potential

$$V(x) = \mu^2/4\sinh^2(\mu x/2), \quad \mu > 0.$$

- The $N = 2$ Schrödinger equation can be solved via the conical function, a specialization of the Gauss hypergeometric function.

- Associated integrable system (N commuting PDOs):

$$H_1 = -i\hbar \sum_{j=1}^{N} \partial_{x_j}, \; H_2 = -\hbar^2 \sum_{1\le j_1<j_2\le N} \partial_{x_{j_1}}\partial_{x_{j_2}} - g(g-\hbar) \sum_{1\le j<l\le N} V(x_j - x_l),$$

$$H_k = (-i\hbar)^k \sum_{1\le j_1<\cdots<j_k\le N} \partial_{x_{j_1}} \cdots \partial_{x_{j_k}} + \mathrm{l.\ o.}, \quad k = 3, \dots, N,$$

  where l.o. = lower order in partials. Thus, the defining Hamiltonian is given by

$$H = H_1^2/2 - H_2.$$

- Integrable versions exist for Lie algebras $B_N, \dots, E_8, F_4, G_2$ (Olshanetsky/Perelomov, Oshima) and BC_N (Inozemtsev, Oshima).
- $N > 2$ eigenfunctions: Harish-Chandra, Heckman/Opdam, Felder/Varchenko, Chalykh,...

Introduction

Goal

- By proceeding recursively in N, construct joint eigenfunctions $\Psi_N(x,p)$ of the Hamiltonians H_k:

$$H_k(x)\Psi_N(x,p) = S_k(p)\Psi_N(x,p), \quad k = 1,\ldots,N,$$

where

$$S_k(p) = \sum_{1\leq j_1<\cdots<j_k\leq N} p_{j_1}\cdots p_{j_N}.$$

- For convenience, we rewrite $\Psi_N(x,p)$ as

$$\begin{aligned}\Psi_N(g;(x_1,\ldots,x_N),(p_1,\ldots,p_N)) &= W_N(g/\hbar;\mu x/2)^{1/2}\\ &\quad\times F_N(g/\hbar;(\mu x_1/2,\ldots,\mu x_N/2),(2p_1/\hbar\mu,\ldots,2p_N/\hbar\mu))\end{aligned}$$

with

$$W_N(\lambda;t) \equiv \prod_{1\leq j<k\leq N}[4\sinh^2(t_j-t_k)]^\lambda.$$

Introduction

Main results

- Assuming $\mathrm{Re}\,\lambda \geq 1$, $u \in \mathbb{R}^N$ and $|\mathrm{Im}\, t_j| < \pi/2$, we obtain

$$F_N(\lambda; t, u) = \int_{\mathbb{R}^{N(N-1)/2}} \prod_{n=1}^{N-1} \frac{\prod_{1 \leq j < k \leq n} [4 \sinh^2(t_{nj} - t_{nk})]^\lambda}{n! \prod_{j=1}^{n+1} \prod_{k=1}^{n} \left[2 \cosh(t_{n+1,j} - t_{nk})\right]^\lambda} \times \exp\left(i \sum_{n=1}^{N} u_n \left(\sum_{j=1}^{n} t_{nj} - \sum_{j=1}^{n-1} t_{n-1,j} \right) \right) \prod_{n=1}^{N-1} \prod_{j=1}^{n} dt_{nj},$$

  where

$$t_{Nj} \equiv t_j, \quad j = 1, \dots, N.$$

- This integral can also be written

$$\exp\left(i u_N \sum_{j=1}^{N} t_j \right) \prod_{n=1}^{N-1} \int_{t_{nn} < \cdots < t_{n1}} \exp\left(i(u_n - u_{n+1})(t_{n1} + \cdots + t_{nn}) \right) \times \frac{\prod_{1 \leq j < k \leq n} [2 \sinh(t_{nj} - t_{nk})]^{2\lambda}}{\prod_{j=1}^{n+1} \prod_{k=1}^{n} \left[2 \cosh(t_{n+1,j} - t_{nk})\right]^\lambda} \prod_{j=1}^{n} dt_{nj}.$$

Introduction

Tools

A crucial ingredient is an explicit description of the eigenvalue equations for F_N.

- The starting point consists of the Lax matrix

$$\mathcal{L}(t,u)_{jk} \equiv \delta_{jk} u_j + (1-\delta_{jk}) \frac{i\lambda}{\sinh(t_j - t_k)}$$

and the diagonal matrix

$$\mathcal{E}(t) \equiv \mathrm{diag}\,(w_1(t), \ldots, w_N(t))$$

with

$$w_j(t) \equiv -i\lambda \sum_{k\neq j} \coth(t_j - t_k).$$

- We let $:\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t):$ denote the normal-ordered PDOs obtained from the symmetric functions

$$\Sigma_k(\mathcal{L}(t,u)+\mathcal{E}(t)) \equiv \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \det(\mathcal{L}(t,u)+\mathcal{E}(t))_I$$

by performing the substitutions

$$u_j \to -i\partial_{t_j}, \quad j=1,\ldots,N.$$

- The Hamiltonians $\mathcal{H}_k(\lambda;t) \equiv (2/\hbar\mu)^k H_k(\lambda\hbar; 2t/\mu)$ are given by (S. R.)

$$\mathcal{H}_k(\lambda;t) = W(t)^{1/2} : \hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t) : W(t)^{-1/2}.$$

- It follows that $F_N(t,u)$ should satisfy the eigenvalue equations

$$:\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t): F_N(t,u) = S_k(u)F_N(t,u), \quad k=1,\ldots,N.$$

Another key ingredient is given by so-called kernel functions.

- Given a pair of operators $H_1(v)$ and $H_2(w)$, a kernel function is a function $\Psi(v,w)$ satisfying

  $$H_1(v)\Psi(v,w) = H_2(w)\Psi(v,w).$$

  Here, v and w may vary over spaces of different dimension.
- There exist elementary kernel functions that connect the PDOs $:\hat{\Sigma}_k(\mathcal{L}+\mathcal{E})(t):$ to a sum of PDOs in variables $s_1,\ldots,s_{N-\ell}$. (Langmann for k=2, Hallnäs/S. R. for k>2.)
- For $\ell = 1$ this connection can be used to set up a recursive scheme yielding the above explicit integral representations of the joint eigenfunctions F_N.
- For $\lambda = 1/2$ recursive H-eigenfunctions were previously found by Gerasimov/Kharchev/Lebedev, and for $\lambda = -1, -2, \ldots$ by Felder/Veselov. (Relation unclear to date.)

$N = 2$ case

From $N = 1$ to $N = 2$

- For $N = 1$ we set
$$F_1(t, u) \equiv \exp(itu),$$
which obviously satisfies
$$-i\partial_t F_1(t, u) = uF_1(t, u).$$
- Now consider
$$F_2(\lambda; t, u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^\sharp(\lambda; t, s) F_1(s, u_1 - u_2)$$
with kernel function
$$\mathcal{K}_2^\sharp(\lambda; t, s) \equiv \prod_{j=1}^{2} [2\cosh(t_j - s)]^{-\lambda}.$$

- If $\operatorname{Re}\lambda > 0$ and $u \in \mathbb{R}^2$, then the integrand decays exponentially as $|s| \to \infty$. It has singularities only at

$$s = t_j \pm \frac{i\pi}{2}(2n+1), \quad j = 1,2, \quad n \in \mathbb{N}.$$

- Hence $F_2(\lambda; t, u)$ is well defined as long as

$$\operatorname{Re}\lambda > 0, \quad u \in \mathbb{R}^2,$$

  and $t \in \mathbb{C}^2$ satisfies

$$|\operatorname{Im} t_j| < \pi/2, \quad j = 1,2.$$

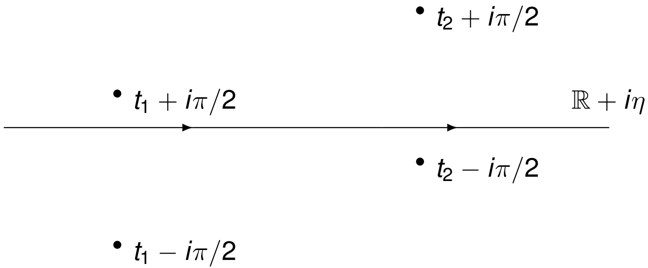
$N = 2$ case

Holomorphy

- $F_2(\lambda; t, u)$ has analytic continuation in (λ, t) to

$$\{\lambda \in \mathbb{C} | \mathrm{Re}\,\lambda > 0\} \times \{t \in \mathbb{C}^2 | |\mathrm{Im}\,(t_1 - t_2)| < \pi\}.$$

- Follows via contour shifts:



where we can choose $\eta = \mathrm{Im}\,(t_1 + t_2)/2$.

- Can allow $u \in \mathbb{C}^2$ such that $|\mathrm{Im}\,(u_1 - u_2)| < 2\mathrm{Re}\,\lambda$.

$N = 2$ case

Eigenfunction property

We claim that $F_2(\lambda; t, u)$ is a joint eigenfunction of the PDOs

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L} + \mathcal{E})(t) := -i(\partial_{t_1} + \partial_{t_2}),$$

$$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) := -\partial_{t_1}\partial_{t_2} + \lambda \coth(t_1 - t_2)(\partial_{t_1} - \partial_{t_2}) + \lambda^2.$$

- Key point: eigenfunction identity

  $$: \hat{\Sigma}_2^{(2)}(\mathcal{L} + \mathcal{E})(t) : \mathcal{K}_2^{\sharp}(t, s) = 0,$$

  and kernel identity

  $$: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : \mathcal{K}_2^{\sharp}(t, s) =: \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) : \mathcal{K}_2^{\sharp}(t, s) = i\partial_s \mathcal{K}_2^{\sharp}(t, s).$$
- By analyticity, need only consider $t \in \mathbb{R}$ and $\lambda > 0$ (say).

To establish the eigenfunction property for $:\hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t):$ (e. g.), we use the following 7 steps.

1. Recall that

$$F_2(\lambda;t,u) \equiv e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

2. Act with $:\hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t):$, and shift through plane wave:

$$e^{iu_2(t_1+t_2)} \ : \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E}+u_2\mathbf{1}_2)(t) \ : \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

3. Note the expansion

$$:\hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E}+u_2\mathbf{1}_2)(t) :=: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : +2u_2.$$

4. Act with PDO under the integral sign and invoke kernel identity:

$$e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds F_1(s,u_1-u_2) : \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) : \mathcal{K}_2^{\sharp}(\lambda;t,s)$$
$$+ 2u_2 e^{iu_2(t_1+t_2)} \int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda;t,s) F_1(s,u_1-u_2).$$

5. Recall that $:\hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(-s) := i\partial_s$, and integrate by parts:

$$e^{iu_2(t_1+t_2)}\int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) : \hat{\Sigma}_1^{(1)}(\mathcal{L}+\mathcal{E})(s) : F_1(s, u_1-u_2) \\ + 2u_2 e^{iu_2(t_1+t_2)}\int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) F_1(s, u_1-u_2).$$

6. Use eigenfunction property for F_1:

$$(u_1-u_2+2u_2)e^{iu_2(t_1+t_2)}\int_{\mathbb{R}} ds \mathcal{K}_2^{\sharp}(\lambda; t, s) F_1(s, u_1-u_2).$$

7. Conclude that

$$: \hat{\Sigma}_1^{(2)}(\mathcal{L}+\mathcal{E})(t) : F_2(t, u) = S_1^{(2)}(u) F_2(t, u),$$

where $S_1^{(2)}(a_1, a_2) \equiv a_1 + a_2$.

□

$N = 2$ case

A bound

- Let $u \in \mathbb{R}^2$. For $\mathrm{Re}\,\lambda > 0$ and $|\mathrm{Im}\,(t_1 - t_2)| < \pi$, we have the F_2-bound

$$|F_2(\lambda; t, u)| < C(\lambda, |\mathrm{Im}\,(t_1 - t_2)|) \\ \times \exp(-\mathrm{Im}\,(t_1 + t_2)(u_1 + u_2)/2) \frac{\mathrm{Re}\,(t_1 - t_2)}{\sinh(\mathrm{Re}\,\lambda \mathrm{Re}\,(t_1 - t_2))}.$$

- This bound readily follows from the integral evaluation

$$\int_{\mathbb{R}} \frac{ds}{\prod_{\delta=+,-} 2\cosh(s + \delta z/2)} = \frac{z}{2\sinh z}.$$

Recursion scheme

Kernel function

- The function

  $$\mathcal{K}_N^{\sharp}(\lambda; t, s) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} [2\cosh(t_j - s_k)]^{-\lambda}, \quad N > 1,$$

  satisfies the eigenfunction identity

  $$:\hat{\Sigma}_N^{(N)}(\mathcal{L}+\mathcal{E})(t):\mathcal{K}_N^{\sharp}(t,s) = 0,$$

  and the kernel identities

  $$\left(:\hat{\Sigma}_k^{(N)}(\mathcal{L}+\mathcal{E})(t): - :\hat{\Sigma}_k^{(N-1)}(\mathcal{L}+\mathcal{E})(-s):\right)\mathcal{K}_N^{\sharp}(t,s) = 0, \quad k < N.$$

- This connection between the N and $N-1$ variable cases can be used to recursively construct the joint eigenfunctions F_N of the N PDOs $:\hat{\Sigma}_k^{(N)}(\mathcal{L}+\mathcal{E})(t):$, $k = 1, \ldots, N$.

Recursion scheme

Formal structure

- Assume the function $F_{N-1}(t,u)$, $t,u \in \mathbb{C}^{N-1}$, has been constructed.
- Then $F_N(t,u)$, $t,u \in \mathbb{C}^N$, is formally given by

$$F_N(t,u) \equiv \frac{e^{iu_N \sum_{j=1}^N t_j}}{(N-1)!} \int_{\mathbb{R}^{N-1}} ds W_{N-1}(s) \mathcal{K}_N^\sharp(t,s) \\ \times F_{N-1}(s,(u_1-u_N,\ldots,u_{N-1}-u_N)).$$

- Have shown $F_N(\lambda;t,u)$ is well defined for $\mathrm{Re}\,\lambda \geq 1$ and $t \in \mathbb{C}^N$ such that $|\mathrm{Im}\, t_j| < \pi/2$ (and $u \in \mathbb{R}^N$), and continues analytically to

$$\{\lambda \in \mathbb{C} | \mathrm{Re}\,\lambda > 1\} \times \{t \in \mathbb{C}^N | \max_{1\leq j<k\leq N} |\mathrm{Im}\,(t_j - t_k)| < \pi\}.$$

A_{N-1} Heckman-Opdam hypergeometric function

- $F_{A_{N-1}}(\tilde{\lambda}, k; h)$ depends on three types of parameters:
  - coupling parameter $k \in \mathbb{C}$,
  - eigenvalue vector $\tilde{\lambda} \in \mathbb{C}^N$,
  - a quantity h, which can be viewed as a diagonal $N \times N$ matrix with $\det(h) = 1$.
- For $t \in \mathbb{C}^N$ such that $\sum t_j = 0$, let

$$h(t) = \mathrm{diag}\left(e^{2t_1}, \ldots, e^{2t_N}\right).$$

- Comparing eigenvalue equations and normalisations, we deduced

$$F_{A_{N-1}}(iu/2, \lambda; h(t)) = \frac{F_N(\lambda; t, u)}{F_N(\lambda; 0, u)},$$

where $\sum t_j = \sum u_j = 0$.

Relativistic generalisation

Reminders

- Given by $2N$ commuting analytic difference operators

$$A_{k,\delta} = \sum_{|I|=k} \prod_{\substack{m \in I \\ n \notin I}} \frac{s_\delta(x_m - x_n - ib)}{s_\delta(x_m - x_n)} \prod_{m \in I} \exp(-ia_{-\delta}\partial_{x_m}),$$

where $k = 1, \ldots, N$, $\delta = +, -$, and

$$s_\delta(z) \equiv \sinh(\pi z / a_\delta).$$

- Physical picture: The imaginary periods and shift step sizes are determined by the length parameters

$$a_+ \equiv 2\pi/\mu, \quad \text{(interaction length)},$$

$$a_- \equiv \hbar/mc, \quad \text{(Compton wave length)},$$

with $\hbar$ Planck's constant, m particle mass and c speed of light.

Relativistic generalisation

Tools

- The hyperbolic gamma function

  $$G(a_+, a_-; z) \equiv \exp(ig(a_+, a_-; z)),$$

  with

  $$g(z) \equiv \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2 \sinh(a_+ y) \sinh(a_- y)} - \frac{z}{a_+ a_- y} \right),$$

  for $|\mathrm{Im}\, z| < (a_+ + a_-)/2$.
- The kernel function

  $$\mathcal{S}_N^\sharp(b; x, y) \equiv \prod_{j=1}^{N} \prod_{k=1}^{N-1} \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)},$$

  which satisfies the $2N$ identities

  $$A_{k,\delta}^{(N)}(x) \mathcal{S}_N^\sharp(b; x, y) = \left(A_{k,\delta}^{(N-1)}(-y) + A_{k-1,\delta}^{(N-1)}(-y) \right) \mathcal{S}_N^\sharp(b; x, y).$$

Relativistic generalisation

Sketch of results

- For $N = 1$ we set

$$J_1(x,y) \equiv \exp(i\alpha xy), \quad \alpha = \frac{2\pi}{a_+ a_-}.$$

- For $N > 1$ we construct joint eigenfunctions $J_N(x,y)$, $x, y \in \mathbb{C}^N$, recursively according to

$$\begin{aligned} J_N(x,y) = e^{i\alpha y_N(x_1+\cdots+x_N)} \int_{\mathbb{R}^{N-1}} & dz W_{N-1}(z) \mathcal{S}_N^\sharp(x,z) \\ & \times J_{N-1}(z,(y_1-y_N,\ldots,y_{N-1}-y_N)) \end{aligned}$$

  with

$$W_M(z) \equiv \prod_{1\leq m<n\leq M} w(z_m - z_n),$$

$$w(z) \equiv 1/c(z)c(-z), \quad c(z) \equiv G(z+ia-ib)/G(z+ia).$$

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