Spin Chains, Structure Factors and Vertex Operators

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Introduction: The XXZ Spin Chain

The anisotropic spin-$\frac{1}{2}$ Heisenberg model is an interacting, one-dimensional quantum integrable system with Hamiltonian

$$H_{XXZ} = -J\sum_{k=1}^{N}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \triangle(\sigma_k^z\sigma_{k+1}^z - 1)\right).$$

In certain magnetic materials microscopic interactions are very strong in a particular direction and so they can be modeled by such one dimensional systems.

The XXZ spin chain in the anti-ferromagnetic massive regime ($1 < \triangle, 0 < J$), for example, accurately describes $CsCoCl_3$ and so not only is the model quantum integrable, it can also be realised experimentally. This gives us the exciting opportunity to directly compare theoretical and experimental results.

The Dynamical Structure Factor

- The dynamical structure factors of the XXZ spin chain are Fourier transforms of spin-spin dynamical correlation functions. For example,

$$S^{zz}(k,\omega) = \sum_{j\in\mathbb{Z}} e^{-ijk}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle \text{vac}|\, S_j^z(t)S_0^z(0)\,|\text{vac}\rangle, \quad S^z = \frac{1}{2}\sigma^z,$$

defines the so-called longitudinal structure factor [4] .

- Computing such an object directly using the vertex operator approach to be described is not feasible as there will be j integrals to compute. However, inserting a complete set of spinon states $\mathbb{I} = \sum_\alpha |\alpha\rangle\langle\alpha|$ allows the structure factor to be computed as

$$S^{zz}(k,\omega) = \sum_\alpha (2\pi)^2\delta(k - K(\alpha))\delta(w - W(\alpha))|\,\langle\text{vac}|\, S_0^z\,|\alpha\rangle\,|^2,$$

where $W(\alpha)$ and $K(\alpha)$ are the energy and momentum of state $|\alpha\rangle$.

- The structure factor $S^{zz}(k,\omega)$ is measurable in inelastic neutron scattering experiments, therefore if we can compute the form factors $\langle\text{vac}|\, S_0^z\,|\alpha\rangle$ using the symmetries of the model, we will have exact results to compare with reality.
- Results for dynamical structure factors (at finite size) have been obtained through the algebraic Bethe Ansatz in e.g. [2], but we will focus on another entirely algebraic approach applicable in the thermodynamic limit.

Background picture is from [1] showing the two-spinon contribution to the transverse dynamical structure factor $S^{-+}(k,w)$ at zero field for $\triangle = 2$.

The Vertex Operator Approach for Spin-1/2

The general idea, following [1], is to exploit the quantum group symmetry of the model and identify all of the key ingredients needed for computation with objects in representation theory.

- The main players are the vertex operators (VOs). These are $U_q(\hat{sl}_2)$ intertwiners,

$$\Phi(\xi) \;:\; V(\Lambda_i) \to V(\Lambda_{1-i})\otimes V_\xi, \quad i = 0,1$$
$$\Phi(\xi) = \sum_{\epsilon=\pm}\Phi_\epsilon(\xi)\otimes v_\epsilon.$$

The Λ_i, are the fundamental weights of $U_q(\hat{sl}_2)$ and $V(\Lambda_i)$ are the corresponding highest weight modules. The evaluation module V_ξ is defined by $V_\xi = V\otimes\mathbb{C}[\xi,\xi^{-1}]$, where V is the two dimensional vector space with basis vectors, v_+ and v_-.

- We then have a dictionary between lattice objects and mathematical objects:

Lattice Object	Mathematical Object
space of states	$\mathcal{F} = \oplus_{i,j=0,1}V(\Lambda_i)\otimes V(\Lambda_j)^*$
translation operator	$T = \sum_\epsilon \Phi_\epsilon(1)\otimes(\Phi_{-\epsilon}(1))^t$
transfer matrix	$T(\xi) = \sum_\epsilon \Phi_\epsilon(\xi)\otimes(\Phi_{-\epsilon}(\xi))^t$
local operators	$E_{\epsilon\epsilon'} = \Phi_\epsilon^*(1)\Phi_{\epsilon'}(1)\otimes\mathbb{I}$,

where $E_{\epsilon\epsilon'}$ are unit matrix operators from which we can construct any desired local operator.

- Creation/annihilation operators are constructed using analogous objects called type II VOs.
- Correlation functions and form factors are expressed as traces of vertex operators over irreducible highest weight representations (IHWR) of $U_q(\hat{sl}_2)$. For example, the one-point function

$$\frac{\langle\text{vac}|\,\sigma_1^+\,|\text{vac}\rangle}{\langle\text{vac}|\text{vac}\rangle},$$

can be realised in terms of the following traces

$$\text{Tr}_{V(\Lambda_i)}\left(q^{2D}\Phi_-(-q^{-1}\xi)\Phi_+(\xi)\right), \quad i = 0,1,$$

where ξ is a spectral parameter and D is the grading operator of the algebra.

- The final step is to use the free field realisation of these algebraic objects in terms of free bosons a_n satisfying

$$[a_n, a_m] = \delta_{n+m,0}\frac{[2n][n]}{n}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

After doing this, such expressions become traces of bosonic fields over Fock spaces - something that can be readily computed in order to give final integral expressions.

Existing Results

The free field realisation of vertex operators has been used in [3,4] in order to calculate two-spinon contributions to the dynamical structure factors of spin-$\frac{1}{2}$ XXZ in the thermodynamic limit in different regimes. The results are compared with finite-size Bethe Ansatz results and the two very different approaches agree for $N \to \infty$. In [3], the exact expressions for physical observables have then been compared with experimental data.

Current Work - VOA for Spin-1

My recent work has been focused on generalising the vertex operator approach to the spin-1 case in order to calculate form factors and thus structure factors of this higher spin model.

- Finite size results for dynamical observables in the spin-1 case do exist [7] as do partial integral expressions for form-factors and correlation functions [6], but the explicit calculation of two particle form-factors using the VOA is yet to be achieved.
- The initial construction parallels that of the spin-$\frac{1}{2}$ case and correlation functions and form-factors can again be obtained as trace expressions over IHWR of $U_q(\hat{sl}_2)$. It is in the free-field realisation of higher spin VOs that technical differences and difficulties arise.

Free Field Realisation

We use an adaptation of the q-Wakimoto bosonisation of $U_q(\hat{sl}_2)$ outlined in [6].

- We require three types of free bosons, a_n, b_n and c_n, satisfying

$$[a_n, a_m] = \delta_{n+m,0}\frac{[2n][4n]}{n}, \quad [b_n, b_m] = -\delta_{n+m,0}\frac{[2n][2n]}{n}, \quad [c_n, c_m] = \delta_{n+m,0}\frac{[2n][2n]}{n}.$$

Our VOs are then realised as combinations of exponents of associated bosonic fields.

- We also need to include additional objects called screening charges, Q, in the bosonisation of our VOs. Their role mirrors that of screening charges in CFT.

The standard practice in this setting is to realise the screening charges in terms of Jackson integrals of bosonic operators. Instead, we are using a novel construction based on work in the elliptic case [8] and define Q by

$$Q = \oint \frac{dz}{2\pi i z}S(z)\frac{[u - \frac{1}{2} + P]_{k+2}}{[u - \frac{1}{2}]_{k+2}},$$

where $z = q^{2u}$, $[u]_x$ is the theta function $[u]_x = \vartheta_1\left(\frac{u}{x}|\tau_x\right)$ and $S(z)$ is a bosonic operator called the screening current. The operator P is a combination of zero modes.

The motivation behind this is that we would much rather deal with integrals in our final calculations than infinite sums.

- The final issue to consider in using the q-Wakimoto bosonisation is that the Fock space F on which our VOs act is reducible. This issue is resolved by obtaining the irreducible part as the BRST cohomology group in a complex of Fock modules. The Fock modules in the complex are defined through a certain restriction of F called the q-Wakimoto module.

The application of the vertex operator approach followed by this refined q-Wakimoto bosonisation scheme will yield exact integral expressions for the spin-1 XXZ dynamical structure factors.

References

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