The high-dimensional cohomology of the moduli space of curves with level structures II: punctures and boundary

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Abstract

We give two proofs that appropriately defined congruence subgroups of the mapping class group of a surface with punctures/boundary have enormous amounts of rational cohomology in their virtual cohomological dimension. In particular we give bounds that are super-exponential in each of three variables: number of punctures, number of boundary components, and genus, generalizing work of Fullarton–Putman. Along the way, we give a simplified account of a theorem of Harer explaining how to relate the homotopy type of the curve complex of a multiply-punctured surface to the curve complex of a once-punctured surface through a process that can be viewed as an analogue of a Birman exact sequence for curve complexes.

As an application, we prove upper and lower bounds on the coherent cohomological dimension of the moduli space of curves with marked points. For $g \leq 5$, we compute this coherent cohomological dimension for any number of marked points. In contrast to our bounds on cohomology, when the surface has $n \geq 1$ marked points, these bounds turn out to be independent of n, and depend only on the genus.

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1 Introduction

Let $\Sigma_{g,n}^b$ be an oriented genus g surface with n punctures and b boundary components. If n or b vanishes, then we will omit them from our notation. The (pure) *mapping class group* of $\Sigma_{g,n}^b$, denoted $\mathrm{PMod}_{g,n}^b$, is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,n}^b$ that fix $\partial\Sigma_{g,n}^b$ pointwise and do not permute the punctures.[1] The cohomology of $\mathrm{PMod}_{g,n}^b$ plays an important role in many areas of mathematics. One fundamental reason for this is that $\mathrm{PMod}_{g,n}^b$ is the orbifold fundamental group of the moduli space $\mathcal{M}_{g,n}^b$ of finite-volume hyperbolic metrics on $\Sigma_{g,n}^b$ with geodesic boundary. The orbifold universal cover of $\mathcal{M}_{g,n}^b$ is Teichmüller space, which is diffeomorphic to a ball and is thus contractible. This implies that $\mathcal{M}_{g,n}^b$ is an orbifold classifying space for $\mathrm{PMod}_{g,n}^b$, and hence that

$$\mathrm{H}^{\bullet}(\mathrm{PMod}_{g,n}^b;\mathbb{Q}) \cong \mathrm{H}^{\bullet}(\mathcal{M}_{g,n}^b;\mathbb{Q}).$$

See [10] for a survey.

Stable cohomology. There are two important families of cohomology classes for $\mathrm{PMod}_{g,n}^b$:

- The Miller–Morita–Mumford classes $\kappa_i \in \mathrm{H}^{2i}(\mathrm{PMod}_{g,n}^b;\mathbb{Q})$.
- For $1 \leq i \leq n$, the Euler class $u_i \in \mathrm{H}^2(\mathrm{PMod}_{g,n}^b;\mathbb{Q})$ of the two-dimensional subbundle of the tangent bundle of $\mathcal{M}_{g,n}^b$ corresponding to tangent directions where the i^{th} puncture moves.

A fundamental theorem of Madsen–Weiss [27] says that the corresponding map

$$\mathbb{Q}[u_1,\ldots,u_n,\kappa_1,\kappa_2,\ldots] \longrightarrow \mathrm{H}^{\bullet}(\mathrm{PMod}_{g,n}^b;\mathbb{Q})$$

of graded rings is an isomorphism in degree k if $g \gg k$ (the reference [27] only deals with surfaces without punctures – see [26, Proposition 2.1] for how to deal with punctures). The portion of the cohomology ring identified by this theorem is the *stable cohomology*. Building on work of Harer–Zagier [15] in the closed case, Bini–Harer [2] proved that the Euler characteristic of $\mathcal{M}_{g,n}^b$ is enormous (at least if $b = 0$), so there must be a large amount of unstable cohomology. However, very little unstable cohomology is known; indeed, there is only one known infinite family of nonzero unstable classes (found by Chan–Galatius–Payne [5], disproving conjectures of Church–Farb–Putman [7] and Kontsevich [24]).

Top-degree cohomology. Our main theorem concerns the "most unstable" possible cohomology groups. Harer [14, Theorem 4.1] proved that $\mathrm{PMod}_{g,n}^b$ is a virtual duality group with virtual cohomological dimension

$$\mathrm{vcd}(\mathrm{PMod}_{g,n}^b) = \begin{cases} 4g-5 & \text{if } n=b=0 \text{ and } g \geq 2, \\ 4g+2b+n-4 & \text{if } n+b \geq 1 \text{ and } g \geq 1. \end{cases} \tag{1.1}$$

This identifies the top degree in which $\mathrm{PMod}_{g,n}^b$ might have some rational cohomology. However, Church–Farb–Putman [6] and Morita–Sakasai–Suzuki [28] independently proved

[1] When $n \in \{0,1\}$, it is common to use the notation Mod rather than PMod; however, we will use PMod consistently throughout the paper for the sake of streamlining certain statements.

that for $n+b \leq 1$ no rational cohomology can be found in this degree:

$$\mathrm{H}^{4g-5}(\mathrm{PMod}_g;\mathbb{Q}) = \mathrm{H}^{4g-3}(\mathrm{PMod}_{g,1};\mathbb{Q}) = \mathrm{H}^{4g-2}(\mathrm{PMod}_g^1;\mathbb{Q}) = 0 \quad \text{for } g \geq 2.$$

Level subgroups without punctures or boundary. For $\ell \geq 2$, let $\mathrm{PMod}_g[\ell]$ be the level-ℓ subgroup of PMod_g, that is, the kernel of the action of PMod_g on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$. This action preserves the algebraic intersection pairing, which is a $\mathbb{Z}/\ell$-valued symplectic form. It thus induces a homomorphism $\mathrm{PMod}_g \to \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ that is classically known to be surjective. This is all summarized in the short exact sequence

$$1 \longrightarrow \mathrm{PMod}_g[\ell] \longrightarrow \mathrm{PMod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell) \longrightarrow 1.$$

Since $\mathrm{PMod}_g[\ell]$ is a finite-index subgroup of PMod_g, it is also a virtual duality group (with the same virtual cohomological dimension). However, unlike for PMod_g its top rational cohomology group is not zero. In fact, a recent theorem of Fullarton–Putman [12] says that if p is a prime dividing ℓ, then the dimension of the rational cohomology of $\mathrm{PMod}_g[\ell]$ in its virtual cohomological dimension is at least

$$\frac{1}{g} p^{2g-1} \prod_{k=1}^{g-1} (p^{2k}-1) p^{2k-1}. \tag{1.2}$$

This is super-exponential in g; its leading term is $\frac{1}{g} p^{\binom{2g}{2}}$.

Level subgroups with punctures or boundary. Our main theorem extends this to surfaces with punctures and boundary. Let $\mathrm{PMod}_{g,n}^b[\ell]$ be the kernel of the action of $\mathrm{PMod}_{g,n}^b$ on $\mathrm{H}_1(\Sigma_{g,n}^b;\mathbb{Z}/\ell)$. The intersection pairing on $\mathrm{H}_1(\Sigma_{g,n}^b;\mathbb{Z}/\ell)$ is degenerate if $n+b \geq 2$, so in these cases $\mathrm{PMod}_{g,n}^b$ does not act via the symplectic group. To clarify this, let $Q \subset \Sigma_{g,n}^b$ be a set consisting of $\partial \Sigma_{g,n}^b$ along with a closed regular neighborhood of each puncture. Poincaré–Lefshetz duality implies that

$$\mathrm{H}^1(\Sigma_{g,n}^b;\mathbb{Z}/\ell) \cong \mathrm{H}_1(\Sigma_{g,n}^b, Q;\mathbb{Z}/\ell).$$

To simplify this even more, let $P \subset \Sigma_g$ be a set consisting of n points and b embedded discs, all pairwise disjoint. We can identify $\Sigma_{g,n}^b$ with a subsurface of Σ_g with $P \cap \Sigma_{g,n}^b = Q$, and

$$\mathrm{H}_1(\Sigma_{g,n}^b, Q;\mathbb{Z}/\ell) = \mathrm{H}_1(\Sigma_g, P;\mathbb{Z}/\ell).$$

In $\mathrm{H}_1(\Sigma_g, P;\mathbb{Z}/\ell)$, we have the homology classes of both loops and arcs connecting points of P. The group $\mathrm{PMod}_{g,n}^b$ can be identified with the group of isotopy classes of orientation-preserving diffeomorphisms $f\colon \Sigma_g \to \Sigma_g$ with $f|_P = \mathrm{id}$. Combining all this, we see that $\mathrm{PMod}_{g,n}^b[\ell]$ is the kernel of the action of $\mathrm{PMod}_{g,n}^b$ on $\mathrm{H}_1(\Sigma_g, P;\mathbb{Z}/\ell)$.

Remark 1.1. It is also common to consider the kernel of the action of $\mathrm{PMod}_{g,n}^b$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$. This is different from $\mathrm{PMod}_{g,n}^b[\ell]$ when $n+b \geq 2$, and our theorems do not apply to it.

In recent work, the third author [33] has proved that

$$\mathrm{H}^k(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q}) \cong \mathrm{H}^k(\mathrm{PMod}_{g,n}^b;\mathbb{Q}) \qquad \text{when } g \gg k.$$

In other words, when you pass to $\mathrm{PMod}_{g,n}^b[\ell]$ no new rational cohomology appears in the stable range. When $n+b \leq 1$, this was earlier proved for $k=1$ by Hain [13] and for $k=2$ by the third author [31].

Main theorem. With this definition, our main theorem is as follows.

Theorem A. *Fix some $g \geq 1$ and $n, b \geq 0$ such that $n+b \geq 1$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}^b$. For all $\ell \geq 2$, the following then hold:*

- *If $n+b=1$ and p is a prime dividing ℓ, then the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q})$ is at least*
$$\frac{1}{g}p^{2g-1}\prod_{k=1}^{g-1}(p^{2k}-1)p^{2k-1}.$$
- *If $n+b \geq 2$ and ν' is the virtual cohomological dimension of $\mathrm{PMod}_{g,1}$, then the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q})$ is at least*
$$\left(\prod_{k=1}^{b+n-1}\left(k\ell^{2g}-1\right)\right)\cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,1}[\ell];\mathbb{Q}).$$

Note the latter expression is super-exponential in n and b as well as g.

Remark 1.2. Our bound when $n+b=1$ is the same as Fullarton–Putman's bound in the closed case, and in fact follows easily from their work. The main point of this paper is to deal with the case where $n+b \geq 2$.

Remark 1.3. It is natural to wonder also whether the rational cohomology of $\mathrm{PMod}_{g,n}^b$ vanishes in its virtual cohomological dimension when $n+b \geq 2$ (as was proved by Church–Farb–Putman [6] and Morita–Sakasai–Suzuki [28] when $n+b \leq 1$). In unpublished work, we can prove this in many cases, and we hope in the near future to be able to prove the general result.

Proof I: Birman exact sequence. We actually give two proofs of Theorem A. The first is more direct, but gives less ancillary information. It makes use of the Birman exact sequence, which is a short exact sequence
$$1 \longrightarrow \pi_1(\Sigma_{g,n-1}^b) \longrightarrow \mathrm{PMod}_{g,n}^b \longrightarrow \mathrm{PMod}_{g,n-1}^b \longrightarrow 1. \tag{1.3}$$
Here the map $\mathrm{PMod}_{g,n}^b \to \mathrm{PMod}_{g,n-1}^b$ arises from filling in the n^{th} puncture and the subgroup $\pi_1(\Sigma_{g,n-1}^b)$ of $\mathrm{PMod}_{g,n}^b$ is the "point-pushing subgroup" consisting of mapping classes that drag the n^{th} puncture around loops in the surface. We develop an analogue of (1.3) for $\mathrm{PMod}_{g,n}^b[\ell]$ and study the cohomology of the resulting extension.

Steinberg module. Our second proof uses the Steinberg module for the mapping class group. Recall from above that Harer [14] proved that $\mathrm{PMod}_{g,n}^b$ is a virtual duality group. In particular, it is a $\mathbb{Q}$-duality group. Letting ν be its virtual cohomological dimension, this implies that the cohomology of $\mathrm{PMod}_{g,n}^b$ satisfies the following Poincaré-duality-like relation:
$$\mathrm{H}^{\nu-i}(\mathrm{PMod}_{g,n}^b;\mathbb{Q}) \cong \mathrm{H}_i(\mathrm{PMod}_{g,n}^b;\mathrm{St}(\Sigma_{g,n}^b)).$$

The term $\mathrm{St}(\Sigma_{g,n}^b)$ (called the *Steinberg module*) is the *dualizing module* for the mapping class group. Harer gave a beautiful description of $\mathrm{St}(\Sigma_{g,n}^b)$ in terms of the curve complex, which we now discuss.

Curve complex. The curve complex $\mathcal{C}_{g,n}^b$ is the simplicial complex whose k-simplices are collections $\{\gamma_0, \ldots, \gamma_k\}$ of isotopy classes of nontrivial simple closed curves on $\Sigma_{g,n}^b$ that can be realized disjointly. Here nontrivial means that γ_i is not homotopic to any of the punctures or boundary components of $\Sigma_{g,n}^b$. As is traditional in the subject, we will usually not distinguish between a simple closed curve and its isotopy class. Harer proved that $\mathcal{C}_{g,n}^b$ is homotopy equivalent to a wedge of spheres of dimension

$$\lambda = \begin{cases} 2g-2 & \text{if } g \geq 1 \text{ and } n = b = 0, \\ 2g-3+n+b & \text{if } g \geq 1 \text{ and } n+b \geq 1, \\ n+b-4 & \text{if } g = 0. \end{cases} \tag{1.4}$$

The only nonzero reduced homology group of $\mathcal{C}_{g,n}^b$ thus lies in degree λ. The group $\mathrm{PMod}_{g,n}^b$ acts on $\mathcal{C}_{g,n}^b$, and thus acts on its homology. Harer proved that $\mathrm{St}(\Sigma_{g,n}^b) = \widetilde{\mathrm{H}}_\lambda(\mathcal{C}_{g,n}^b; \mathbb{Q})$ is the dualizing module of $\mathrm{PMod}_{g,n}^b$.

Proof II. Since groups $\mathrm{PMod}_{g,n}^b[\ell]$ are finite-index subgroups of the virtual duality groups $\mathrm{PMod}_{g,n}^b$, they are also virtual duality groups with the same dualizing module. In particular, letting ν be their virtual cohomological dimension, we have

$$\mathrm{H}^\nu(\mathrm{PMod}_{g,n}^b[\ell]; \mathbb{Q}) \cong \mathrm{H}_0(\mathrm{PMod}_{g,n}^b[\ell]; \mathrm{St}(\Sigma_{g,n}^b)) = (\mathrm{St}(\Sigma_{g,n}^b))_{\mathrm{PMod}_{g,n}^b[\ell]},$$

where the subscript indicates that we are taking coinvariants. To understand these coinvariants and prove Theorem A, we must relate $\mathrm{St}(\Sigma_{g,n}^b)$ to $\mathrm{St}(\Sigma_{g,1})$. This is the subject of our next main theorem.

Inductive description of curve complex. It is clear that $\mathcal{C}_{g,n}^b \cong \mathcal{C}_{g,n+b}$, so to simplify our notation we will focus on the surfaces with punctures but no boundary components. In the spirit of the Birman exact sequence, we can try to relate $\mathcal{C}_{g,n}$ and $\mathcal{C}_{g,n-1}$. In [14], Harer proved that the map $\mathcal{C}_{g,1} \to \mathcal{C}_g$ obtained by removing the puncture is a homotopy equivalence for all $g \geq 1$. See [20, Proposition 4.7] and [23, Corollary 1.1] for alternate proofs. We will thus focus on relating $\mathcal{C}_{g,n}$ to $\mathcal{C}_{g,n-1}$ for $n \geq 2$.

In these cases, the map $\Sigma_{g,n} \to \Sigma_{g,n-1}$ that fills in the n^{th} puncture *almost* induces a map $\mathcal{C}_{g,n} \to \mathcal{C}_{g,n-1}$. The only problem is that under this map curves γ that bound a twice-punctured disc one of whose punctures is the n^{th} one become trivial. Let $\mathrm{N}\mathcal{C}_{g,n}$ be the set of such curves, and let $\mathcal{X}_{g,n}$ be the full subcomplex of $\mathcal{C}_{g,n}$ spanned by the vertices that do not lie in $\mathrm{N}\mathcal{C}_{g,n}$. Any two curves in $\mathrm{N}\mathcal{C}_{g,n}$ must intersect, so none of them are joined by an edge in $\mathcal{C}_{g,n}$. Regarding $\mathrm{N}\mathcal{C}_{g,n}$ as a discrete set and letting $*$ denote the simplicial join, we thus have

$$\mathcal{C}_{g,n} \subset \mathrm{N}\mathcal{C}_{g,n} * \mathcal{X}_{g,n}.$$

Filling in the n^{th} puncture does induce a map $\mathcal{X}_{g,n} \to \mathcal{C}_{g,n-1}$. This map is not an isomorphism, but it is implicit in Harer's work that the map $\mathcal{X}_{g,n} \to \mathcal{C}_{g,n-1}$ is a homotopy equivalence.

Just like in the case where $n = 1$, this can also be proved as in [20, Proposition 4.7] and [23, Corollary 1.1].

In fact, even more is true:

Theorem B. *Fix some $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$. Then there is a $\mathrm{PMod}_{g,n}$-equivariant homotopy equivalence $\mathcal{C}_{g,n} \simeq \mathrm{N}\mathcal{C}_{g,n} * \mathcal{C}_{g,n-1}$.*

Theorem B was essentially proved by Harer, but we give a self-contained and simplified account of it using the idea of "Hatcher flows" introduced in [18].

Inductive description of Steinberg. Let λ be as in (1.4). Since $\mathrm{N}\mathcal{C}_{g,n}$ is a discrete set, Theorem B implies that

$$\mathrm{St}(\Sigma_{g,n}) = \widetilde{\mathrm{H}}_{\lambda}(\mathcal{C}_{g,n}; \mathbb{Q}) \cong \widetilde{\mathrm{H}}_{0}(\mathrm{N}\mathcal{C}_{g,n}; \mathbb{Q}) \otimes \widetilde{\mathrm{H}}_{\lambda-1}(\mathcal{C}_{g,n-1}; \mathbb{Q}) = \widetilde{\mathrm{H}}_{0}(\mathrm{N}\mathcal{C}_{g,n}; \mathbb{Q}) \otimes \mathrm{St}(\Sigma_{g,n-1}).$$

This is an isomorphism of $\mathrm{PMod}_{g,n}$-modules, where $\mathrm{PMod}_{g,n}$ acts on $\mathrm{N}\mathcal{C}_{g,n}$ via its action on $\Sigma_{g,n}$ and on $\mathrm{St}(\Sigma_{g,n-1})$ via the surjection $\mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$ that fills in the n^{th} puncture. For a set S, let $\mathbb{Q}[S]$ be the $\mathbb{Q}$-vector space with basis S and let $\widetilde{\mathbb{Q}}[S]$ be the kernel of the augmentation map $\mathbb{Q}[S] \to \mathbb{Q}$ taking each element of S to 1. The above discussion is summarized in the following corollary.

Corollary C. *Fix some $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$. We then have an isomorphism*

$$\mathrm{St}(\Sigma_{g,n}) \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}] \otimes \mathrm{St}(\Sigma_{g,n-1})$$

of $\mathrm{PMod}_{g,n}$-modules.

Alternate proof of corollary. In addition to the proof of Corollary C via Theorem B described above, we also give an alternate proof using the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,n-1}) \longrightarrow \mathrm{PMod}_{g,n} \longrightarrow \mathrm{PMod}_{g,n-1} \longrightarrow 1.$$

The free group $\pi_1(\Sigma_{g,n-1})$ is a duality group, and the key to our second proof of Corollary C is showing that the dualizing module for $\pi_1(\Sigma_{g,n-1})$ can be identified with $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$. This alternate proof of Corollary C is more direct than our proof via Theorem B, but it does not give the same space-level information that Theorem B does. We think that both proofs are enlightening.

Applications to algebraic geometry. The moduli space $\mathcal{M}_{g,n}$ is a quasi-projective complex variety of dimension $3g - 3 + n$. In [12], Fullarton–Putman applied their theorem (1.2) to deduce an interesting result about the algebraic geometry of $\mathcal{M}_g$. As we describe now, our Theorem A allows a generalization of this to $\mathcal{M}_{g,n}$.

For a variety X, the *coherent cohomological dimension* of X, denoted $\mathrm{CohCD}(X)$, is the maximal dimension k such that there exists a quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathrm{H}^k(X; \mathcal{F}) \neq 0$. This measures the geometric complexity of X. For instance, Serre [17, Theorem 3.7] proved that X is affine if and only if $\mathrm{CohCD}(X) = 0$. More generally, if X can be covered by m open affine subspaces, then Serre's result together with the Mayer–Vietoris spectral sequence associated to this affine cover implies that $\mathrm{CohCD}(X) \leq m-1$.

Looijenga [9] conjectured that for $g \geq 2$ the variety $\mathcal{M}_g$ can be covered by $(g-1)$ open affine subsets, and in particular that $\mathrm{CohCD}(\mathcal{M}_g) \leq g-2$. Fullarton–Putman showed how to derive the opposite inequality $\mathrm{CohCD}(\mathcal{M}_g) \geq g-2$ from (1.2). In particular $\mathcal{M}_g$ cannot be covered by fewer than $(g-1)$ open affine subsets. Our generalization of this is as follows:

Theorem D. *For $g \geq 2$ and $n \geq 1$, we have* $\mathrm{CohCD}(\mathcal{M}_{g,n}) \geq g-1$.

On first glance, it is striking that this bound is independent of n, unlike the bound of Theorem A. However, we will prove the following.

Theorem E. *For $g \geq 2$ and $n \geq 1$, we have* $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq \mathrm{CohCD}(\mathcal{M}_g) + 1$.

Combining Theorems D and E, we see that if $\mathrm{CohCD}(\mathcal{M}_g) = g-2$, then $\mathrm{CohCD}(\mathcal{M}_{g,n}) = g-1$ for all $n \geq 1$. For instance, in light of Fullarton–Putman's aforementioned result this holds if Looijenga's conjecture is true. Since Fontanari–Pascolutti [11] have proven Looijenga's conjecture for $2 \leq g \leq 5$, we get the following corollary.

Corollary F. *For $g \geq 2$, if* $\mathrm{CohCD}(\mathcal{M}_g) = g-2$, *then* $\mathrm{CohCD}(\mathcal{M}_{g,n}) = g-1$ *for all $n \geq 1$. In particular, if $2 \leq g \leq 5$, then equality holds:* $\mathrm{CohCD}(\mathcal{M}_{g,n}) = g-1$.

Outline. In §2, we make some initial reductions. In §3 we give our proof of Theorem A using the Birman exact sequence. Next, in §4 we give two proofs of Corollary C (which inductively identifies the Steinberg module), the first by proving Theorem B and the second using the Birman exact sequence. This is followed by §5, which proves Theorem A using our Corollary C. Finally, in §6 we prove Theorems D and E.

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2 Initial reductions: removing the boundary and setting up the induction

The goal of this section is to reduce Theorem A to the following result.

Theorem A′. *Fix some $g \geq 1$ and $n \geq 2$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}$. For all $\ell \geq 2$, the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q})$ is at least*

$$\big((n-1)\,\ell^{2g}-1\big)\cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathbb{Q}).$$

We will give two proofs of Theorem A′, one in §3 and the other in §5. The rest of this section is devoted to explaining why it implies Theorem A. The first step is the following lemma, which reduces us to the case of surfaces without boundary.

Lemma 2.1. *Fix some $g \geq 1$ and $n, b \geq 0$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}^b$ and let ν' be the virtual cohomological dimension of $\mathrm{PMod}_{g,n+b}$. Then for all $\ell \geq 2$ we have $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q}) \cong \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q})$.*

Proof. There is a surjective map $\rho\colon \mathrm{PMod}_{g,n}^b \to \mathrm{PMod}_{g,n+b}$ obtained by gluing punctured discs to each component of $\partial\Sigma_{g,n}^b$ and extending mapping classes by the identity. The kernel of ρ is the central subgroup $\mathbb{Z}^b$ generated by Dehn twists about the components of $\partial\Sigma_{g,n}^b$. This is where we use that $g \geq 1$; indeed, if $g = 0$ and $n + b \leq 2$, then there are cases where some of these Dehn twists are either trivial or equal to each other. Each of these Dehn twists lies in $\mathrm{PMod}_{g,n}^b[\ell]$, and ρ restricts to a surjection $\mathrm{PMod}_{g,n}^b[\ell] \to \mathrm{PMod}_{g,n+b}[\ell]$. In other words, we have a central extension

$$1 \longrightarrow \mathbb{Z}^b \longrightarrow \mathrm{PMod}_{g,n}^b[\ell] \longrightarrow \mathrm{PMod}_{g,n+b}[\ell] \longrightarrow 1.$$

The associated Hochschild–Serre spectral sequence converging to $\mathrm{H}^{\bullet}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q})$ has

$$E_2^{pq} = \mathrm{H}^p(\mathrm{PMod}_{g,n+b}[\ell];\mathrm{H}^q(\mathbb{Z}^b;\mathbb{Q})) \cong \mathrm{H}^p(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q}) \otimes \wedge^q\mathbb{Q}^b. \tag{2.1}$$

By Harer's computation of the virtual cohomological dimension of the mapping class group (1.1), we have $\nu = \nu' + b$. The only term of (2.1) with $p + q = \nu' + b$ that can possibly be nonzero is

$$E_2^{\nu',b} = \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q}) \otimes \wedge^b\mathbb{Q}^b \cong \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q}).$$

No nonzero differentials can come into or out of this, so it survives to the E_∞-page. We conclude that

$$\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q}) = \mathrm{H}^{\nu'+b}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q}) \cong \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q}). \qquad \square$$

Proof of Theorem A, assuming Theorem A′. We start by recalling the setup. Let $g \geq 1$ and $n, b \geq 0$ be such that $n + b \geq 1$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}^b$. Fix some $\ell \geq 2$. We must prove the following two things:

- If $n + b = 1$ and p is a prime dividing ℓ, then the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q})$ is at least
$$\frac{1}{g}p^{2g-1}\prod_{k=1}^{g-1}(p^{2k}-1)p^{2k-1}. \tag{2.2}$$
- If $n + b \geq 2$ and ν' is the virtual cohomological dimension of $\mathrm{PMod}_{g,1}$, then the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q})$ is at least
$$\left(\prod_{k=1}^{b+n-1}\left(k\ell^{2g}-1\right)\right)\cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,1}[\ell];\mathbb{Q}).$$

For the cases where $n + b = 1$, Harer's computation (1.1) of the virtual cohomological dimension of the mapping class group says that it is $4g-3$ for $\mathrm{PMod}_{g,1}$, is $4g-2$ for PMod_g^1, and is $4g-5$ for PMod_g. Lemma 2.1 along with Bieri–Eckmann duality shows that

$$\begin{aligned}\mathrm{H}^{4g-2}(\mathrm{PMod}_g^1[\ell];\mathbb{Q}) &\cong \mathrm{H}^{4g-3}(\mathrm{PMod}_{g,1}[\ell];\mathbb{Q}) \\ &\cong \mathrm{H}_0(\mathrm{PMod}_{g,1}[\ell];\mathrm{St}(\Sigma_{g,1})) = (\mathrm{St}(\Sigma_{g,1}))_{\mathrm{PMod}_{g,1}[\ell]},\end{aligned}$$

where the subscript indicates that we are taking coinvariants. Harer [14] also proved that the map $\mathcal{C}_{g,1} \to \mathcal{C}_g$ induced by the map that deletes the puncture is a homotopy equivalence. This implies that we have a $\mathrm{PMod}_{g,1}$-equivariant isomorphism $\mathrm{St}(\Sigma_{g,1}) \cong \mathrm{St}(\Sigma_g)$, where $\mathrm{PMod}_{g,1}$ acts on $\mathrm{St}(\Sigma_g)$ via the induced map $\mathrm{PMod}_{g,1} \to \mathrm{PMod}_g$. Continuing the previous calculation and applying Bieri–Eckmann duality again, we see that

$$(\mathrm{St}(\Sigma_{g,1}))_{\mathrm{PMod}_{g,1}[\ell]} \cong (\mathrm{St}(\Sigma_g))_{\mathrm{PMod}_g[\ell]} \cong \mathrm{H}^{4g-5}(\mathrm{PMod}_g[\ell];\mathbb{Q}).$$

Fullarton–Putman [12] proved that the dimension of this is at least the quantity in (2.2).

We now turn to the cases where $n + b \geq 2$. Let ν'' be the virtual cohomological dimension of $\mathrm{PMod}_{g,n+b}$. Harer's computation (1.1) of the virtual cohomological dimension of the mapping class group implies that adding a puncture to a non-closed surface causes the virtual cohomological dimension to go up by 1. It follows that

$$\nu'' = \nu' + (b + n - 1).$$

The inequality we must prove now follows from Lemma 2.1 along with repeated applications of Theorem A′ as follows:

$$\begin{aligned}\dim_{\mathbb{Q}} \mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}^b[\ell];\mathbb{Q}) &= \dim_{\mathbb{Q}} \mathrm{H}^{\nu'+(b+n-1)}(\mathrm{PMod}_{g,n+b}[\ell];\mathbb{Q}) \\ &\geq \big((n+b-1)\,\ell^{2g}-1\big)\cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu'+(b+n-2)}(\mathrm{PMod}_{g,n+b-1}[\ell];\mathbb{Q}) \\ &\geq \big((n+b-1)\,\ell^{2g}-1\big)\cdot\big((n+b-2)\,\ell^{2g}-1\big) \\ &\qquad\qquad \cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu'+(b+n-3)}(\mathrm{PMod}_{g,n+b-2}[\ell];\mathbb{Q}) \\ &\geq \cdots \geq \left(\prod_{k=1}^{b+n-1}\big(k\ell^{2g}-1\big)\right)\cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu'}(\mathrm{PMod}_{g,1}[\ell];\mathbb{Q}).\end{aligned}$$ □

3 Proof I: via the Birman exact sequence

This section contains our first proof of Theorem A′. The proof is in §3.2, which is preceded by the preliminary §3.1 on the Birman exact sequence.

3.1 The Birman exact sequence

We start by reviewing the Birman exact sequence [3]. Fix some $g \geq 0$ and $n \geq 1$ such that $\Sigma_{g,n} \notin \{\Sigma_{1,1}, \Sigma_{0,3}\}$. There is a surjective map $\mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$ that fills in the n^{th} puncture. Its kernel, denoted $\mathrm{PP}_{g,n}$, is the *point-pushing subgroup* and satisfies $\mathrm{PP}_{g,n} \cong \pi_1(\Sigma_{g,n-1})$. This is where we use our assumption on $\Sigma_{g,n}$; indeed, in the degenerate cases the group $\mathrm{PP}_{g,n}$ would be trivial. Informally, elements of $\mathrm{PP}_{g,n}$ "push" the n^{th} puncture around paths in $\Sigma_{g,n-1}$. This is all summarized in the Birman exact sequence

$$1 \longrightarrow \mathrm{PP}_{g,n} \longrightarrow \mathrm{PMod}_{g,n} \longrightarrow \mathrm{PMod}_{g,n-1} \longrightarrow 1.$$

See [10, §4.2] for a textbook reference.

We wish to generalize this to $\mathrm{PMod}_{g,n}[\ell]$. From our definitions, it is clear that the surjection $\mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$ restricts to a map $\rho\colon \mathrm{PMod}_{g,n}[\ell] \to \mathrm{PMod}_{g,n-1}[\ell]$, though it is not clear that this restriction is surjective. Define $\mathrm{PP}_{g,n}[\ell] = \mathrm{PP}_{g,n} \cap \mathrm{PMod}_{g,n}[\ell]$. We thus have an exact sequence

$$1 \longrightarrow \mathrm{PP}_{g,n}[\ell] \longrightarrow \mathrm{PMod}_{g,n}[\ell] \xrightarrow{\rho} \mathrm{PMod}_{g,n-1}[\ell].$$

The following theorem provides a Birman exact sequence for level-ℓ subgroups; in particular it asserts that ρ is indeed surjective and also gives a description of $\mathrm{PP}_{g,n}[\ell]$.

Theorem 3.1. *Fix some $g \geq 0$ and $n \geq 1$ such that $\Sigma_{g,n} \notin \{\Sigma_{1,1}, \Sigma_{0,3}\}$. For all $\ell \geq 2$, we have a short exact sequence*

$$1 \longrightarrow \mathrm{PP}_{g,n}[\ell] \longrightarrow \mathrm{PMod}_{g,n}[\ell] \xrightarrow{\rho} \mathrm{PMod}_{g,n-1}[\ell] \longrightarrow 1.$$

Moreover, $\mathrm{PP}_{g,n}[\ell]$ is the kernel of the map

$$\mathrm{PP}_{g,n} \cong \pi_1(\Sigma_{g,n-1}) \longrightarrow \mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell)$$

arising from the map $\Sigma_{g,n-1} \to \Sigma_g$ that fills in the punctures.

The third author has proven a similar theorem for the Torelli group (on a surface with boundary but no punctures) [29, Theorem 1.2] and for a variant of the level-ℓ subgroup in some special cases in [31, Theorem 2.10]. Since our proof is not dramatically different from these previous proofs, we will omit some routine verifications.

Proof of Theorem 3.1. We first prove that ρ is surjective. Let $\iota\colon \Sigma^1_{g,n-2} \hookrightarrow \Sigma_{g,n}$ be an embedding such that $\Sigma_{g,n} \setminus \iota(\Sigma^1_{g,n-2})$ is an open disc containing the n^{th} and $(n-1)^{\text{st}}$ punctures. We then get an induced map $\iota_*\colon \mathrm{PMod}^1_{g,n-2} \to \mathrm{PMod}_{g,n}$ that extends mapping classes over this twice-punctured disc by the identity. It is immediate from the definitions that ι_* restricts to a map $\mathrm{PMod}^1_{g,n-2}[\ell] \to \mathrm{PMod}_{g,n}[\ell]$. The composition

$$\mathrm{PMod}^1_{g,n-2}[\ell] \xrightarrow{\iota_*} \mathrm{PMod}_{g,n}[\ell] \xrightarrow{\rho} \mathrm{PMod}_{g,n-1}[\ell]$$

can be identified with the surjective map whose kernel is generated by the Dehn twist about $\partial\Sigma^1_{g,n-2}$ that arose in Lemma 2.1. It follows that ρ is surjective.

We now prove that

$$\mathrm{PP}_{g,n}[\ell] = \ker\left(\mathrm{PP}_{g,n} \cong \pi_1\left(\Sigma_{g,n-1}\right) \longrightarrow \mathrm{H}_1\left(\Sigma_g; \mathbb{Z}/\ell\right)\right). \tag{3.1}$$

Let $P \subset \Sigma_g$ be a set of n distinct points, which we identify with the punctures of $\Sigma_{g,n}$. By definition, $\mathrm{PP}_{g,n}[\ell]$ is the kernel of the action of $\mathrm{PP}_{g,n}$ on $\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$. To understand this action, consider the group $\widehat{\mathrm{PMod}}_{g,n}[\ell] = \rho^{-1}(\mathrm{PMod}_{g,n-1}[\ell])$, which fits into a short exact sequence

$$1 \longrightarrow \mathrm{PP}_{g,n} \longrightarrow \widehat{\mathrm{PMod}}_{g,n}[\ell] \xrightarrow{\rho} \mathrm{PMod}_{g,n-1}[\ell] \longrightarrow 1.$$

As we will see, the action of $\widehat{\mathrm{PMod}}_{g,n}[\ell]$ on $\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$ induces a representation whose image is abelian and rather simple.

Let $P' \subset P$ be the first $(n-1)$ punctures. We have the following isomorphisms:

$$\begin{aligned} \mathrm{H}_1(P, P'; \mathbb{Z}/\ell) &\cong 0, \\ \mathrm{H}_0(P, P'; \mathbb{Z}/\ell) &\cong \mathbb{Z}/\ell, \\ \mathrm{H}_0(\Sigma_g, P'; \mathbb{Z}/\ell) &\cong 0. \end{aligned}$$

The long exact sequence in homology for the triple (Σ_g, P, P') thus contains the segment

$$0 \longrightarrow \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell) \longrightarrow \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell) \xrightarrow{\partial} \mathbb{Z}/\ell \longrightarrow 0. \tag{3.2}$$

Since $\mathrm{PMod}_{g,n}$ does not permute the points in P, its action on $\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$ preserves this exact sequence. Its action on $\mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell)$ factors through $\rho\colon \mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$, and its action on $\mathbb{Z}/\ell$ is trivial.

It follows that the image of $\widehat{\mathrm{PMod}}_{g,n}[\ell]$ in $\mathrm{Aut}(\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell))$ lies in the subgroup Λ consisting of automorphisms that preserve the exact sequence (3.2) and that act trivially on its kernel and cokernel. For $\zeta \in \mathrm{Hom}(\mathbb{Z}/\ell, \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell))$, there is a corresponding automorphism in $\Lambda \subset \mathrm{Aut}(\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell))$ defined via the formula

$$x \mapsto x + \zeta(\partial(x)) \qquad (x \in \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)).$$

This correspondence gives rise to an isomorphism, and hence we have

$$\Lambda \cong \mathrm{Hom}(\mathbb{Z}/\ell, \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell)) \cong \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell).$$

We now return to the subgroup $\mathrm{PP}_{g,n}$ of $\widehat{\mathrm{PMod}}_{g,n}[\ell]$. Tracing through the definitions, we see that the restriction of the map

$$\widehat{\mathrm{PMod}}_{g,n}[\ell] \longrightarrow \Lambda \cong \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell)$$

to the point-pushing subgroup $\mathrm{PP}_{g,n}$ is precisely the map

$$\mathrm{PP}_{g,n} \longrightarrow \mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell) \subset \mathrm{H}_1(\Sigma_g, P'; \mathbb{Z}/\ell)$$

featured in (3.1). The identity (3.1) follows. $\square$

3.2 High-dimensional cohomology and the Birman exact sequence

We now use Theorem 3.1 to prove Theorem A′.

Proof of Theorem A′. We start by recalling the statement. Consider some $g \geq 1$ and $n \geq 2$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}$. Finally, fix some $\ell \geq 2$. We must prove that $\dim_{\mathbb{Q}} \mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}[\ell]; \mathbb{Q})$ is at least

$$\big((n-1)\,\ell^{2g} - 1\big) \cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathbb{Q}). \tag{3.3}$$

We will prove this using the Birman exact sequence from Theorem 3.1.

This Birman exact sequence is of the form

$$1 \longrightarrow \mathrm{PP}_{g,n}[\ell] \longrightarrow \mathrm{PMod}_{g,n}[\ell] \longrightarrow \mathrm{PMod}_{g,n-1}[\ell] \longrightarrow 1.$$

The associated Hochschild–Serre spectral sequence converging to $\mathrm{H}^{\bullet}(\mathrm{PMod}_{g,n}[\ell]; \mathbb{Q})$ has

$$E_2^{pq} = \mathrm{H}^p(\mathrm{PMod}_{g,n-1}[\ell]; \mathrm{H}^q(\mathrm{PP}_{g,n}[\ell]; \mathbb{Q})). \tag{3.4}$$

Since

$$\mathrm{PP}_{g,n}[\ell] \subset \mathrm{PP}_{g,n} \cong \pi_1(\Sigma_{g,n-1})$$

is a free group, its cohomological dimension is 1. Also, by Harer's computation (1.1) of the virtual cohomological dimension of the mapping class group, the virtual cohomological dimension of $\mathrm{PMod}_{g,n-1}$ is $\nu - 1$. It follows that the only term of (3.4) with $p + q = \nu$ that can possibly be nonzero is

$$E_2^{\nu-1,1} = \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathrm{H}^1(\mathrm{PP}_{g,n}[\ell]; \mathbb{Q})).$$

No nonzero differentials can come into or out of this, so it survives to the E_∞-page. We conclude that

$$\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}[\ell]; \mathbb{Q}) \cong \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathrm{H}^1(\mathrm{PP}_{g,n}[\ell]; \mathbb{Q})).$$

It is enough, therefore, to prove that $\dim_{\mathbb{Q}} \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathrm{H}^1(\mathrm{PP}_{g,n}[\ell]; \mathbb{Q}))$ is at least the quantity (3.3).

By Theorem 3.1, the group $\mathrm{PP}_{g,n}[\ell]$ is the fundamental group of the cover $\widetilde{\Sigma}$ of $\Sigma_{g,n-1}$ corresponding to the surjection

$$\pi_1(\Sigma_{g,n-1}) \to \mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell) \tag{3.5}$$

arising from the map $\Sigma_{g,n-1} \to \Sigma_g$ that fills in the punctures of $\Sigma_{g,n-1}$. We thus have $\mathrm{H}^1(\mathrm{PP}_{g,n}[\ell]; \mathbb{Q}) \cong \mathrm{H}^1(\widetilde{\Sigma}; \mathbb{Q})$.

Since loops around punctures of $\Sigma_{g,n-1}$ lie in the kernel of (3.5), the deck group $\mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell)$ acts freely on the punctures of $\widetilde{\Sigma}$. We deduce that $\widetilde{\Sigma}$ has

$$(n-1) \cdot |\,\mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell)| = (n-1)\ell^{2g}$$

punctures. Let $\widetilde{\Sigma}'$ be the closed surface obtained by filling in the punctures of $\widetilde{\Sigma}$. We then have a short exact sequence

$$0 \longrightarrow \mathrm{H}^1(\widetilde{\Sigma}';\mathbb{Q}) \longrightarrow \mathrm{H}^1(\widetilde{\Sigma};\mathbb{Q}) \longrightarrow \mathbb{Q}^{(n-1)\ell^{2g}-1} \longrightarrow 0 \tag{3.6}$$

of $\mathrm{PMod}_{g,n-1}[\ell]$-modules whose cokernel arises from the action on the punctures. Since $\mathrm{PMod}_{g,n-1}[\ell]$ acts trivially on the deck group $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$, its action on $\mathbb{Q}^{(n-1)\ell^{2g}-1}$ is trivial.

The long exact sequence in $\mathrm{PMod}_{g,n-1}[\ell]$-cohomology associated to the short exact sequence (3.6) contains the segment

$$\begin{aligned}\mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\widetilde{\Sigma};\mathbb{Q})) \longrightarrow \mathrm{H}^{\nu-1}&(\mathrm{PMod}_{g,n-1}[\ell];\mathbb{Q}^{(n-1)\ell^{2g}-1})\\ &\longrightarrow \mathrm{H}^{\nu}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\widetilde{\Sigma}';\mathbb{Q})).\end{aligned}$$

Since $\nu-1$ is the virtual cohomological dimension of $\mathrm{PMod}_{g,n-1}[\ell]$, we have

$$\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\widetilde{\Sigma}';\mathbb{Q}))=0.$$

We conclude that the central arrow of the composition

$$\begin{aligned}\mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\mathrm{PP}_{g,n}[\ell];\mathbb{Q})) &\cong \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\widetilde{\Sigma};\mathbb{Q}))\\ &\to \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathbb{Q}^{(n-1)\ell^{2g}-1})\\ &\cong \left(\mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathbb{Q})\right)^{\oplus((n-1)\ell^{2g}-1)}.\end{aligned}$$

is surjective. It follows that the dimension of $\mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathrm{H}^1(\mathrm{PP}_{g,n}[\ell];\mathbb{Q}))$ (and hence also the dimension of $\mathrm{H}^{\nu}(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q})$) is at least the quantity (3.3), as desired. □

4 Identifying the Steinberg module

This section contains our two proofs of Corollary C, which gives an inductive description of the Steinberg module on a punctured surface. The first is in §4.1, which proves the stronger Theorem B which gives an inductive description of the curve complex. The second is in §4.2, which works directly with the Steinberg module.

4.1 Inductive description of the curve complex

In this section, we prove Theorem B, which relates $\mathcal{C}_{g,n}$ to $\mathcal{C}_{g,n-1}$. As we described in the introduction, this implies Corollary C.

We start by proving a technical result. Stating it will require introducing two pieces of notation.

- If X and Y are simplicial complexes, then $X * Y$ denotes the join of X and Y. By definition, this is the simplicial complex whose vertices are $X^{(0)} \sqcup Y^{(0)}$ and whose simplices correspond to sets σ of vertices such that $\sigma \cap X^{(0)}$ (resp. $\sigma \cap Y^{(0)}$) is either $\emptyset$ or a simplex of X (resp. Y).

- If Z is a simplicial complex and $v \in Z^{(0)}$ is a vertex of Z, then $\mathrm{lk}_Z(v)$ is the subcomplex of Z whose simplices correspond to sets σ of vertices such that $\sigma \cup \{v\}$ is a simplex of Z but such that $v \notin \sigma$.

Our technical result is then as follows.

Lemma 4.1. *Let X be a simplicial complex, let I be a discrete set, and let Y be a subcomplex of the join $I * X$ such that $I, X \subset Y$. Assume that for all $i \in I$, the inclusion $\mathrm{lk}_Y(i) \hookrightarrow X$ is a homotopy equivalence. Then the inclusion $Y \hookrightarrow I * X$ is a homotopy equivalence.*

Proof. It is enough to find open covers $\{U_j\}_{j \in J}$ of Y and $\{V_j\}_{j \in J}$ of $I * X$ with the following two properties [19, Corollary 4K.2] :

1. For all $j \in J$, we have $U_j \subset V_j$.
2. For all $k \geq 1$ and all distinct $j_1, \ldots, j_k \in J$, the inclusion $U_{j_1} \cap \cdots \cap U_{j_k} \hookrightarrow V_{j_1} \cap \cdots \cap V_{j_k}$ is a homotopy equivalence

For $i \in I$, let $X_i = \mathrm{lk}_Y(i)$. Set $J = I \sqcup \{0\}$. We define the covers $\{U_j\}$ of Y and $\{V_j\}$ of $I * X$ as follows. To begin, we set $U_0 = Y \setminus I$ and $V_0 = (I * X) \setminus I$. Next, for $j \in I$, we set $U_j = (\{j\} * X_j) \setminus X_j$ and $V_j = (\{j\} * X) \setminus X$. We emphasize in these definitions that the U_j and V_j are open sets and not subcomplexes.

It follows immediately from the definitions that $U_j \subset V_j$ for all $j \in J$, so it remains to check the second condition above. First, we note that for all distinct $i_1, i_2 \in I$, we have $U_{i_1} \cap U_{i_2} = \emptyset$ and $V_{i_1} \cap V_{i_2} = \emptyset$. There are thus just three cases to check.

Case 1: The inclusion $U_0 \hookrightarrow V_0$ is a homotopy equivalence since both U_0 and V_0 deformation retract to X.

Case 2: For $j \in I$, the inclusion $U_j \hookrightarrow V_j$ is a homotopy equivalence since both U_j and V_j are contractible.

Case 3: For $j \in I$, the inclusion $U_0 \cap U_j \hookrightarrow V_0 \cap V_j$ can be seen to be a homotopy equivalence as follows. By construction, $U_0 \cap U_j \cong X_j \times (0,1)$ and $V_0 \cap V_j \cong X \times (0,1)$. Under these identifications, the inclusion $U_0 \cap U_j \hookrightarrow V_0 \cap V_j$ is identified with the inclusion $X_j \times (0,1) \hookrightarrow X \times (0,1)$ induced by inclusion $X_j \hookrightarrow X$. This is a homotopy equivalence by assumption. $\square$

Proof of Theorem B. Let us first recall what we must prove. Fix some $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$. Recall that $\mathrm{N}\mathcal{C}_{g,n}$ is the set of vertices of $\mathcal{C}_{g,n}$ consisting of simple closed curves γ on $\Sigma_{g,n}$ that bound a twice-punctured disc one of whose punctures is the n^{th} one. We remark that our assumption $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$ ensures that such curves are nontrivial. Our goal is to construct a $\mathrm{PMod}_{g,n}$-equivariant homotopy equivalence $\mathcal{C}_{g,n} \simeq \mathrm{N}\mathcal{C}_{g,n} * \mathcal{C}_{g,n-1}$.

We start with the following observation. Consider some $\gamma \in \mathrm{N}\mathcal{C}_{g,n}$. Let $D \subset \Sigma_{g,n}$ be the closed disc bounded by γ, so D contains two punctures. Since $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma)$ is the full subcomplex of $\mathcal{C}_{g,n}$ spanned by nontrivial curves in $\Sigma_{g,n} \setminus D \cong \Sigma_{g,n-1}$, we have that $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma) \cong \mathcal{C}_{g,n-1}$.

Define $\mathcal{X}_{g,n}$ to be the full subcomplex of $\mathcal{C}_{g,n}$ spanned by the vertices that do *not* lie in $\mathrm{N}\mathcal{C}_{g,n}$. Since any two curves in $\mathrm{N}\mathcal{C}_{g,n}$ must intersect and thus are not joined by an edge in $\mathcal{C}_{g,n}$, we have

$$\mathcal{C}_{g,n} \subset \mathrm{N}\mathcal{C}_{g,n} * \mathcal{X}_{g,n}. \tag{4.1}$$

Below we will prove that for all $\gamma \in \mathrm{N}\mathcal{C}_{g,n}$, the inclusion $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma) \hookrightarrow \mathcal{X}_{g,n}$ is a homotopy equivalence. Having done this, we can apply Lemma 4.1 and conclude that the inclusion (4.1) is a homotopy equivalence.

From this, the theorem can be deduced as follows. Deleting the n^{th} puncture defines a map $\pi\colon \mathcal{X}_{g,n} \to \mathcal{C}_{g,n-1}$. Choose some $\gamma \in \mathrm{N}\mathcal{C}_{g,n}$. Since the inclusion $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma) \hookrightarrow \mathcal{X}_{g,n}$ is a homotopy equivalence and the composition

$$\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma) \hookrightarrow \mathcal{X}_{g,n} \xrightarrow{\pi} \mathcal{C}_{g,n-1} \tag{4.2}$$

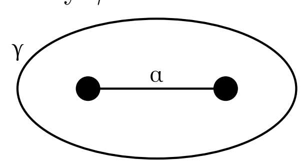
is an isomorphism, the map $\pi\colon \mathcal{X}_{g,n} \to \mathcal{C}_{g,n-1}$ is a homotopy equivalence. We thus have a sequence of homotopy equivalences

$$\mathcal{C}_{g,n} \simeq \mathrm{N}\mathcal{C}_{g,n} * \mathcal{X}_{g,n} \simeq \mathrm{N}\mathcal{C}_{g,n} * \mathcal{C}_{g,n-1},$$

as desired.

Fixing some $\gamma \in \mathrm{N}\mathcal{C}_{g,n}$, it remains to prove that the inclusion $\iota\colon \mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma) \hookrightarrow \mathcal{X}_{g,n}$ is a homotopy equivalence. Since the composition (4.2) is an isomorphism, the map ι induces an injection on homotopy groups. It is thus enough to prove that it induces a surjection as well. Fixing some $m \geq 0$ and some simplicial complex structure on S^m, let $\psi\colon S^m \to \mathcal{X}_{g,n}$ be a simplicial map. Our goal is to homotope ψ such that its image lies in $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma)$.

Let α be an arc in the disc bounded by γ that connects the two punctures in that disc:



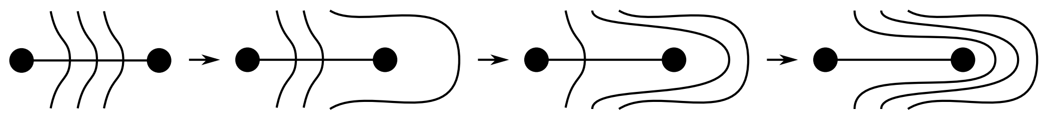
The curve γ is thus the boundary of a regular neighborhood of α, and a simple closed curve on $\Sigma_{g,n}$ is isotopic to a curve that is disjoint from γ if and only if it is isotopic to a curve that is disjoint from α. It follows that it is enough to homotope ψ such that for all vertices v of S^m, the curve $\psi(v)$ can be isotoped so as to be disjoint from α. We will do this using the "Hatcher flow" idea introduced in [18].

Let $v_1, \ldots, v_r$ be the vertices of S^m. For each $1 \leq i \leq r$, pick a simple closed curve representative δ_i of the isotopy class $\psi(v_i)$ with the following key property:

- For all distinct $1 \leq i, j \leq r$, if $\psi(v_i)$ and $\phi(v_j)$ are isotopic to disjoint simple closed curves, then δ_i and δ_j are disjoint. This holds in particular if v_i and v_j are adjacent in S^m.

For instance, fixing a hyperbolic metric on $\Sigma_{g,n}$, we can let δ_i be the geodesic in the isotopy class $\psi(v_i)$ and then if $\delta_i = \delta_j$ for some $i \neq j$ perturb them slightly so as to be disjoint. Perturbing the δ_i slightly, we can assume that they intersect α transversely and that no two of the δ_i intersect α in the same point.

Let $p_1, \ldots, p_s$ be the intersection points of the δ_i with α, enumerated in their natural order starting from the one closest to one of the endpoints of α. For $1 \leq j \leq s$, let δ_{i_j} be the simple closed curve containing p_j. As in the following figure, we can "slide" these intersection points off of the arc α one at a time:



When p_j is slid off, the curve δ_{i_j} is replaced with a new curve. Set $\psi_0 = \psi$, and for $1 \leq j \leq s$, let $\psi_j \colon S^m \to \mathcal{X}_{g,n}$ be the simplicial map obtained from ψ by sliding off $p_1, \ldots, p_j$ as above (and thus changing $\delta_{i_1}, \ldots, \delta_{i_j}$; of course, some of the δ_i will be changed multiple times in this process).

It is clear from our construction that $\psi_j \colon S^m \to \mathcal{X}_{g,n}$ is homotopic to $\psi_{j+1} \colon S^m \to \mathcal{X}_{g,n}$ for $0 \leq j < s$, so $\psi = \psi_0$ is homotopic to ψ_s. Since each simple closed curve in the image of ψ_s is disjoint from α, the image of ψ_s lies in $\mathrm{lk}_{\mathcal{C}_{g,n}}(\gamma)$, as desired. $\square$

4.2 Directly identifying the Steinberg module

In this section, we give a direct proof of Corollary C. Recall that this asserts that for $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$, we have

$$\mathrm{St}(\Sigma_{g,n}) \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}] \otimes \mathrm{St}(\Sigma_{g,n-1}).$$

Here $\mathrm{St}(\Sigma_{g,n})$ and $\mathrm{St}(\Sigma_{g,n-1})$ are the dualizing modules for the $\mathbb{Q}$-duality groups $\mathrm{PMod}_{g,n}$ and $\mathrm{PMod}_{g,n-1}$, respectively. We will deduce this from the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,n-1}) \longrightarrow \mathrm{PMod}_{g,n} \longrightarrow \mathrm{PMod}_{g,n-1} \longrightarrow 1 \tag{4.3}$$

discussed in §3.1.

Reduction. The free group $\pi_1(\Sigma_{g,n-1})$ is a $\mathbb{Q}$-duality group (in fact, even a $\mathbb{Z}$-duality group) of dimension 1. Letting D be the dualizing module for the $\mathbb{Q}$-duality group $\pi_1(\Sigma_{g,n-1})$, we can apply a basic theorem of Bieri–Eckmann (see [1, Theorem 9.10]) about extensions of $\mathbb{Q}$-duality groups to (4.3) and deduce that the dualizing module $\mathrm{St}(\Sigma_{g,n})$ of $\mathrm{PMod}_{g,n}$ is isomorphic to $D \otimes \mathrm{St}(\Sigma_{g,n-1})$. Here $\mathrm{PMod}_{g,n}$ acts on $\mathrm{St}(\Sigma_{g,n-1})$ via the surjection $\mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$ and on D via the conjugation action of $\mathrm{PMod}_{g,n}$ on its normal subgroup $\pi_1(\Sigma_{g,n-1})$. It follows that to prove Corollary C, it is enough to prove the following key lemma.

Lemma 4.2. *Fix some $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$. There is then a $\mathrm{PMod}_{g,n}$-equivariant isomorphism between the dualizing module of $\pi_1(\Sigma_{g,n-1})$ and $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$.*

Dualizing modules. Before we prove Lemma 4.2, we need to discuss some general facts about duality groups for which [1, Chapter 3] and [4, Chapter VIII] form appropriate textbook references. Fix a group G. The group G is a $\mathbb{Q}$-duality group of dimension n if and only if $\mathrm{H}^k(G;\mathbb{Q}[G]) = 0$ for all $k \neq n$. The dualizing module for G is then $\mathrm{H}^n(G;\mathbb{Q}[G])$, on which G acts via its right action on $\mathbb{Q}[G]$.

Topological interpretation. We now give a topological description of $\mathrm{H}^n(G;\mathbb{Q}[G])$. Assume that (X, x_0) is a based compact simplicial complex that forms an Eilenberg–MacLane space for G, so $\pi_1(X, x_0) = G$ and the based universal cover $(\widetilde{X}, \widetilde{x}_0)$ is contractible. The $\pi_1(X, x_0)$-module $\mathbb{Q}[G]$ is a local system on X, and $\mathrm{H}^n(G;\mathbb{Q}[G]) \cong \mathrm{H}^n(X;\mathbb{Q}[G])$. Since X is a compact simplicial complex, there is a natural isomorphism between $\mathrm{H}^n(X;\mathbb{Q}[G])$ and the compactly supported cohomology $\mathrm{H}^n_c(\widetilde{X};\mathbb{Q})$ of $\widetilde{X}$.

Action of automorphisms. The cohomology of a group with twisted coefficients forms a bifunctor which is contravariant in the group and covariant in the coefficient system. More precisely, if for $i = 1, 2$ we have groups G_i equipped with coefficient systems M_i, then given a group homomorphism $f\colon G_2 \to G_1$ and a morphism $f'\colon M_1 \to M_2$ such that

$$f'(f(\gamma) \cdot m) = \gamma \cdot f'(m) \qquad (\gamma \in G_2, m \in M_1),$$

we get induced maps

$$(f, f')_*\colon \mathrm{H}^n(G_1; M_1) \to \mathrm{H}^n(G_2; M_2)$$

for all n. In particular, the group $\mathrm{Aut}(G)$ acts on $\mathrm{H}^n(G;\mathbb{Q}[G])$ via the formula

$$\phi_*(x) = (\phi^{-1}, \phi)_*(x) \qquad (\phi \in \mathrm{Aut}(G), x \in \mathrm{H}^n(G;\mathbb{Q}[G]).$$

Topological interpretation of action. Again let (X, x_0) be a based compact simplicial complex that forms an Eilenberg–MacLane space for G and let $(\widetilde{X}, \widetilde{x}_0)$ be the based universal cover of (X, x_0). The group $\mathrm{Aut}(G)$ is isomorphic to the group of homotopy classes of basepoint-preserving self-homotopy-equivalences of (X, x_0). Consider $\phi \in \mathrm{Aut}(G)$, and let $\psi\colon (X, x_0) \to (X, x_0)$ be a homotopy equivalence realizing ϕ^{-1}. We can then lift ψ to a proper map $\widetilde{\psi}\colon (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{x}_0)$. The map

$$\phi_* = (\phi^{-1}, \phi)_*\colon \mathrm{H}^n(G;\mathbb{Q}[G]) \to \mathrm{H}^n(G;\mathbb{Q}[G])$$

described above can then be identified with $\widetilde{\psi}^*\colon \mathrm{H}^n_c(\widetilde{X};\mathbb{Q}) \to \mathrm{H}^n_c(\widetilde{X};\mathbb{Q})$ in the sense that the diagram

$$\begin{tikzcd} \mathrm{H}^n_c(\widetilde{X};\mathbb{Q}) \arrow[r, "\widetilde{\psi}^*"] \arrow[d, "\cong"'] & \mathrm{H}^n_c(\widetilde{X};\mathbb{Q}) \arrow[d, "\cong"'] \\ \mathrm{H}^n(G;\mathbb{Q}[G]) \arrow[r, "(\phi^{-1},\phi)_*"] & \mathrm{H}^n(G;\mathbb{Q}[G]) \end{tikzcd}$$

commutes.

The proof. We can now prove Lemma 4.2.

Proof of Lemma 4.2. We first recall the statement, using the notation we introduced above. Fix some $g \geq 0$ and $n \geq 2$ such that $\Sigma_{g,n} \notin \{\Sigma_{0,2}, \Sigma_{0,3}\}$. We must prove that there is an isomorphism

$$\mathrm{H}^1(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})]) \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$$

of $\mathrm{PMod}_{g,n}$-modules. We will do this in two steps:

Step 1. *We construct an isomorphism*

$$\eta\colon\ \mathrm{H}^1(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})]) \xrightarrow{\cong} \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$$

of $\pi_1(\Sigma_{g,n-1})$-modules.

Let $p_n \in \Sigma_{g,n-1}$ be the basepoint for π_1. The group $\mathrm{PMod}_{g,n}$ acts on $\pi_1(\Sigma_{g,n-1}, p_n)$ by identifying p_n with the n^{th} puncture of $\Sigma_{g,n}$. Our assumptions on g and n imply that $\Sigma_{g,n-1}$ can be endowed with a hyperbolic metric of finite volume. Fixing such a metric identifies the based universal cover of $(\Sigma_{g,n-1}, p_n)$ with $\rho\colon (\mathbb{H}^2, \widetilde{p}_n) \to (\Sigma_{g,n-1}, p_n)$ for some basepoint $\widetilde{p}_n \in \mathbb{H}^2$.

Let $C_1, \ldots, C_{n-1} \subset \Sigma_{g,n-1}$ be open neighborhoods of the cusps of $\Sigma_{g,n-1}$ satisfying the following properties:

- For all $1 \leq i \leq n-1$, the preimage $\widetilde{C}_i \subset \mathbb{H}^2$ of C_i is a disjoint union of open horoballs.
- For distinct $1 \leq i, j \leq n-1$, the closures $\overline{C}_i$ and $\overline{C}_j$ of C_i and C_j in $\Sigma_{g,n-1}$ are disjoint.
- For all $1 \leq i \leq n-1$, we have $p_n \notin \overline{C}_i$.

Define

$$X = \Sigma_{g,n-1} \setminus \bigcup_{i=1}^{n-1} C_i \quad \text{and} \quad \widetilde{X} = \mathbb{H}^2 \setminus \bigcup_{i=1}^{n-1} \widetilde{C}_i,$$

so X is a compact aspherical 2-dimensional manifold with boundary such that $\pi_1(X, p_n) = \pi_1(\Sigma_{g,n-1}, p_n)$ and $\rho\colon (\widetilde{X}, \widetilde{p}_n) \to (X, p_n)$ is its based universal cover.

As was discussed before the proof, since X is a compact Eilenberg–MacLane space for $\pi_1(\Sigma_{g,n-1}, p_n)$ we have an isomorphism

$$\mathrm{H}^1(\pi_1(\Sigma_{g,n-1}, p_n); \mathbb{Q}[\pi_1(\Sigma_{g,n-1}, p_n)]) \cong \mathrm{H}^1_c(\widetilde{X}; \mathbb{Q}) \tag{4.4}$$

of $\pi_1(\Sigma_{g,n-1}, p_n)$-modules. By Poincaré-Lefschetz duality and the long exact sequence of the pair $(\widetilde{X}, \partial\widetilde{X})$, we have

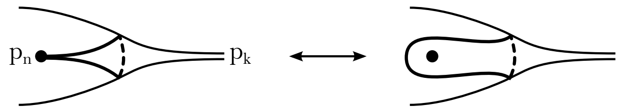
$$\mathrm{H}^1_c(\widetilde{X}; \mathbb{Q}) \cong \mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q}) \cong \widetilde{\mathrm{H}}_0(\partial\widetilde{X}; \mathbb{Q}). \tag{4.5}$$

Let B be the set of components of $\partial\widetilde{X}$, so

$$\widetilde{\mathrm{H}}_0(\partial\widetilde{X}; \mathbb{Q}) \cong \widetilde{\mathbb{Q}}[B]. \tag{4.6}$$

The elements of B are in bijection with the connected components of $\widetilde{C}_1 \cup \cdots \cup \widetilde{C}_{n-1}$, which themselves are in bijection with the maximal parabolic subgroups of $\pi_1(\Sigma_{g,n-1}, p_n) \subset \mathrm{Isom}(\mathbb{H}^2)$. These are precisely the subgroups generated by elements of $\pi_1(\Sigma_{g,n-1}, p_n)$ that are freely homotopic to loops surrounding one of the punctures.

These loops are simple, and each maximal parabolic subgroup is generated by two such loops, one with the cusp on its left and the other with the cusp on its right. It follows that B is in bijection with homotopy classes of p_n-based *unoriented* loops on $\Sigma_{g,n-1}$, and as the following picture shows, these are in bijection with elements of $\mathrm{N}\mathcal{C}_{g,n}$ (which recall is the set of homotopy classes of simple closed curves on $\Sigma_{g,n}$ bounding twice-punctured discs one of whose punctures is the n^{th} one, which we identify with p_n):



In this figure, the k^{th} puncture is suggestively labeled p_k, and the loop on the right bounds a disc containing the k^{th} and n^{th} punctures (as in the definition of $\mathrm{N}\mathcal{C}_{g,n}$). We conclude from this discussion that we have an isomorphism

$$\widetilde{\mathbb{Q}}[B] \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}] \tag{4.7}$$

of $\pi_1(\Sigma_{g,n-1}, p_n)$-modules.

Combining (4.4), (4.5), (4.6), and (4.7), we obtain an isomorphism

$$\eta\colon \mathrm{H}^1(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})]) \xrightarrow{\cong} \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$$

of $\pi_1(\Sigma_{g,n-1}, p_n)$-modules.

Step 2. *We prove that the map η is an isomorphism of* $\mathrm{PMod}_{g,n}$*-modules.*

We will continue using the notation defined in the previous step. Consider some $\phi \in \mathrm{PMod}_{g,n}$. Our goal is to prove that η identifies the action of ϕ on $\mathrm{H}^1(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})])$ induced by the conjugation action of $\mathrm{PMod}_{g,n}$ on its normal subgroup $\pi_1(\Sigma_{g,n-1})$ with the evident action of ϕ on $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$.

As was discussed before the proof, let $\psi\colon (X, p_n) \to (X, p_n)$ be an orientation-preserving diffeomorphism representing the restriction of the inverse of ϕ to $X \subset \Sigma_{g,n-1}$. Let $\widetilde{\psi}\colon (\widetilde{X}, \widetilde{p}_n) \to (\widetilde{X}, \widetilde{p}_n)$ be the lift of $\widetilde{\psi}$. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q}) \arrow[d, "\cong"'] & & \mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q}) \arrow[ll, "\widetilde{\psi}_*"'] \arrow[d, "\cong"] \\
\mathrm{H}^n_c(\widetilde{X}; \mathbb{Q}) \arrow[rr, "\widetilde{\psi}^*"] \arrow[d, "\cong"'] & & \mathrm{H}^n_c(\widetilde{X}; \mathbb{Q}) \arrow[d, "\cong"] \\
\mathrm{H}^n(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})]) \arrow[rr, "(\phi^{-1},\phi)_*"] & & \mathrm{H}^n(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})]).
\end{tikzcd}$$

In other words, under our isomorphisms the action of $\widetilde{\psi}_*^{-1}$ on $\mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q})$. is identified with the action of ϕ on $\mathrm{H}^n(\pi_1(\Sigma_{g,n-1}); \mathbb{Q}[\pi_1(\Sigma_{g,n-1})])$. Under the isomorphism

$$\mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q}) \cong \widetilde{\mathrm{H}}_0(\partial\widetilde{X}) \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}],$$

the action of $\widetilde{\psi}_*^{-1}$ on $\mathrm{H}_1(\widetilde{X}, \partial\widetilde{X}; \mathbb{Q})$ is identified with the action of ϕ on $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]$ (here we are using the fact that $\psi\colon (X, p_n) \to (X, p_n)$ represents ϕ^{-1}). The lemma follows. $\square$

5 Proof II: via the Steinberg module

This section uses Corollary C to give our second proof of Theorem A$'$. There are two sections: the actual proof is in §5.1, and §5.2 comments on a possible way to improve our bounds.

5.1 High-dimensional cohomology and the Steinberg module

The heart of our proof of Theorem A$'$ via Corollary C is the following lemma. For $1 \leq k \leq n-1$, let $\mathrm{N}\mathcal{C}_{g,n}^{k}$ be the set of isotopy classes of simple closed curves on $\Sigma_{g,n}$ that bound a twice-punctured disc whose punctures are the k^{th} and n^{th} ones, so $\mathrm{N}\mathcal{C}_{g,n} = \sqcup_{k=1}^{n-1} \mathrm{N}\mathcal{C}_{g,n}^{k}$.

Lemma 5.1. *Fix some $g \geq 1$ and $n \geq 2$. Then for all $1 \leq k \leq n-1$ and $\ell \geq 2$, the action of $\mathrm{PMod}_{g,n}[\ell]$ on $\mathrm{N}\mathcal{C}_{g,n}^{k}$ has ℓ^{2g} orbits.*

Proof. Just like in the proof of Theorem B in §4.1, there is a bijection between $\mathrm{N}\mathcal{C}_{g,n}^{k}$ and the set of isotopy classes of embedded arcs in $\Sigma_{g,n}$ connecting the k^{th} puncture to the n^{th} puncture. This bijection takes such an arc α to the simple closed curve γ forming the boundary of a regular neighborhood of α. We can therefore regard $\mathrm{N}\mathcal{C}_{g,n}^{k}$ as being the set of such arcs.

Let $P = \{p_1, \ldots, p_n\}$ be a set of n distinct points on Σ_g, which we identify with the punctures of $\Sigma_{g,n}$. Define a set map

$$\phi\colon\ \mathrm{N}\mathcal{C}_{g,n}^{k} \to \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$$

by letting $\phi(\alpha)$ be the homology class of the arc $\alpha \in \mathrm{N}\mathcal{C}_{g,n}^{k}$. We have a short exact sequence

$$0 \longrightarrow \mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell) \longrightarrow \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell) \xrightarrow{\partial} \widetilde{\mathrm{H}}_0(P; \mathbb{Z}/\ell) \longrightarrow 0,$$

and the image of ϕ lies in the set

$$K = \{x \in \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell) \mid \partial(x) = p_n - p_k\}.$$

The set K has ℓ^{2g} elements; indeed, $\mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell)$ acts freely and transitively on it. The map ϕ is $\mathrm{PMod}_{g,n}[\ell]$-invariant, so it induces a map

$$\Phi\colon\ \mathrm{N}\mathcal{C}_{g,n}^{k} \,/\, \mathrm{PMod}_{g,n}[\ell] \to K.$$

To prove the lemma, it is enough to prove that Φ is a bijection.

Let $\mathrm{PP}_{g,n} \cong \pi_1(\Sigma_{g,n})$ be the point-pushing subgroup of $\mathrm{PMod}_{g,n}$ (c.f. §3.1). The action of $\mathrm{PP}_{g,n}$ on $\mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$ preserves K, and as we observed in the proof of Theorem 3.1 its action on K is given by the following formula:

$$\gamma \cdot x = x + [\gamma] \qquad (\gamma \in \mathrm{PP}_{g,n}, x \in K). \tag{5.1}$$

Here $[\gamma] \in \mathrm{H}_1(\Sigma_g; \mathbb{Z}/\ell) \subset \mathrm{H}_1(\Sigma_g, P; \mathbb{Z}/\ell)$ is the mod-ℓ homology class of γ.

Fix $\alpha_0 \in \mathrm{N}\mathcal{C}_{g,n}^k$, and set $\kappa_0 = \phi(\alpha_0)$. The equation (5.1) implies that $\mathrm{PP}_{g,n}$ acts transitively on K, so

$$K = \{\gamma \cdot \kappa_0 \mid \gamma \in \mathrm{PP}_{g,n}\} = \{\phi(\gamma \cdot \alpha_0) \mid \gamma \in \mathrm{PP}_{g,n}\}.$$

We deduce that ϕ is surjective and hence that Φ is surjective.

To see that Φ is injective, consider arbitrary elements $\alpha_1, \alpha_2 \in \mathrm{N}\mathcal{C}_{g,n}^k$ such that $\phi(\alpha_1) = \phi(\alpha_2)$. We must prove that there exists some $f \in \mathrm{PMod}_{g,n}[\ell]$ such that $f \cdot \alpha_2 = \alpha_1$. The point-pushing subgroup $\mathrm{PP}_{g,n}$ acts transitively on $\mathrm{N}\mathcal{C}_{g,n}^k$, so we can write $\alpha_1 = \gamma_1 \cdot \alpha_0$ and $\alpha_2 = \gamma_2 \cdot \alpha_0$ for some $\gamma_1, \gamma_2 \in \mathrm{PP}_{g,n}$. By (5.1), we have

$$\phi(\alpha_1) = \kappa_0 + [\gamma_1] \quad \text{and} \quad \phi(\alpha_2) = \kappa_0 + [\gamma_2],$$

so $[\gamma_1] = [\gamma_2]$. This implies that $[\gamma_1 \gamma_2^{-1}] = 0$, so by Theorem 3.1 we have

$$f := \gamma_1 \gamma_2^{-1} \in \mathrm{PP}_{g,n}[\ell] = \mathrm{PP}_{g,n} \cap \mathrm{PMod}_{g,n}[\ell].$$

This f satisfies $f \cdot \alpha_2 = \alpha_1$, as desired. □

Proof of Theorem A′. We start by recalling the statement. Fix some $g \geq 1$ and $n \geq 2$. Let ν be the virtual cohomological dimension of $\mathrm{PMod}_{g,n}$. Finally, fix some $\ell \geq 2$. We must prove that the dimension of $\mathrm{H}^\nu(\mathrm{PMod}_{g,n}[\ell]; \mathbb{Q})$ is at least

$$\big((n-1)\,\ell^{2g} - 1\big) \cdot \dim_{\mathbb{Q}} \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathbb{Q}). \tag{5.2}$$

Applying Bieri–Eckmann duality, we see that

$$\mathrm{H}^\nu(\mathrm{PMod}_{g,n}[\ell]; \mathbb{Q}) \cong \mathrm{H}_0(\mathrm{PMod}_{g,n}[\ell]; \mathrm{St}(\Sigma_{g,n})) = (\mathrm{St}(\Sigma_{g,n}))_{\mathrm{PMod}_{g,n}[\ell]},$$

where the subscript indicates that we are taking coinvariants. So what we must do is produce a $\mathrm{PMod}_{g,n}[\ell]$-invariant epimorphism from $\mathrm{St}(\Sigma_{g,n})$ to a vector space whose dimension is the quantity (5.2).

Corollary C says that

$$\mathrm{St}(\Sigma_{g,n}) \cong \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}] \otimes \mathrm{St}(\Sigma_{g,n-1}).$$

Here the action of $\mathrm{PMod}_{g,n}$ on $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}(\Sigma_{g,n})]$ is induced by the permutation action of $\mathrm{PMod}_{g,n}$ on $\mathrm{N}\mathcal{C}_{g,n}$ and the action of $\mathrm{PMod}_{g,n}$ on $\mathrm{St}(\Sigma_{g,n-1})$ is the one that factors through the projection $\mathrm{PMod}_{g,n} \to \mathrm{PMod}_{g,n-1}$ that deletes the n^{th} puncture.

By Harer's computation (1.1) of the virtual cohomological dimension of the mapping class group, the virtual cohomological dimension of $\mathrm{PMod}_{g,n-1}$ is $\nu - 1$. Let

$$\pi\colon\ \mathrm{St}(\Sigma_{g,n-1}) \to (\mathrm{St}(\Sigma_{g,n-1}))_{\mathrm{PMod}_{g,n-1}[\ell]} \cong \mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell]; \mathbb{Q})$$

be the evident projection.

We have $\mathrm{N}\mathcal{C}_{g,n} = \sqcup_{k=1}^{n-1} \mathrm{N}\mathcal{C}_{g,n}^k$. Lemma 5.1 says that $\mathrm{N}\mathcal{C}_{g,n}^k \,/\, \mathrm{PMod}_{g,n}[\ell]$ has ℓ^{2g} elements, so $\mathrm{N}\mathcal{C}_{g,n} \,/\, \mathrm{PMod}_{g,n}[\ell]$ has $(n-1)\ell^{2g}$ elements. It follows that $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n} \,/\, \mathrm{PMod}_{g,n}[\ell]]$ is $((n-1)\ell^{2g} - 1)$-dimensional. Let

$$\rho\colon\ \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}] \to \widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n} \,/\, \mathrm{PMod}_{g,n}[\ell]]$$

be the projection.

Combining ρ and π, we obtain a map

$$\rho\otimes\pi\colon\ \mathrm{St}(\Sigma_{g,n})\cong\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]\otimes\mathrm{St}(\Sigma_{g,n-1})\to\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}\,/\,\mathrm{PMod}_{g,n}[\ell]]\otimes\mathrm{H}^{\nu-1}(\mathrm{PMod}_{g,n-1}[\ell];\mathbb{Q}).$$

This map is clearly $\mathrm{PMod}_{g,n}[\ell]$-invariant, and its image has dimension equal to the quantity (5.2), as desired. $\square$

5.2 Some remarks on coinvariants

In the proof of Theorem A′ in §5.1, we made use of the map

$$\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]\longrightarrow\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}\,/\,\mathrm{PMod}_{g,n}[\ell]].$$

The target of this map is a quotient of the coinvariants $\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]_{\mathrm{PMod}_{g,n}[\ell]}$, and one might think that a stronger result could be proven by using these coinvariants instead, which a priori might be bigger. However, the following lemma shows that they are not, at least for $g\geq 3$. We do not know whether a better result for $g=2$ could be obtained using the coinvariants.

Lemma 5.2. *Fix $g\geq 3$ and $n\geq 2$. For all $\ell\geq 2$, we have*

$$\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]_{\mathrm{PMod}_{g,n}[\ell]}\cong\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}\,/\,\mathrm{PMod}_{g,n}[\ell]].$$

Proof. Recall that

$$\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]_{\mathrm{PMod}_{g,n}[\ell]}=\mathrm{H}_0(\mathrm{PMod}_{g,n}[\ell];\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]).$$

The long exact sequence in homology associated to the short exact sequence

$$0\longrightarrow\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]\longrightarrow\mathbb{Q}[\mathrm{N}\mathcal{C}_{g,n}]\longrightarrow\mathbb{Q}\longrightarrow 0$$

of $\mathrm{PMod}_{g,n}[\ell]$-modules contains the segment

$$\mathrm{H}_1(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q})\to\mathrm{H}_0(\mathrm{PMod}_{g,n}[\ell];\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}])\to\mathrm{H}_0(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q}[\mathrm{N}\mathcal{C}_{g,n}])\to\mathbb{Q}\to 0.$$

Since $\mathrm{PMod}_{g,n}[\ell]$ contains all Dehn twists about separating curves, a result of the third author [30, Corollary C] implies that $\mathrm{H}_1(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q})=0$; we note that the hypothesis of the result used here include the requirement that $g\geq 3$. We conclude that

$$\begin{aligned}\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}]_{\mathrm{PMod}_{g,n}[\ell]}&\cong\ker(\mathrm{H}_0(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q}[\mathrm{N}\mathcal{C}_{g,n}])\to\mathbb{Q})\\&=\ker(\mathbb{Q}[\mathrm{N}\mathcal{C}_{g,n}]_{\mathrm{PMod}_{g,n}[\ell]}\to\mathbb{Q})\\&=\ker(\mathbb{Q}[\mathrm{N}\mathcal{C}_{g,n}\,/\,\mathrm{PMod}_{g,n}[\ell]]\to\mathbb{Q})\\&=\widetilde{\mathbb{Q}}[\mathrm{N}\mathcal{C}_{g,n}\,/\,\mathrm{PMod}_{g,n}[\ell]].\end{aligned}$$

$\square$

Remark 5.3. The fact that $\mathrm{H}_1(\mathrm{PMod}_{g,n}[\ell];\mathbb{Q})=0$ for $g\geq 3$ is a special case of a conjecture of Ivanov [21] saying that $\mathrm{H}_1(\Gamma;\mathbb{Q})=0$ for all finite-index subgroup Γ of $\mathrm{PMod}_{g,n}$ with $g\geq 3$. This conjecture has been verified in many cases – in addition to the paper [30] we cited above, other recent results on it includes work of Ershov-He [8] and the third author with Wieland [34]. We also remark that it is important that we are working over $\mathbb{Q}$ since the abelianization of $\mathrm{PMod}_{g,n}[\ell]$ contains a large amount of exotic torsion (see [32, 35]).

6 Applications to algebraic geometry

In this final section, we prove Theorems D and E. The actual proofs are in §6.2, which is preceded by the preliminary §6.1 surveying basic properties of coherent cohomological dimension.

6.1 Coherent cohomological dimension

In this section, all varieties will be defined over $\mathbb{C}$.

Recall that for a variety X, the coherent cohomological dimension of X, denoted $\mathrm{CohCD}(X)$, is the maximal k such that there exists some quasi-coherent sheaf $\mathcal{F}$ on X such that $\mathrm{H}^k(X; \mathcal{F}) \neq 0$. This is always finite; indeed, we have the following:

Lemma 6.1. *For any variety X, we have $\mathrm{CohCD}(X) \leq \dim(X)$.*

Proof. Grothendieck's vanishing theorem [17, Theorem 2.7] says that for *any* sheaf $\mathcal{F}$ of abelian groups on X, we have $\mathrm{H}^k(X; \mathcal{F}) = 0$ for $k > \dim(X)$. □

The most familiar quasi-coherent sheaves are the finite-rank locally free ones, that is, sections of finite-rank algebraic vector bundles on X. The following lemma shows that in most cases it is enough to consider only these sheaves:

Lemma 6.2. *For a quasi-projective variety X, we have that $\mathrm{CohCD}(X)$ is the maximal k such that there exists a finite-rank locally free sheaf $\mathcal{F}$ on X such that $\mathrm{H}^k(X; \mathcal{F}) \neq 0$.*

Proof. Set $k = \mathrm{CohCD}(X)$. By Lemma 6.1, we have $k < \infty$. We must prove that there exists some finite-rank locally free sheaf $\mathcal{F}$ on X such that $\mathrm{H}^k(X; \mathcal{F}) \neq 0$. By definition, there exists a quasi-coherent sheaf $\mathcal{G}$ on X such that $\mathrm{H}^k(X; \mathcal{G}) \neq 0$. Since every quasi-coherent sheaf is the direct limit of its coherent subsheaves [17, Exercise II.5.15c] and sheaf cohomology commutes with direct limits [17, Proposition III.2.9], we can assume that $\mathcal{G}$ is coherent. Since X is quasi-projective, there exists some finite-rank locally free sheaf $\mathcal{F}$ along with a surjection $\pi\colon \mathcal{F} \to \mathcal{G}$; indeed, this holds for projective X [17, Corollary 5.18], and in the quasi-projective case we can embed X as an open subvariety of a projective variety Y and extend $\mathcal{G}$ to a coherent sheaf on Y [17, Exercise 5.15b]. Letting $\mathcal{G}' = \ker(\pi)$, the long exact sequence in cohomology associated to the short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{F} \xrightarrow{\pi} \mathcal{G} \longrightarrow 0$$

contains the segment

$$\mathrm{H}^k(X; \mathcal{F}) \longrightarrow \mathrm{H}^k(X; \mathcal{G}) \longrightarrow \mathrm{H}^{k+1}(X; \mathcal{G}').$$

Since $k = \mathrm{CohCD}(X)$, we have $\mathrm{H}^{k+1}(X; \mathcal{G}') = 0$. Since $\mathrm{H}^k(X; \mathcal{G}) \neq 0$, this implies that $\mathrm{H}^k(X; \mathcal{F}) \neq 0$, as desired. □

The following says in particular that passing to a finite cover does not change the coherent cohomological dimension:

Theorem 6.3 (Hartshorne, [16, Proposition 1.1]). *If $f\colon X \to X'$ is a finite surjective map between varieties, then* $\mathrm{CohCD}(X) = \mathrm{CohCD}(X')$.

We now turn to some calculations. The first is as follows.

Lemma 6.4. *We have* $\mathrm{CohCD}(\mathbb{P}^n) = n$.

Proof. By Lemma 6.1, we have $\mathrm{CohCD}(\mathbb{P}^n) \leq n$. This inequality is an equality since $\mathrm{H}^n(\mathbb{P}^n; \mathcal{O}(-n-1)) \cong \mathbb{C}$; see [17, Theorem III.5.1]. □

Corollary 6.5. *If X is a projective variety of dimension n, then* $\mathrm{CohCD}(X) = n$.

Proof. By a sequence of linear projections from points of projective space not lying in X we can construct a finite surjective morphism $X \to \mathbb{P}^n$ (this is a geometric version of Noether Normalization). The corollary now follows from Theorem 6.3 and Lemma 6.4. □

The following result shows that affine varieties behave very differently from projective ones.

Theorem 6.6 (Serre, [17, Theorem 3.7]). *A variety X is affine if and only if* $\mathrm{CohCD}(X) = 0$.

If X is a smooth projective curve, then Corollary 6.5 implies that $\mathrm{CohCD}(X) = 1$. Our next lemma shows that removing any finite set of points from such an X causes the coherent cohomological dimension to drop to 0:

Lemma 6.7. *Let X be a smooth projective curve and let $P \subset X$ be a nonempty finite set. Then $X \setminus P$ is affine. In other words,* $\mathrm{CohCD}(X \setminus P) = 0$.

Proof. The Riemann–Roch theorem implies that we can find a finite map $f\colon X \to \mathbb{P}^1$ such that $f^{-1}(\infty) = P$ (as a set, not as a divisor – we are ignoring multiplicities). It follows that $f|_{X \setminus P}$ is a finite surjective map to $\mathbb{A}^1$, which has coherent cohomological dimension 0 by Theorem 6.6. The lemma now follows from Theorem 6.3. An alternate way of concluding the proof is to use the fact that a finite morphism is affine, so $f^{-1}(\mathbb{A}^1) = X \setminus P$ is affine. □

Our final two results concern the effect of morphisms on coherent cohomological dimension. The first is as follows:

Lemma 6.8. *Let $f\colon X \to Y$ be an affine morphism between varieties. Then* $\mathrm{CohCD}(X) \leq \mathrm{CohCD}(Y)$.

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Since f is affine, the Leray spectral sequence of f degenerates (see [17, Exercise III.8.2]; the key point here is Theorem 6.6) to show that

$$\mathrm{H}^k(X; \mathcal{F}) \cong \mathrm{H}^k(Y; f_* \mathcal{F}) \qquad \text{for all } k.$$

The sheaf $f_*\mathcal{F}$ on Y is quasi-coherent, so for $k > \mathrm{CohCD}(Y)$ we have $\mathrm{H}^k(X;\mathcal{F}) = 0$. The lemma follows. □

Remark 6.9. Recalling that finite morphisms are affine, one might hope that like in Theorem 6.3, equality always hold in Lemma 6.8. Unfortunately, this is false. Indeed, by the "Jouanolou trick" (see [22, Lemma 1.5]), for any quasi-projective variety Y, there exists a surjective affine morphism $X \to Y$ with X an affine variety (so $\mathrm{CohCD}(X) = 0$ by Theorem 6.6, while $\mathrm{CohCD}(Y)$ could be anything).

Our final result concerns the coherent cohomological dimension of flat families:

Lemma 6.10. *Let $f\colon X \to Y$ be a flat projective morphism between quasi-projective varieties. Letting r be the dimension of the fibers of f, we then have* $\mathrm{CohCD}(X) \leq \mathrm{CohCD}(Y) + r$.

Proof. Set $k = \mathrm{CohCD}(X)$. By Lemma 6.2, we can find a finite-rank locally free sheaf $\mathcal{F}$ on X such that $\mathrm{H}^k(X;\mathcal{F}) \neq 0$. We then have the Leray spectral sequence

$$E_2^{pq} = \mathrm{H}^p(Y; R^q f_*(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X;\mathcal{F}).$$

Since the higher direct images $R^q f_*(\mathcal{F})$ are quasi-coherent, we have $E_2^{pq} = 0$ for $p > \mathrm{CohCD}(Y)$. Below we will prove that $R^q f_*(\mathcal{F}) = 0$ for $q > r$, so $E_2^{pq} = 0$ for $q > r$. Since $\mathrm{H}^k(X;\mathcal{F}) \neq 0$, we must therefore have $k \leq \mathrm{CohCD}(Y) + r$, as desired.

Fix some $q > r$. It remains to prove that $R^q f_*(\mathcal{F}) = 0$. Consider a point $y_0 \in Y$. We will prove that

$$(R^q f_*(\mathcal{F})) \otimes k(y_0) = 0.$$

Letting $X_{y_0} = f^{-1}(y_0)$, there is a natural map

$$\eta : (R^q f_*(\mathcal{F})) \otimes k(y_0) \to \mathrm{H}^q(X_{y_0}; \mathcal{F}_{y_0});$$

see [17, §III.12]. Since X_{y_0} has dimension r, Lemma 6.1 implies that $\mathrm{H}^q(X_{y_0}; \mathcal{F}_{y_0}) = 0$. It is thus enough to prove that the map η is an isomorphism.

Since $\mathcal{F}$ is locally free on X, it is flat over X and thus flat over Y. What is more, since its target is 0, the map η is trivially surjective. Under these assumptions (f a projective morphism, $\mathcal{F}$ a coherent sheaf on X that is flat over Y, surjectivity of η), we can apply [17, Theorem 12.11 (Cohomology and base change)] to deduce that η is an isomorphism, as desired. □

6.2 Algebraic geometry proofs

We finally prove Theorems D and E.

Proof of Theorem D. We first recall the statement. Fix some $g \geq 2$ and $n \geq 1$. We must prove that $\mathrm{CohCD}(\mathcal{M}_{g,n}) \geq g - 1$. Assume for the sake of contradiction that $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq g - 2$. Harer's computation of the virtual cohomological dimension of the mapping class group (1.1) says that the virtual cohomological dimension of $\mathrm{PMod}_{g,n}$ is $4g-4+n$. We will prove below that our assumption that $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq g-2$ implies that $\mathrm{H}^{4g-4+n}(\mathrm{PMod}_{g,n}[\ell];\mathbb{C}) = 0$ for all $\ell \geq 3$. However, Theorem A implies that $\mathrm{H}^{4g-4+n}(\mathrm{PMod}_{g,n}[\ell];\mathbb{C}) \neq 0$, a contradiction.

It remains to prove that our assumption $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq g-2$ implies that

$$\mathrm{H}^{4g-4+n}(\mathrm{PMod}_{g,n}[\ell];\mathbb{C}) = 0 \qquad \text{for all } \ell \geq 3. \tag{6.1}$$

Fix some $\ell \geq 3$ and let $\mathcal{M}_{g,n}[\ell]$ be[2] the finite cover of $\mathcal{M}_{g,n}$ corresponding to the subgroup $\mathrm{PMod}_{g,n}[\ell]$ of its (orbifold) fundamental group $\mathrm{PMod}_{g,n}$. Recall that the singularities of $\mathcal{M}_{g,n}$ exist due to the presence of curves with automorphisms (for instance, from the point of view of Teichmüller theory, the singularities come from the fixed points of the action of the mapping class group on Teichmüller space). Our assumption $\ell \geq 3$ implies that the group $\mathrm{PMod}_{g,n}[\ell]$ is torsion-free, so $\mathcal{M}_{g,n}[\ell]$ is a smooth variety.

The map $\mathcal{M}_{g,n}[\ell] \to \mathcal{M}_{g,n}$ is a finite surjective map, so by Theorem 6.3 we have

$$\mathrm{CohCD}(\mathcal{M}_{g,n}[\ell]) = \mathrm{CohCD}(\mathcal{M}_{g,n}) \leq g-2. \tag{6.2}$$

Since $\mathcal{M}_{g,n}[\ell]$ is smooth, we can calculate its cohomology using de Rham cohomology. The Hodge–de Rham spectral sequence for $\mathcal{M}_{g,n}[\ell]$ converges to $\mathrm{H}^{\bullet}(\mathcal{M}_{g,n}[\ell];\mathbb{C})$ and has

$$E_1^{pq} = \mathrm{H}^p(\mathcal{M}_{g,n}[\ell];\Omega^q).$$

Since the complex dimension of $\mathcal{M}_{g,n}[\ell]$ is $3g-3+n$, we have $\Omega^q = 0$ for $q \geq 3g-2+n$, and thus

$$E_1^{pq} = 0 \qquad (q \geq 3g-2+n). \tag{6.3}$$

Moreover, since Ω^q is a coherent sheaf on $\mathcal{M}_{g,n}[\ell]$ equation (6.2) implies that

$$E_1^{pq} = \mathrm{H}^p(\mathcal{M}_{g,n}[\ell];\Omega^q) = 0 \qquad (p \geq g-1). \tag{6.4}$$

From (6.3) and (6.4), we deduce that $E_1^{pq} = 0$ whenever $p+q = 4g-4+n$. We conclude that (6.1) holds, as desired. □

Before proving Theorem E we require one further lemma. It is a generalization to the relative setting of the familiar fact that if X is a projective variety and $D \subset X$ is an ample divisor, then $X \setminus D$ is affine (quick proof: by the definition of an ample divisor, there exists an embedding $X \hookrightarrow \mathbb{P}^n$ whose hyperplane section at infinity is a positive multiple of D, so $X \setminus D$ is a closed subvariety of $\mathbb{A}^n$).

Lemma 6.11. *Let $f\colon X \to Y$ be a projective morphism of varieties and let D be a divisor on X whose intersection with each fiber $X_y = f^{-1}(y)$ is ample. Then the restriction of f to $X \setminus D$ is affine.*

[2]Warning: this notation is also commonly used to denote a different space, namely, the moduli space of curves with a full level-ℓ structure, which corresponds to the kernel of the action of $\mathrm{PMod}_{g,n}$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$ obtained by filling in the punctures. The space $\mathcal{M}_{g,n}[\ell]$ is a cover of this.

Proof. It is enough to assume that Y is affine and prove that $X \setminus D$ is affine. By [25, Theorem 1.7.8], the divisor D is ample relative to f. This means that for some $m \gg 0$, the divisor mD induces an embedding $X \hookrightarrow \mathbb{P}^n_Y$ for some n such that f factors as $X \hookrightarrow \mathbb{P}^n_Y \to Y$. The hyperplane section at infinity of $X \subset \mathbb{P}^n_Y$ is precisely mD. We conclude that $X \setminus D$ is a closed subvariety of the affine variety $\mathbb{A}^n_Y$, and hence is itself affine. □

Proof of Theorem E. We first recall the statement. Fix some $g \geq 2$ and $n \geq 1$. We must prove that $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq \mathrm{CohCD}(\mathcal{M}_g) + 1$. Set

$$\delta = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

We will prove that $\mathrm{CohCD}(\mathcal{M}_{g,n}) \leq \mathrm{CohCD}(\mathcal{M}_{g,n-1}) + \delta$.

Fix some $\ell \geq 3$. Just like in the previous proof[3], let $\mathcal{M}_{g,n-1}[\ell]$ be the cover of $\mathcal{M}_{g,n-1}$ corresponding to the subgroup $\mathrm{PMod}_{g,n-1}[\ell]$ of its (orbifold) fundamental group $\mathrm{PMod}_{g,n-1}$. Recall that $\mathcal{M}_{g,n-1}$ is only a coarse moduli space rather than a fine moduli space for the same reason it is not smooth: the existence of curves with automorphisms. Since $\ell \geq 3$, the group $\mathrm{PMod}_{g,n-1}[\ell]$ is torsion-free, so $\mathcal{M}_{g,n-1}[\ell]$ is not only smooth, but also a fine moduli space. It thus has a universal curve $U_{g,n-1}[\ell] \to \mathcal{M}_{g,n-1}[\ell]$. Removing the $(n-1)$ distinguished points from each fiber of $U_{g,n-1}[\ell]$, we get a space $\widetilde{\mathcal{M}}_{g,n}$ that fits into a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}}_{g,n} \arrow[r] \arrow[d, "f"] & \mathcal{M}_{g,n} \arrow[d] \\ \mathcal{M}_{g,n-1}[\ell] \arrow[r] & \mathcal{M}_{g,n-1}. \end{tikzcd}$$

Both horizontal maps are finite surjective maps, so by Theorem 6.3 we have

$$\mathrm{CohCD}(\widetilde{\mathcal{M}}_{g,n}) = \mathrm{CohCD}(\mathcal{M}_{g,n}) \quad \text{and} \quad \mathrm{CohCD}(\mathcal{M}_{g,n-1}[\ell]) = \mathrm{CohCD}(\mathcal{M}_{g,n-1}).$$

It is thus enough to prove that $\mathrm{CohCD}(\widetilde{\mathcal{M}}_{g,n}) \leq \mathrm{CohCD}(\mathcal{M}_{g,n-1}[\ell]) + \delta$.

We first deal with the case where $n = 1$. The morphism $\widetilde{\mathcal{M}}_{g,1} \to \mathcal{M}_g[\ell]$ is a flat projective morphism whose fibers are smooth projective curves. We can thus apply Lemma 6.10 to deduce that

$$\mathrm{CohCD}(\widetilde{\mathcal{M}}_{g,1}) \leq \mathrm{CohCD}(\mathcal{M}_g[\ell]) + 1,$$

as desired.

We now deal with the case where $n > 1$. By Lemma 6.8, to prove that $\mathrm{CohCD}(\widetilde{\mathcal{M}}_{g,n}) \leq \mathrm{CohCD}(\mathcal{M}_{g,n-1}[\ell])$, it is enough to prove that the morphism $\widetilde{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n-1}[\ell]$ is affine. Recalling that $U_{g,n-1}[\ell]$ is the universal curve, the morphism $\pi \colon U_{g,n-1}[\ell] \to \mathcal{M}_{g,n-1}[\ell]$ is projective, and $\widetilde{\mathcal{M}}_{g,n}$ is obtained from $U_{g,n-1}[\ell]$ by removing a divisor whose intersection with each fiber $\pi^{-1}(S)$ is a sum of $(n-1)$ points. Since all positive divisors on an algebraic curve are ample, the result now follows from Lemma 6.11. □

[3]Unlike in the previous proof, here we could have simply taken the moduli space of curves with a full level-ℓ structure; however, we have decided not to introduce yet another piece of notation.

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