ERRATUM TO: THE LEVEL FOUR BRAID GROUP

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As in our original paper [2], let ρ be the symplectic representation of B_n, let $\pi_1(D_n', p_1), \ldots, \pi_1(D_n', p_n)$ denote the point pushing subgroups of B_n, and for $1 \leq k \leq n$ set

$$K_{n,k} = \pi_1(D_n', p_1) \cap \cdots \cap \pi_1(D_n', p_k)$$

Also, let $\Gamma_n[m]$ denote $\mathrm{Sp}_{2g}(\mathbb{Z})[m]$ when $n = 2g+1$ and $\left(\mathrm{Sp}_{2g+2}(\mathbb{Z})[m]\right)_{\vec{y}_{g+1}}$ when $n = 2g+2$.

Theorem 5.1 describes $\rho(K_{n,k})$ for $n \geq 5$. The theorem separately addresses the cases where $n = 2g+1$ and $n = 2g+2$. In each case, there are two statements. The first statement is that $\rho(K_{n,k})$ contains $\Gamma_n[4]$ and the second statement describes the quotient of $\rho(K_{n,k})$ by $\Gamma_n[4]$. We refer to these two statements as the containment statement and the quotient statement, respectively.

The proof of the containment statement of Theorem 5.1 is correct for $k = 1$ and incorrect for $k \geq 2$. What our argument for the containment statement actually shows is that each $\rho(\pi_1(D_n', p_i))$ contains $\Gamma_n[4]$ and hence the argument only proves the weaker statement that

$$L_{n,k} = \rho(\pi_1(D_n', p_1)) \cap \cdots \cap \rho(\pi_1(D_n', p_k))$$

contains $\Gamma_n[4]$. Since $L_{n,1} = \rho(K_{n,1})$, the argument for the containment statement is correct for $k = 1$ and $n \geq 5$. For $k \geq 2$ we have $L_{n,k} \supseteq \rho(K_{n,k})$, but this is not an equality in general.

It should be considered an open question as to whether the containment statement of Theorem 5.1 is correct for $k \geq 2$. At the end of the paper, we explain how our proof of Theorem 5.1 can be extended to the case $n = 3$, in particular that $\rho(K_{3,k})$ contains $\Gamma_3[4] = \mathrm{SL}_2(\mathbb{Z})[4]$. This statement, the $n = 3$ version of the containment statement, is not correct. In particular, the last statement in the paper, that $\rho(K_{3,3}) = \Gamma_3[4]$, is not correct. In fact, $\rho(K_{3,3})$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$. To see this, we first note that $K_{3,3}$ is the Brunnian subgroup of B_3. Let Z denote the kernel of $\rho : \mathrm{B}_3 \to \mathrm{SL}_2(\mathbb{Z})$. The group Z is an infinite cyclic group generated by the square of the Dehn twist about the boundary of D_3'. For $m \neq 0$, no element of the coset $\sigma_1^m Z$ is Brunnian, hence no power of the matrix $\rho(\sigma_1)$ lies in $\rho(K_{3,3})$.

The statement and proof of the quotient statement of Theorem 5.1 are correct for $k = 1$. Because of the $n = 3$ case, we expect that the containment statement of Theorem 5.1 is not correct for any $k \geq 2$ and $n \geq 5$. If this is the case, the quotient statement does not make sense for $k \geq 2$.

As in the $n = 3$ case, we expect that $\rho(K_{n,k})$ in fact has infinite index in $\Gamma_n[4]$ for $n \geq 4$ and $k \geq 2$. As in the $n = 3$ case, the $k = n$ version of this statement can be proven by showing that if $h \in \ker(\rho)$ then $\sigma_1^m h$ is not

Brunnian. Since $\ker(\rho)$ is generated by squares of Dehn twists about curves surrounding an odd number of punctures [1], we may assume that h is such a product.

What our argument for the quotient statement of Theorem 5.1 actually shows is that the image of $\rho(K_{n,k})$ in $\Gamma_n[2]/\Gamma_n[4]$ is $(\mathbb{Z}/2)^{2g}$, $\mathbb{Z}/2$, or 1, according to whether k is 1, 2, or greater. In other words, $\rho(K_{n,k})$ modulo $\rho(K_{n,k}) \cap \Gamma_n[4]$ is the abelian group given in the previous sentence. It is also true that $L_{n,k}/\Gamma_n[4]$ is the same abelian group. The given indices of $\rho(K_{n,k})$ in $\Gamma_n[2]$ for $k \geq 2$ are the correct indices for $L_{n,k}$ in $\Gamma_n[2]$.

There are two other incorrect statements in Section 3 that we would like to point out. First, we incorrectly state that $\rho(\mathrm{B}_n)$ is the semi-direct product of a symmetric group with $\Gamma_n[2]$. In fact $\rho(\mathrm{B}_n)$ is a non-split extension of these groups. Also, we incorrectly state that $\psi : \mathrm{Sp}_{2g}(\mathbb{Z}/2) \to \mathfrak{sp}_{2g}(\mathbb{Z}/2)$ is the abelianization map for $\mathrm{Sp}_{2g}(\mathbb{Z}/2)$. We are grateful to David Benson and Nick Salter for these corrections.

References

[1] Tara Brendle, Dan Margalit, and Andrew Putman. Generators for the hyperelliptic Torelli group and the kernel of the Burau representation at $t = -1$. *Invent. Math.*, 200(1):263–310, 2015.

[2] Tara E. Brendle and Dan Margalit. The level four braid group. *J. Reine Angew. Math.*, 735:249–264, 2018.