Lectures
on

Constitutive Modelling of Arteries

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Overview of the Ingredients of Continuum Mechanics needed in Soft Tissue Biomechanics

and the phenomenological description of material properties

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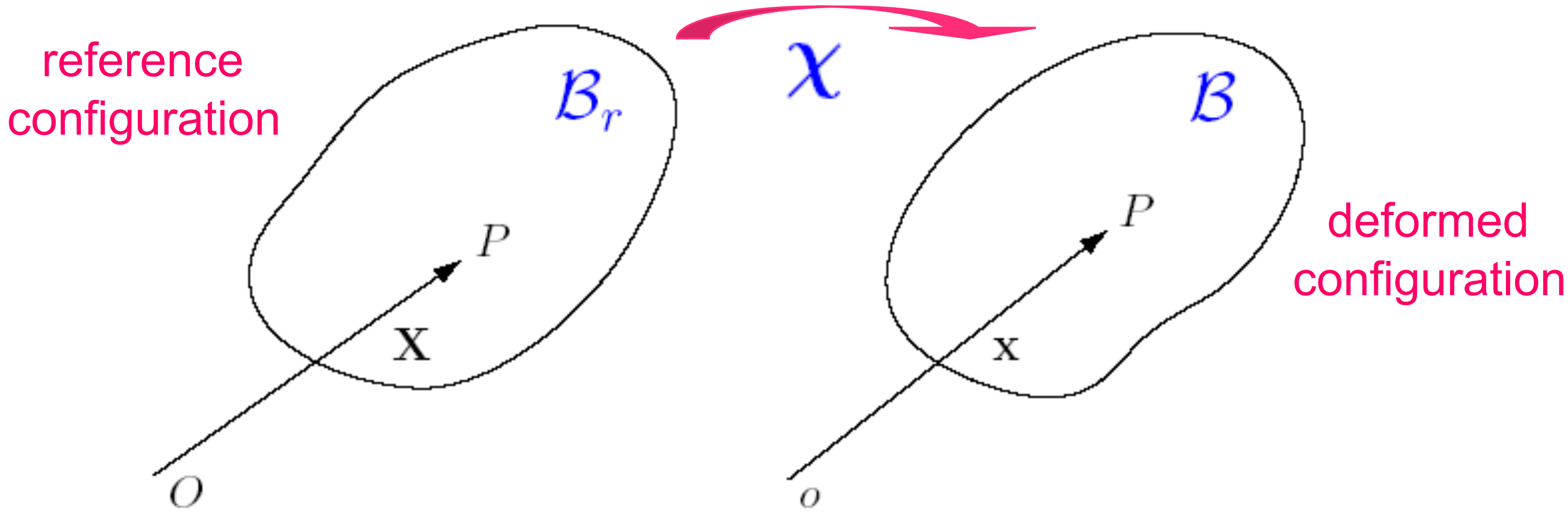
Lecture Contents

Outline of the basics tools from continuum mechanics – kinematics, invariants, stress, elasticity, strain energy, stress-deformation relations

Characterization of material properties – isotropy and anisotropy, fibrous materials

Application to arteries (and the myocardium – if time allows)

Kinematics



Deformation $\mathbf{x} = \chi(\mathbf{X})$

Properties of χ

Continuous
One-to-one and onto – invertible
Differentiable, inverse differentiable
(not necessarily continuously differentiable)

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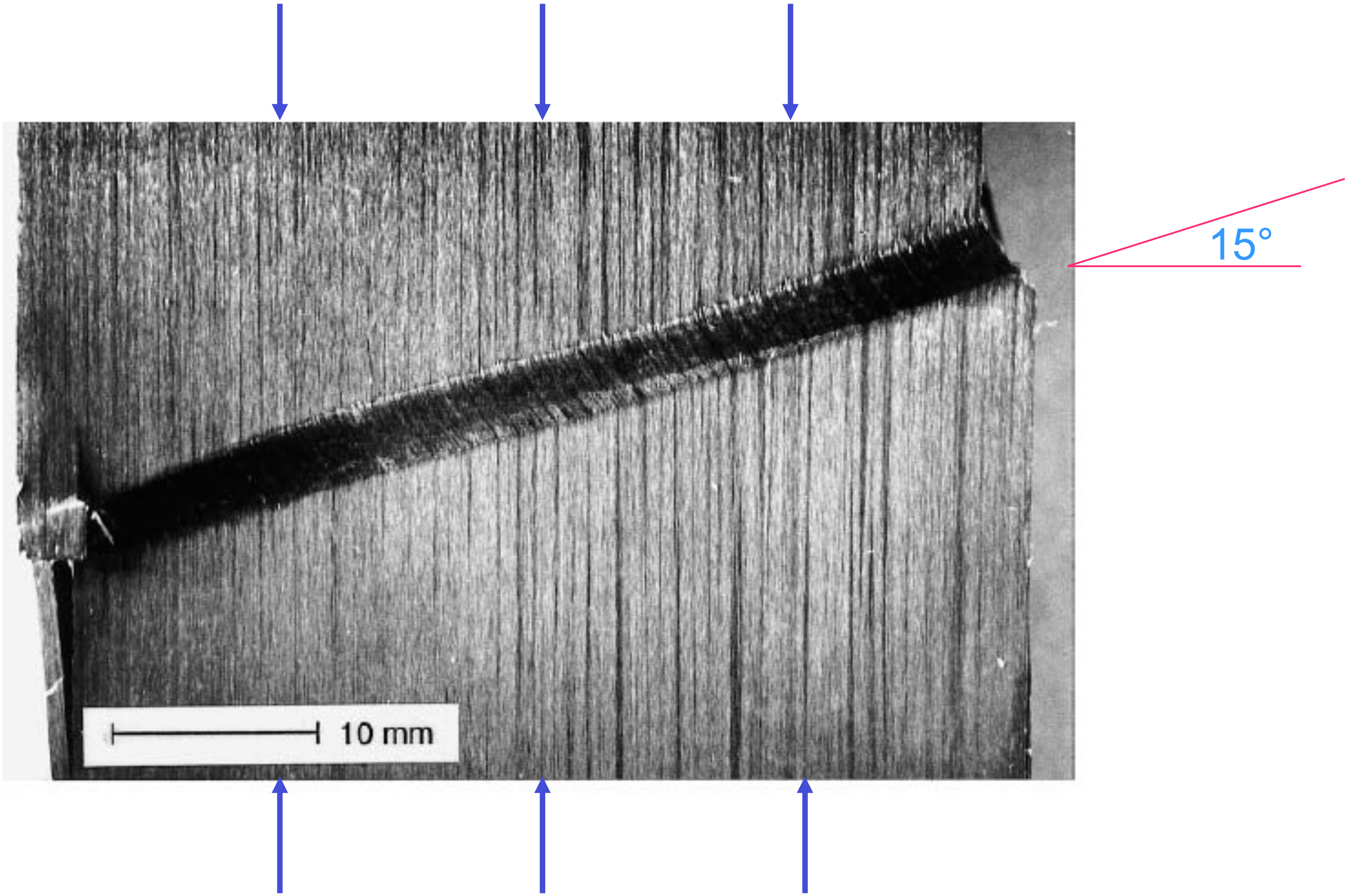
Example of a deformation that is **not** continuously differentiable – a kink band –

from Vogler and Kyriakides (1999)



Deformation gradient $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$

Associated (Cauchy−Green) tensors

left $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ right

Polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

rotation tensor ($\mathbf{R}$)

positive definite symmetric tensors ($\mathbf{U}$, $\mathbf{V}$)

Eigenvalues of $\mathbf{U}$ and $\mathbf{V}$

are the **principal stretches** $\lambda_i > 0 \quad i = 1, 2, 3$

Eigenvalues of $\mathbf{B}$ and $\mathbf{C}$ are λ_i^2

Stretch can be defined for any reference direction $\mathbf{M}$ – unit vector

Square of length of a line element

$$|\mathrm{d}\mathbf{x}|^2 = (\mathbf{F}\mathrm{d}\mathbf{X}) \cdot (\mathbf{F}\mathrm{d}\mathbf{X}) = (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M})|\mathrm{d}\mathbf{X}|^2 = (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{M}) \cdot \mathbf{M}|\mathrm{d}\mathbf{X}|^2$$

unit vector in reference configuration

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^2$$

$$\frac{|\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{X}|} = |\mathbf{F}\mathbf{M}| = [(\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{M}) \cdot \mathbf{M}]^{1/2} \equiv \lambda(\mathbf{M})$$

– stretch in direction $\mathbf{M}$

Inextensibility constraint

$$(\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{M}) \cdot \mathbf{M} = 1$$

Green strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$$

Deformation of

1. line elements $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$

2. area elements $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}A$ – Nanson's formula

3. volume elements $\mathrm{d}v = J\mathrm{d}V$

$$J = \det \mathbf{F} > 0$$

Incompressibility constraint

$$J \equiv \det \mathbf{F} = 1$$

Deformation invariants

Principal invariants of $\mathbf{C}$

$$I_1 = \mathrm{tr}\mathbf{C}$$

$$I_2 = \frac{1}{2}[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)]$$

$$I_3 = \det \mathbf{C} = J^2$$

Invariants associated with a distinguished direction $\mathbf{M}$ in the reference configuration

$$I_4 = \mathbf{M} \cdot (\mathbf{C}\mathbf{M}) = \lambda(\mathbf{M})^2$$ – square of stretch in direction $\mathbf{M}$

$$I_5 = \mathbf{M} \cdot (\mathbf{C}^2\mathbf{M})$$ – no simple physical interpretation

An alternative to I_5 based on Nanson's formula

$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}A$$

$$I_5^* = (\mathbf{C}^*\mathbf{M}) \cdot \mathbf{M}$$ – square of ratio of deformed to undeformed area element initially normal to $\mathbf{M}$

$$\mathbf{C}^* = I_3\mathbf{C}^{-1}$$

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Invariants associated with two distinguished directions $\mathbf{M}$ $\mathbf{M}'$ in the reference configuration

Additional invariants

$$I_6 = \mathbf{M}' \cdot (\mathbf{C}\mathbf{M}') \qquad I_7 = \mathbf{M}' \cdot (\mathbf{C}^2\mathbf{M}')$$

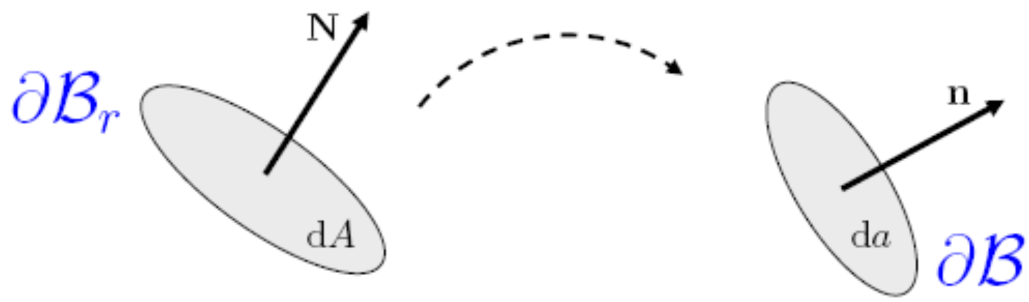
$$I_8 = \mathbf{M} \cdot (\mathbf{C}\mathbf{M}')(\mathbf{M} \cdot \mathbf{M}')$$

Note: another way to write I_4 and similarly for the other invariants is

$$I_4 = \mathrm{tr}(\mathbf{C}\underbrace{\mathbf{M} \otimes \mathbf{M}})$$

structure tensor

Stress tensors



$$\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}A$$

$$\mathbf{t}\,\mathrm{d}a = \boldsymbol{\sigma}\mathbf{n}\,\mathrm{d}a = \boxed{J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}}\mathbf{N}\,\mathrm{d}A \equiv \mathbf{S}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}A$$

traction vector

Cauchy stress tensor (symmetric)

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$$ nominal stress tensor

$$\mathbf{S}^{\mathrm{T}}$$ first Piola-Kirchhoff stress tensor

Equilibrium (no body forces) $\int_{\partial\mathcal{B}_r} \mathbf{S}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}A = \mathbf{0} \;\Rightarrow\; \mathrm{Div}\,\mathbf{S} = \mathbf{0}$

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Introduction of the (elastic) strain energy

consider the virtual work of the surface tractions

$$\int_{\partial \mathcal{B}_r} (\mathbf{S}^{\mathrm{T}}\mathbf{N}) \cdot \delta \mathbf{x} \mathrm{d}A = \int_{\mathcal{B}_r} \mathrm{Div}(\mathbf{S}\delta\mathbf{x})\mathrm{d}V = \int_{\mathcal{B}_r} \mathrm{tr}(\mathbf{S}\underbrace{\mathrm{Grad}\delta\mathbf{x}}_{\delta\mathbf{F}})\mathrm{d}V$$

This is converted into stored (elastic) energy

if there exists a scalar function $W = W(\mathbf{F})$

such that

$$\delta W = \mathrm{tr}(\mathbf{S}\delta\mathbf{F})$$

from which we obtain the stress-deformation relation

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} \qquad \boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}$$

for an unconstrained material

Some properties of $W(\mathbf{F})$

$W(\mathbf{F}) \xrightarrow{\text{objectivity}} W(\mathbf{U})$ or $W(\mathbf{C})$ or $W(\mathbf{E})$

$$\mathbf{C} = \mathbf{U}^2 \qquad \mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I})$$

$W(\mathbf{C}) \xrightarrow{\text{material symmetry}} W(I_1, I_2, \dots)$

For an incompressible material $\det \mathbf{F} = 1$

Lagrange multiplier

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1} \qquad \boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}$$

Material symmetry

$$W \longrightarrow W(I_1, I_2, \ldots, I_N)$$

$$\mathbf{S} = \sum_{i=1}^{N} W_i \frac{\partial I_i}{\partial \mathbf{F}} \underbrace{-p\mathbf{F}^{-1}} \qquad W_i = \frac{\partial W}{\partial I_i}$$

incompressible $I_3 \equiv 1$

Incompressible isotropic elastic materials

Principal invariants $I_1 = \mathrm{tr}\,\mathbf{C}$ $\quad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}\,(\mathbf{C}^2)]$

Strain energy $W = W(I_1, I_2)$ $\quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ right C-G tensor

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2W_1\mathbf{B} + 2W_2\left(I_1\mathbf{B} - \mathbf{B}^2\right)$$

left C-G tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$

Principal stresses

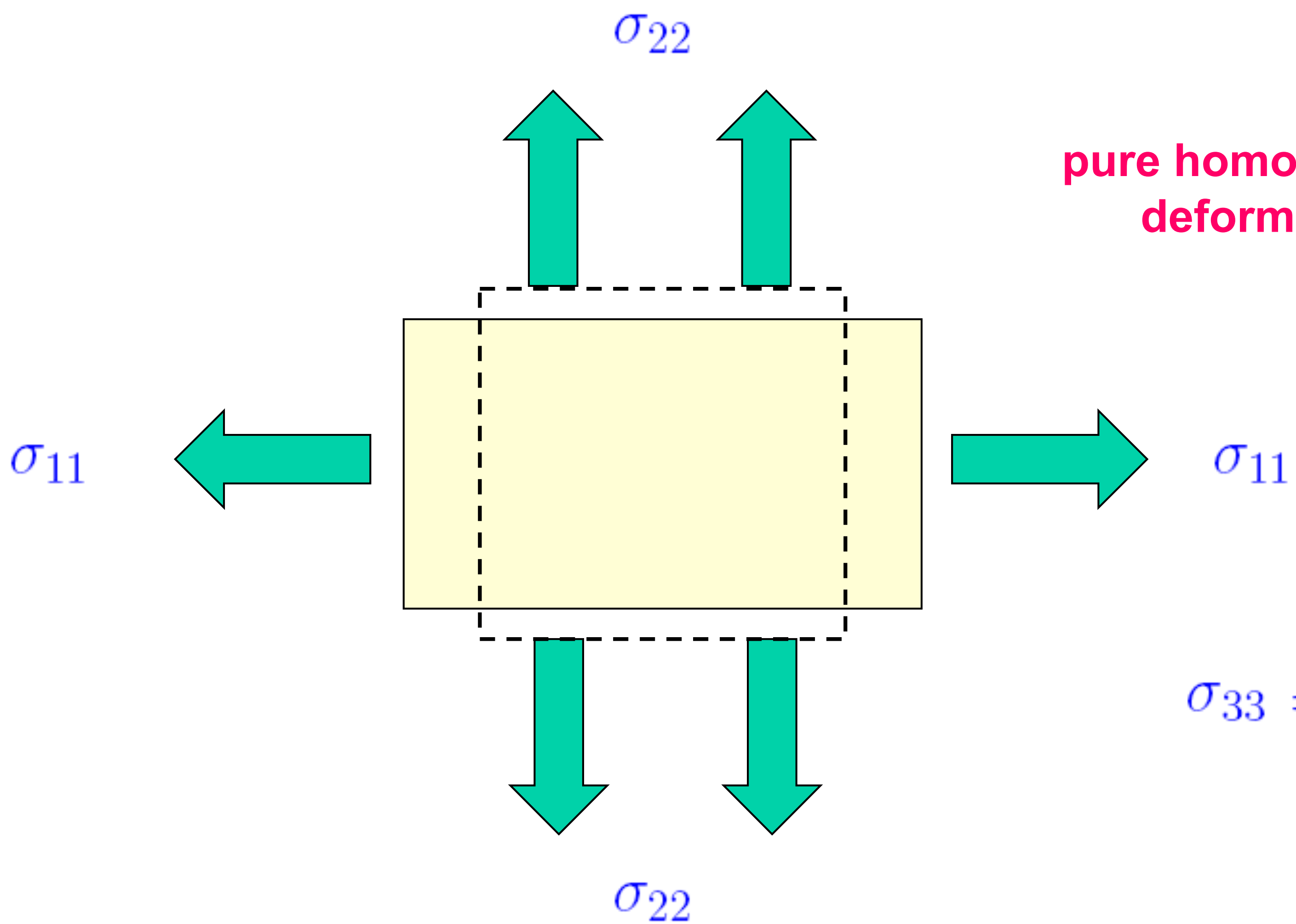
$$\sigma_1 = -p + 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^4)$$

$$\sigma_2 = -p + 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_2^4)$$

$$\sigma_3 = -p + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4) = 0$$

$$\lambda_1\lambda_2\lambda_3 = 1$$

To characterize W it suffices to perform planar biaxial tests on a thin sheet



pure homogeneous deformation

$$\sigma_{33} = 0$$

Eliminate p

$$\sigma_1 = 2(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2})\left(\frac{\partial W}{\partial I_1} + \lambda_2^2\frac{\partial W}{\partial I_2}\right)$$

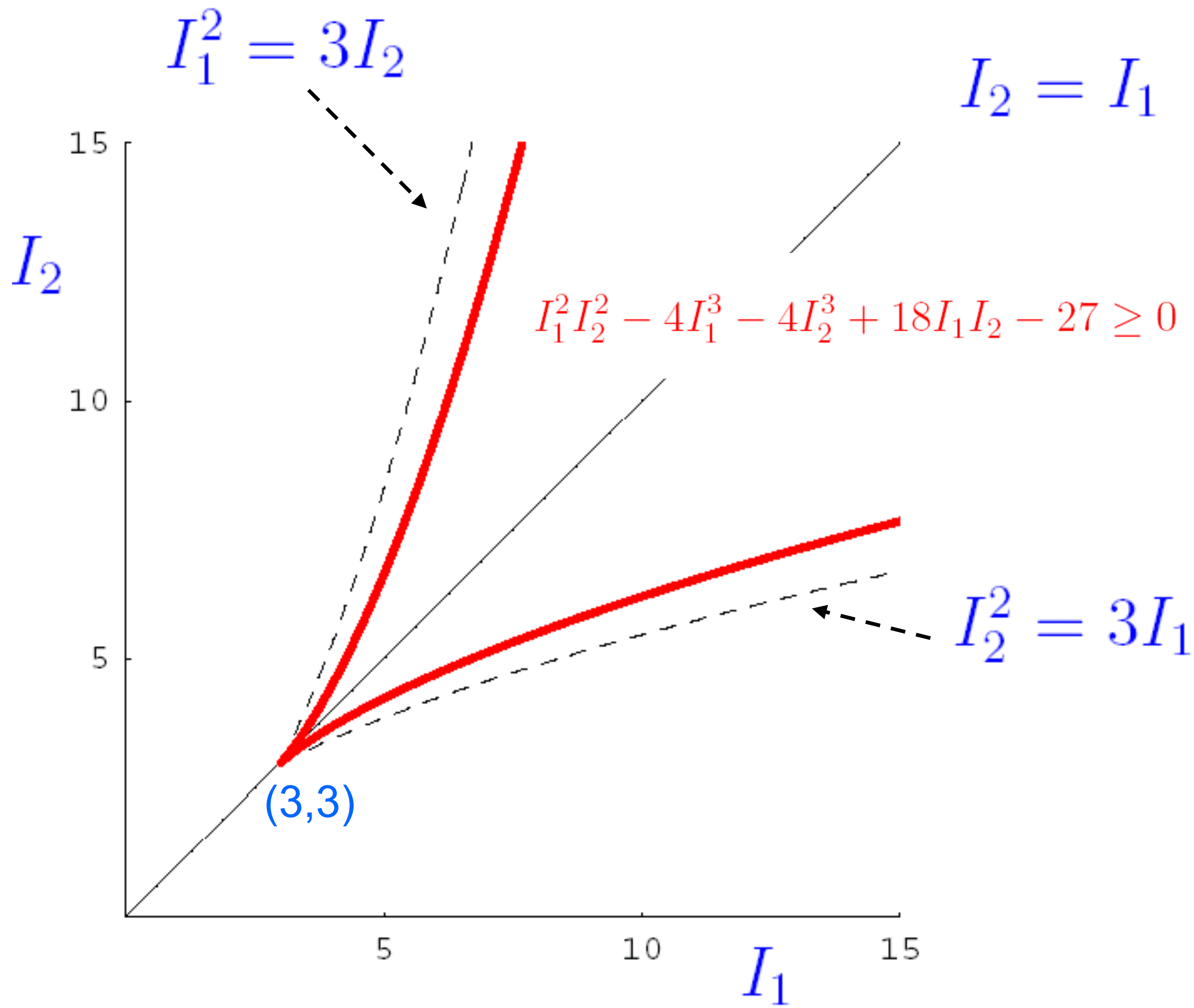
$$\sigma_2 = 2(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2})\left(\frac{\partial W}{\partial I_1} + \lambda_1^2\frac{\partial W}{\partial I_2}\right)$$

Data with $\lambda_1\lambda_2\lambda_3 = 1$ enable

$$\frac{\partial W}{\partial I_1} = \frac{1}{2}\frac{\lambda_1^2\sigma_1}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} - \frac{1}{2}\frac{\lambda_2^2\sigma_2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}$$

$$\frac{\partial W}{\partial I_2} = \frac{1}{2}\frac{\sigma_2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} - \frac{1}{2}\frac{\sigma_1}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}$$

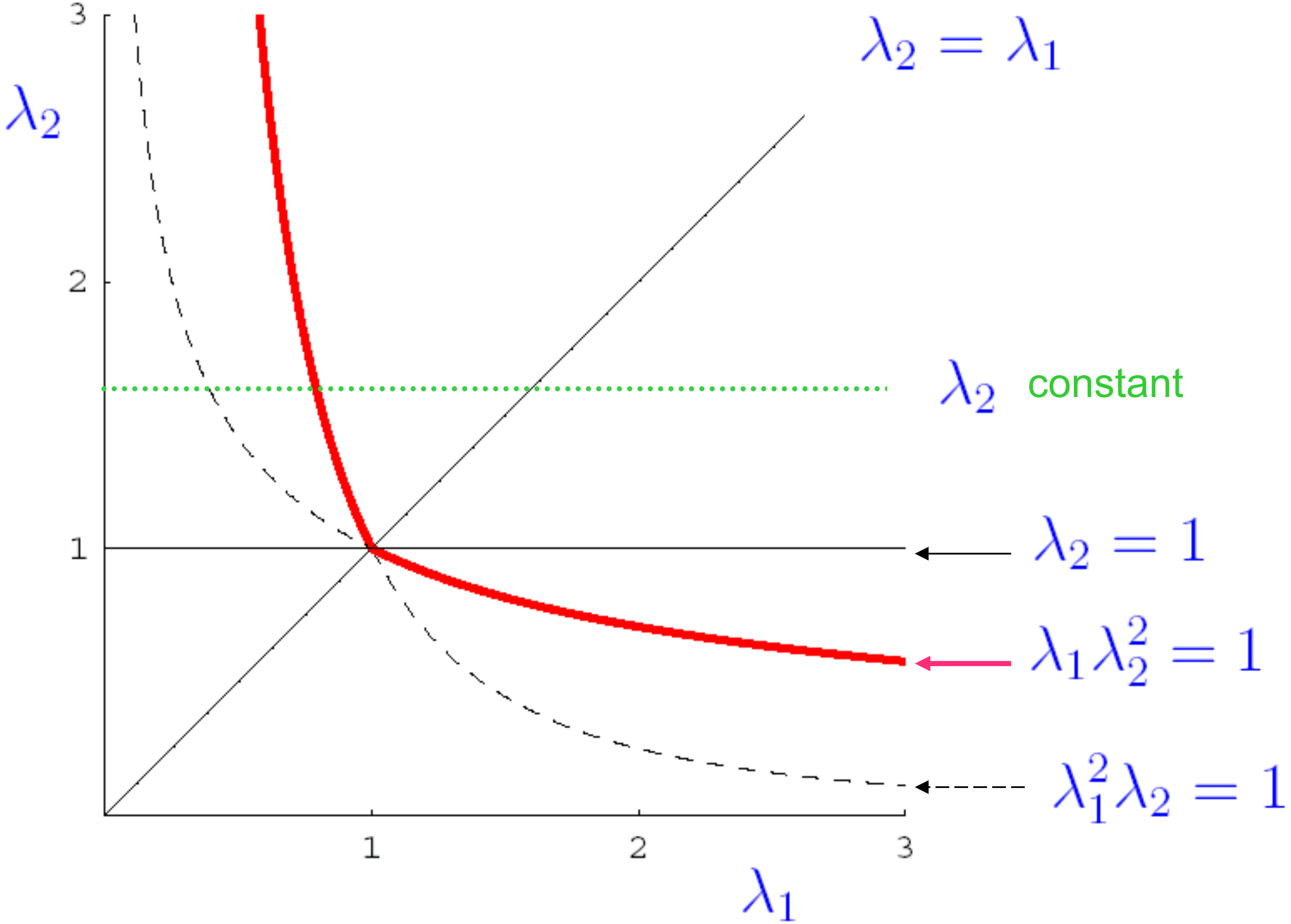
to be determined


$I_1^2 = 3I_2$
$I_2 = I_1$
15
10
5
I_2
$I_1^2 I_2^2 - 4I_1^3 - 4I_2^3 + 18I_1 I_2 - 27 \geq 0$
$I_2^2 = 3I_1$
(3,3)
5
10
15
I_1

Alternatively (and equivalently) –
in terms of the principal stretches

$$W = \hat{W}(\lambda_1, \lambda_2)$$

$$\sigma_1 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} \qquad \sigma_2 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}$$



3
2
1
1
2
3
λ_2
λ_1
$\lambda_2 = \lambda_1$
λ_2 constant
$\lambda_2 = 1$
$\lambda_1 \lambda_2^2 = 1$
$\lambda_1^2 \lambda_2 = 1$

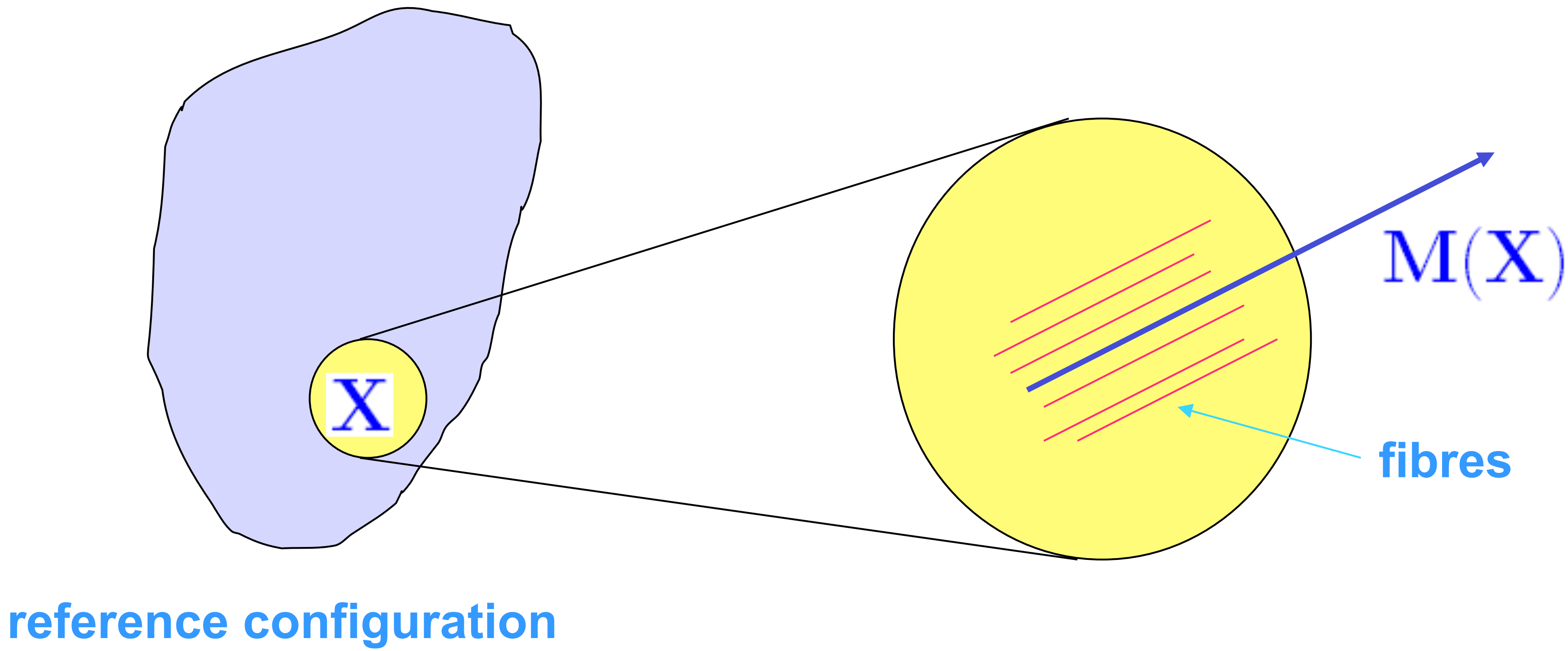
In either case there are two independent deformation quantities and two stress components –

thus, **planar biaxial tests** (or extension-inflation tests) **are sufficient to fully determine the three-dimensional material properties for an incompressible isotropic material**

This is not the case for anisotropic materials

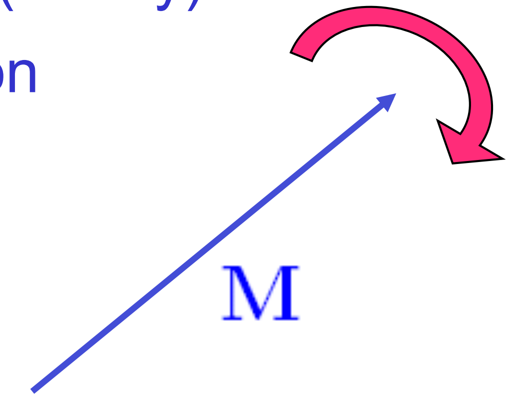
contrary to various claims in the literature

Modelling fibre reinforcement



Fibres characterized in terms of the unit vector field $\mathbf{M}$

One family of fibres – transverse isotropy (locally) – rotational symmetry about direction



$W(\mathbf{F})$ is an isotropic function of

$\mathbf{C}$ and $\mathbf{M} \otimes \mathbf{M}$

structure tensor

$$W(\mathbf{F}) \quad \rightarrow \quad W(I_1, I_2, I_4, I_5)$$

anisotropic (transversely isotropic) invariants

$$I_4 = \mathbf{M} \cdot (\mathbf{C}\mathbf{M}) \qquad I_5 = \mathbf{M} \cdot (\mathbf{C}^2\mathbf{M})$$

Cauchy stress

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2)$$
$$+ 2W_4\mathbf{m} \otimes \mathbf{m} + 2W_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m})$$

$$\mathbf{m} = \mathbf{FM}$$

4 constitutive functions – require 4 independent tests to determine

$$W_1 \quad W_2 \quad W_4 \quad W_5$$

Arterial tissue and characterization of the elastic properties of fibrous materials

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Typical arterial segments

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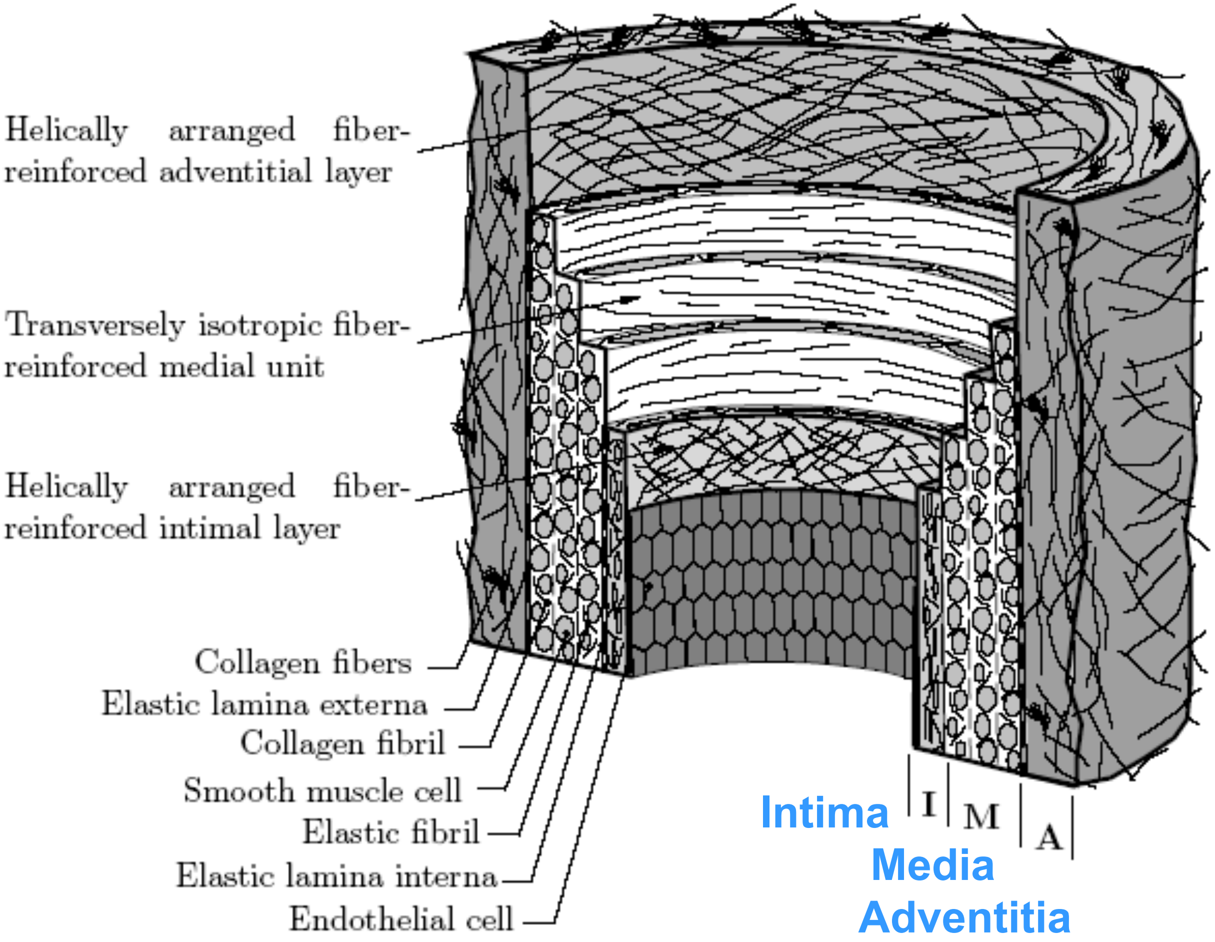
Schematic of arterial wall layered structure



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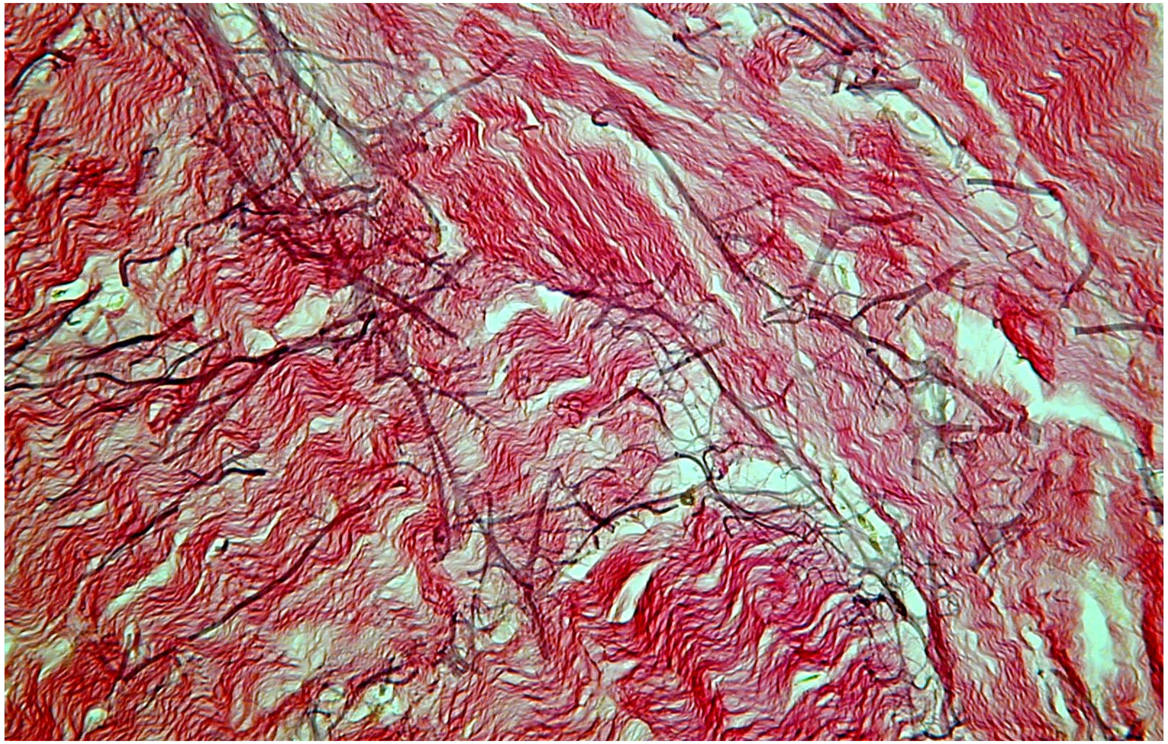
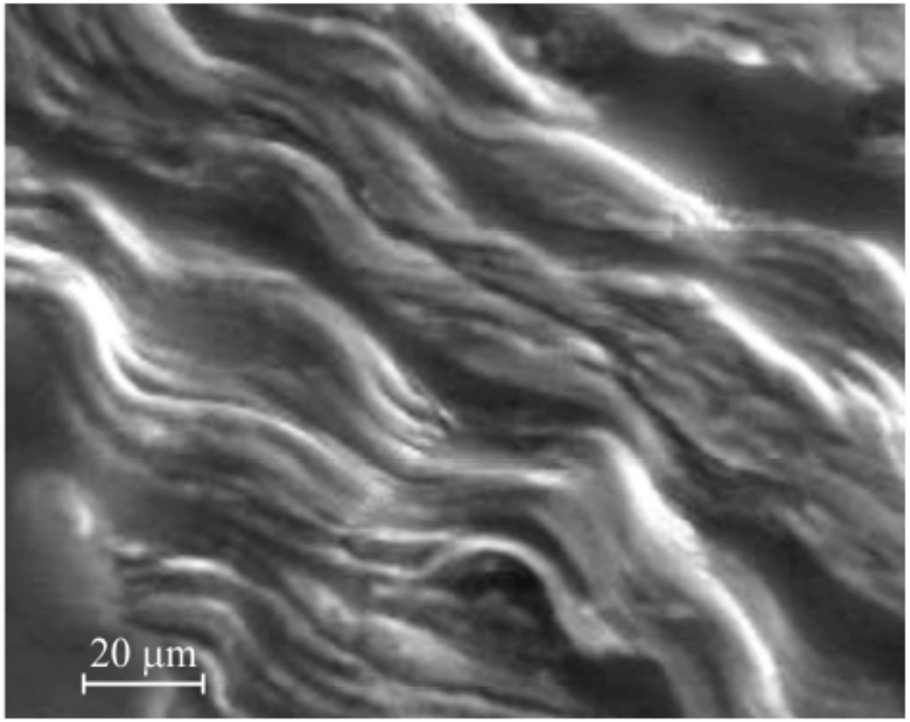


Collagen fibres in an iliac artery (adventitia)

ESEM – adventitia of human aorta



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Rubber and soft tissue elasticity – similarities and differences

Rubber	Soft tissue
Elastic	Elastic
Large deformations	Large deformations
Incompressible	Incompressible
Isotropic	**An**isotropic

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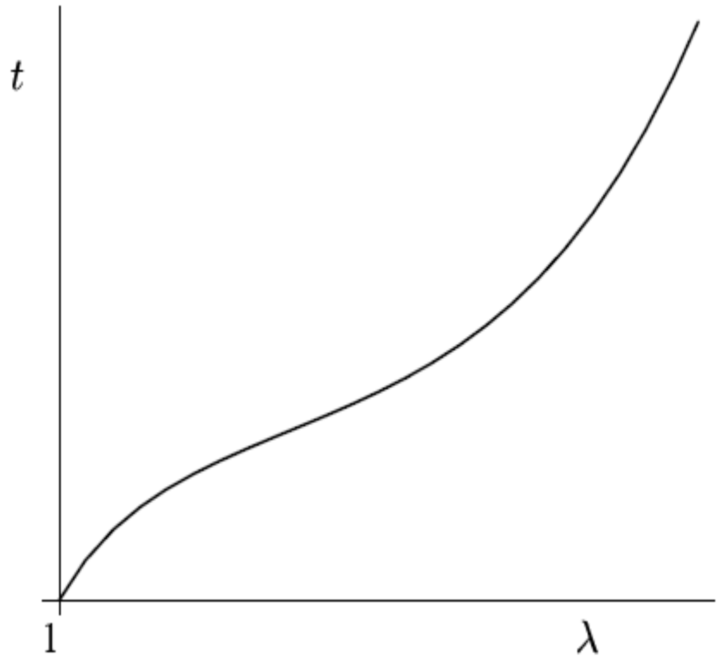
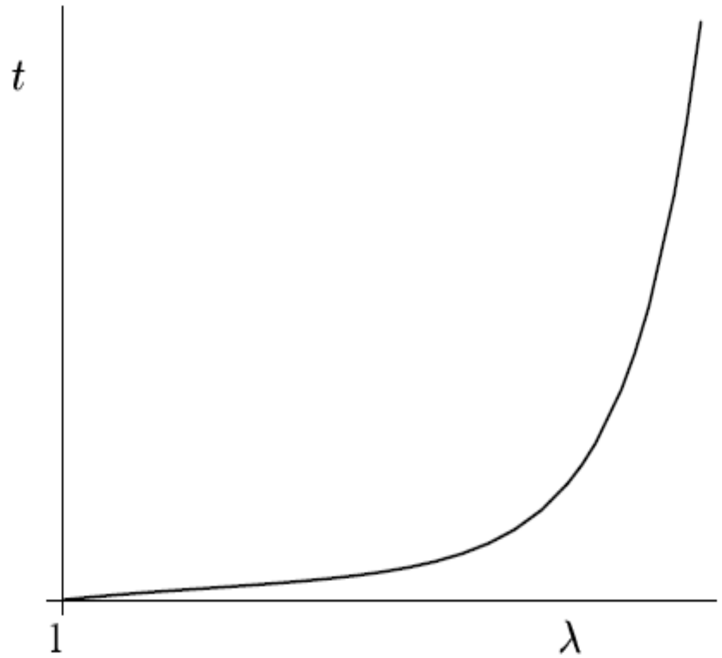
Comparison of responses of rubber and soft tissue

Simple tension (tension vs stretch)

Rubber



Soft tissue



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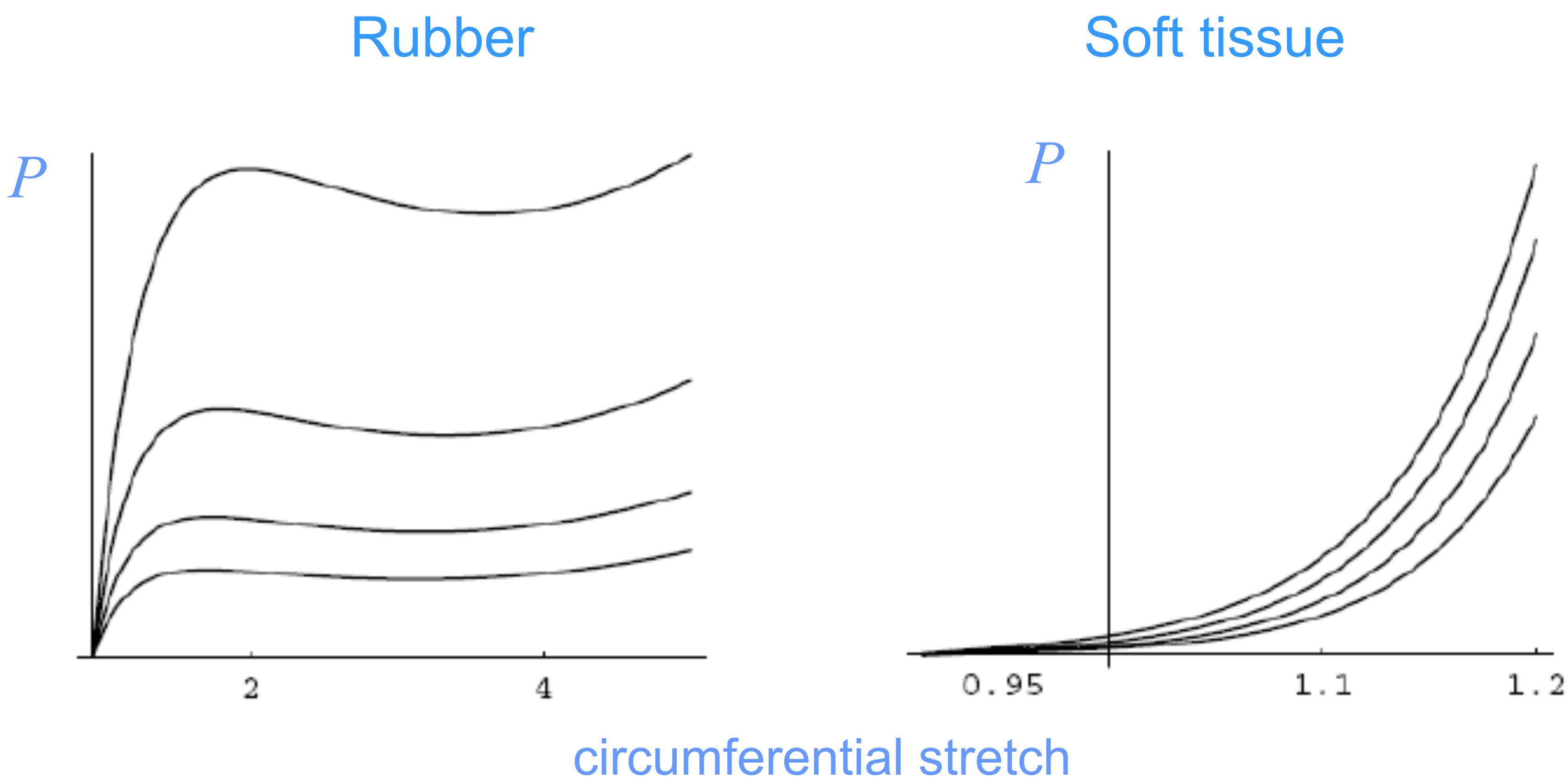
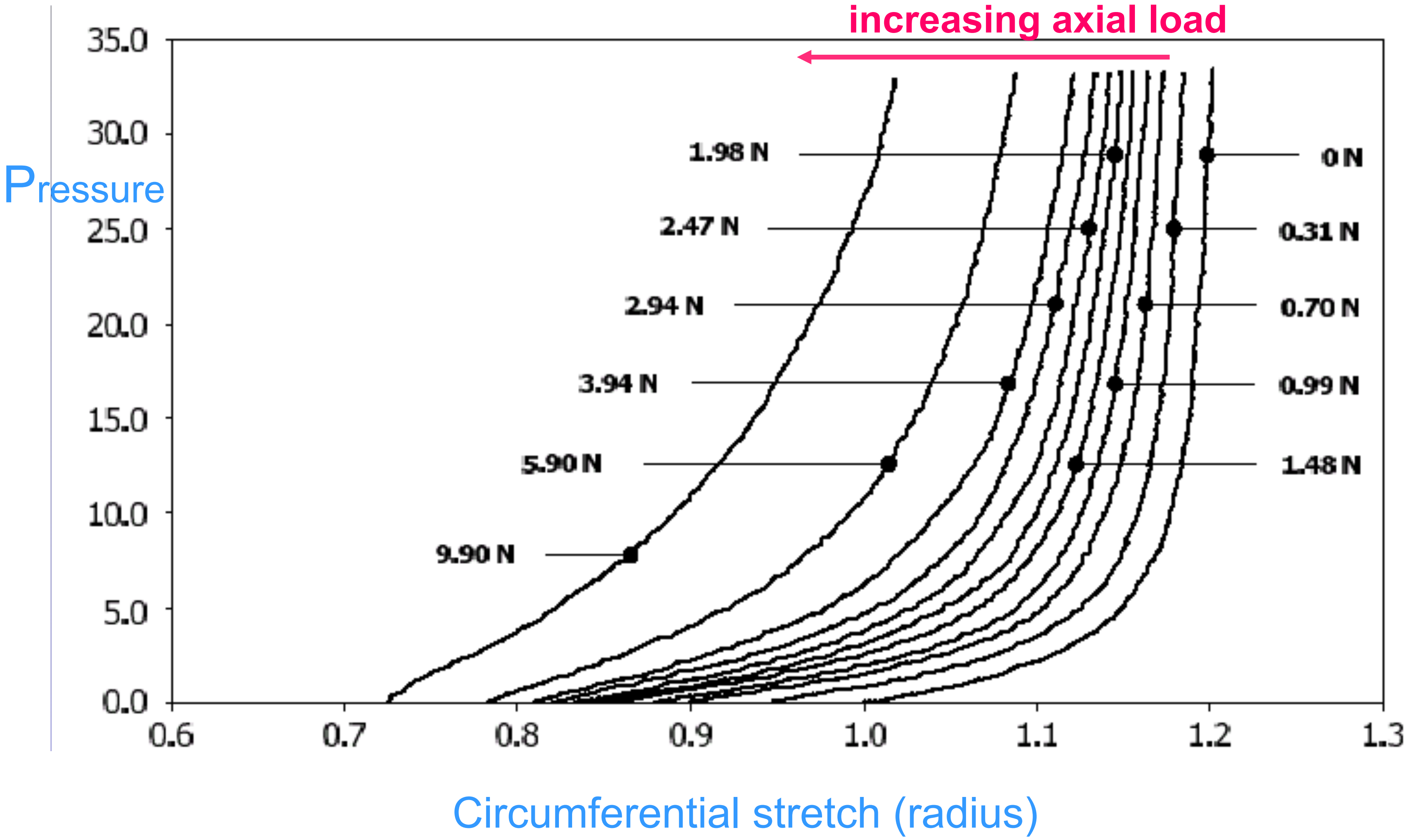
Extension-inflation of a (thin-walled) tube

Pressure vs circumferential stretch



axial pre-stretch 1.2

Typical data for a short arterial length



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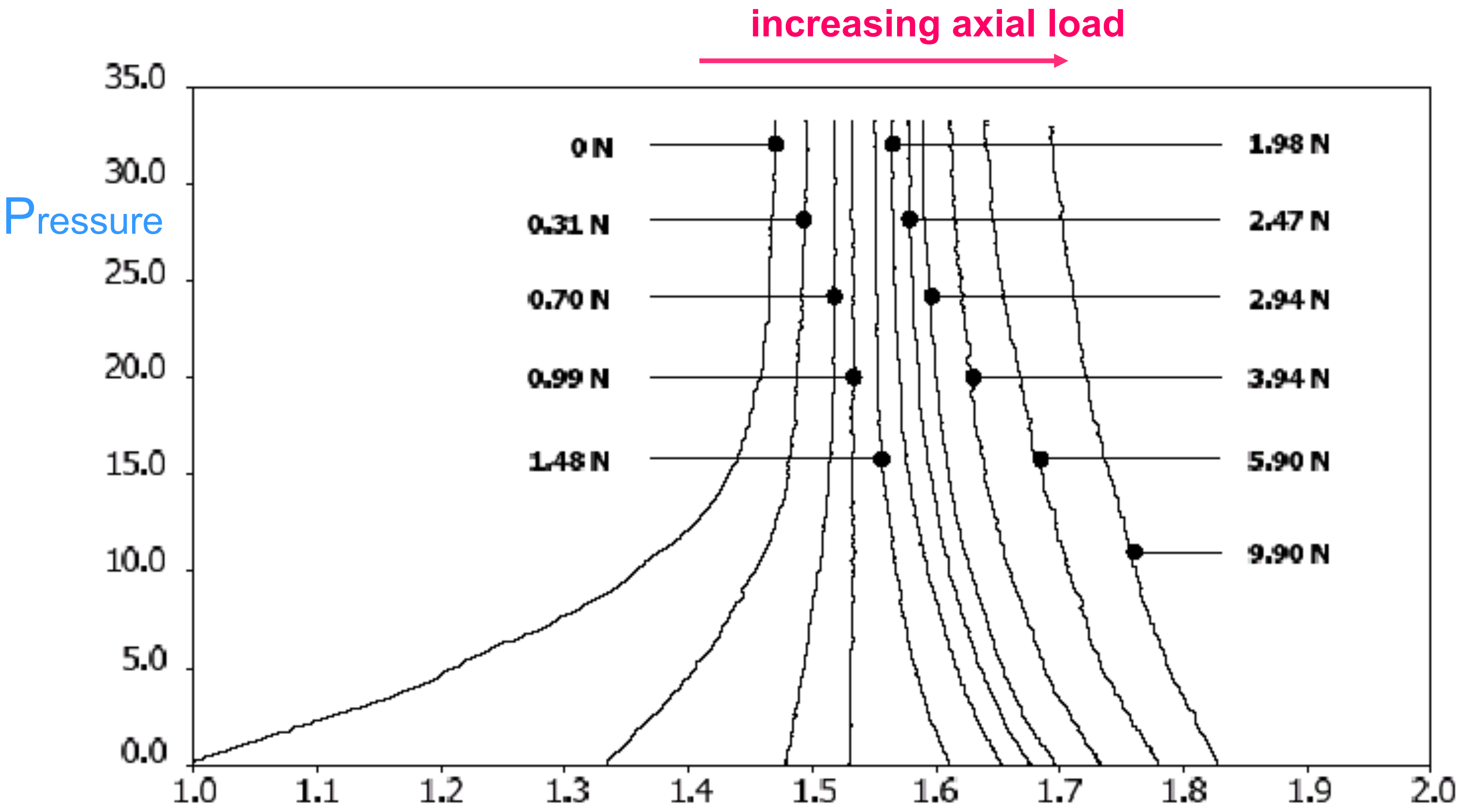

Pressure vs axial stretch (length)

For arteries – two families of fibres – unit vector fields $\mathbf{M}$ $\mathbf{M}'$

Invariants $I_1 = \operatorname{tr}\mathbf{C} \quad I_2 = \operatorname{tr}(\mathbf{C}^{-1})$

$$I_4 = \mathbf{M}\cdot(\mathbf{C}\mathbf{M}) \quad I_5 = \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M})$$

$$I_6 = \mathbf{M}'\cdot(\mathbf{C}\mathbf{M}') \quad I_7 = \mathbf{M}'\cdot(\mathbf{C}^2\mathbf{M}') \quad I_8 = \mathbf{M}\cdot(\mathbf{C}\mathbf{M}')(\mathbf{M}\cdot\mathbf{M}')$$

Cauchy stress

$$\begin{aligned}\boldsymbol{\sigma} = &-p\,\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2)\\ &+ 2W_4\mathbf{m}\otimes\mathbf{m} + 2W_5(\mathbf{m}\otimes\mathbf{B}\mathbf{m} + \mathbf{B}\mathbf{m}\otimes\mathbf{m})\\ &+ 2W_6\mathbf{m}'\otimes\mathbf{m}' + 2W_7(\mathbf{m}'\otimes\mathbf{B}\mathbf{m}' + \mathbf{B}\mathbf{m}'\otimes\mathbf{m}')\\ &+ W_8(\mathbf{m}\otimes\mathbf{m}' + \mathbf{m}'\otimes\mathbf{m})\end{aligned}$$

$$\mathbf{m} = \mathbf{F}\mathbf{M} \quad \mathbf{m}' = \mathbf{F}\mathbf{M}'$$

Stress components

$$\sigma_{11} = -p + 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^4) + 2(W_4 + W_6 + W_8)\lambda_1^2\cos^2\varphi + 4(W_5 + W_7)\lambda_1^4\cos^2\varphi$$

$$\sigma_{22} = -p + 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_2^4) + 2(W_4 + W_6 - W_8)\lambda_2^2\sin^2\varphi + 4(W_5 + W_7)\lambda_2^4\sin^2\varphi$$

$$\sigma_{12} = 2[W_4 - W_6 + (W_5 - W_7)(\lambda_1^2 + \lambda_2^2)]\lambda_1\lambda_2\sin\varphi\cos\varphi$$

$$\sigma_{33} = -p + 2W_1\lambda_3^2 + 2W_2(I_1\lambda_3^2 - \lambda_3^4) \qquad \sigma_{13} = \sigma_{23} = 0$$

Fibre families mechanically equivalent

→ $W_4 = W_6 \qquad W_5 = W_7$

$\sigma_{12} = 0$ **no shear stress**

$$\sigma_{11} = \sigma_1 \quad \sigma_{22} = \sigma_2 \quad \sigma_{33} = \sigma_3$$ – principal stresses

$$W \longrightarrow \hat{W}(\lambda_1, \lambda_2, \varphi)$$

not symmetric in general

$$\sigma_{11} - \sigma_{33} = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1} \qquad \sigma_{22} - \sigma_{33} = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}$$ as in isotropy

These equations are applicable to the extension and inflation of a tube (artery)

$$1 \rightarrow \theta \quad 2 \rightarrow z \quad 3 \rightarrow r$$ cylindrical polars

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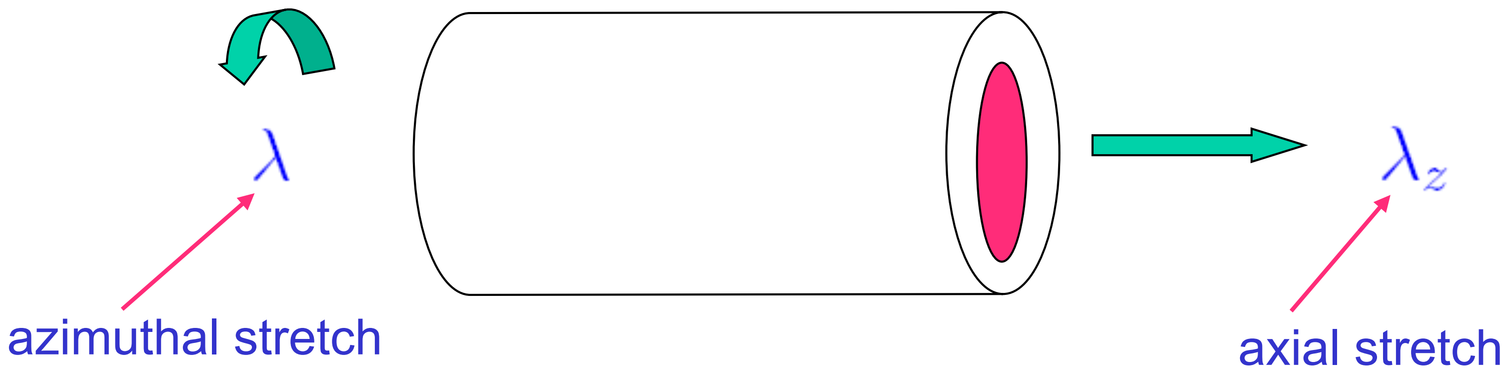
Extension-inflation of a tube

Reference geometry

$$A \le R \le B \quad 0 \le \Theta \le 2\pi \quad 0 \le Z \le L$$

$$(R, \Theta, Z) \quad \rightarrow \quad (r, \theta, z)$$

cylindrical polars



Deformation

$$r^2 - a^2 = \lambda_z^{-1}(R^2 - A^2) \quad \theta = \Theta \quad z = \lambda_z Z$$

Principal stretches

$$\lambda_1 = \frac{r}{R} = \lambda \quad \lambda_2 = \lambda_z \quad \lambda_3 = \lambda^{-1}\lambda_z^{-1}$$

azimuthal axial radial

Strain energy $\hat{W}(\lambda, \lambda_z, \varphi)$

Forms of $\hat{W}$ and φ may be different for different layers

$$\sigma_{\theta\theta} - \sigma_{rr} = \lambda \frac{\partial \hat{W}}{\partial \lambda} \qquad \sigma_{zz} - \sigma_{rr} = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z}$$

Equilibrium

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

Pressure

$$P = \int_a^b \lambda \hat{W}_\lambda \frac{\mathrm{d}r}{r}$$

Axial load

$$N = 2\pi \int_a^b \sigma_{zz} r \mathrm{d}r$$

Illustrative strain energy (Holzapfel, Gasser, Ogden, J. Elasticity, 2000)

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2)$$
$$+ 2W_4\mathbf{m}\otimes\mathbf{m} + 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m})$$
$$+ 2W_6\mathbf{m}'\otimes\mathbf{m}' + 2W_7(\mathbf{m}'\otimes\mathbf{Bm}' + \mathbf{Bm}'\otimes\mathbf{m}')$$
$$+ W_8(\mathbf{m}\otimes\mathbf{m}' + \mathbf{m}'\otimes\mathbf{m})$$

Pressure

$$P = \int_a^b \lambda \hat{W}_\lambda \frac{\mathrm{d}r}{r}$$

Axial load

$$N = 2\pi \int_a^b \sigma_{zz} r \mathrm{d}r$$

Illustrative strain energy (Holzapfel, Gasser, Ogden, J. Elasticity, 2000)

$$\boldsymbol{\sigma} = -p\,\mathbf{I} + 2W_1\mathbf{B}$$
$$+ 2W_4\mathbf{m} \otimes \mathbf{m}$$
$$+ 2W_6\mathbf{m}' \otimes \mathbf{m}'$$

Specifically

$$W = W_{\text{iso}} + W_{\text{aniso}}$$

matrix fibres

with

$$W_{\text{iso}} = \frac{1}{2}\mu_1(I_1 - 3)$$

neo-Hookean

$$W_{\text{aniso}} = \frac{\mu_2}{2\mu_3}\left\{\exp\left[\mu_3(I_4 - 1)^2\right] + \exp\left[\mu_3(I_6 - 1)^2\right] - 2\right\}$$

$$I_4 = \mathbf{M} \cdot (\mathbf{CM}) \quad I_6 = \mathbf{M}' \cdot (\mathbf{CM}')$$

Material constants (positive) $\mu_1,\ \mu_2,\ \mu_3$

W_{aniso} only active if $I_4 > 1$ or $I_6 > 1$

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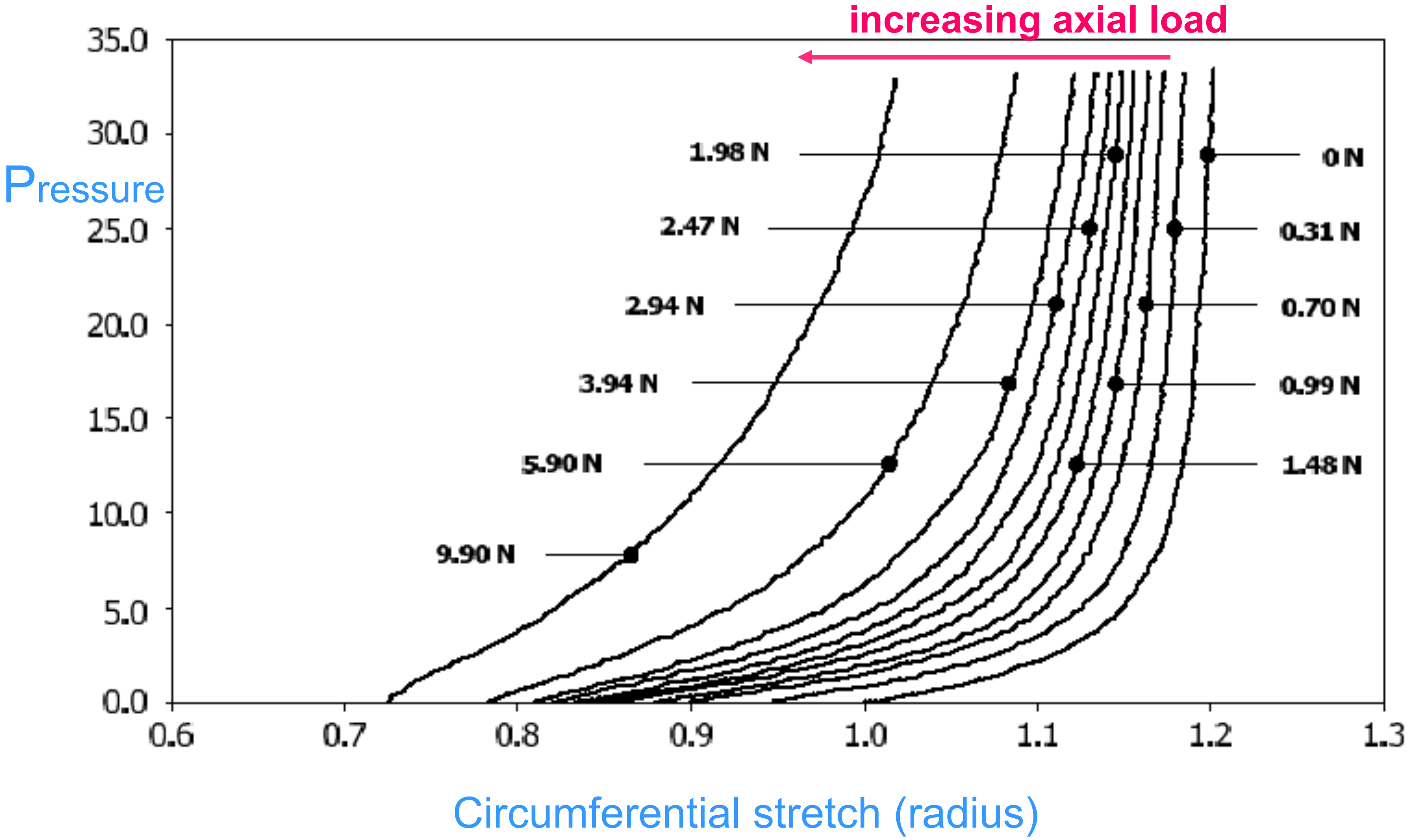
Fits the data well for the overall response of an intact arterial segment

Typical data for a short arterial length



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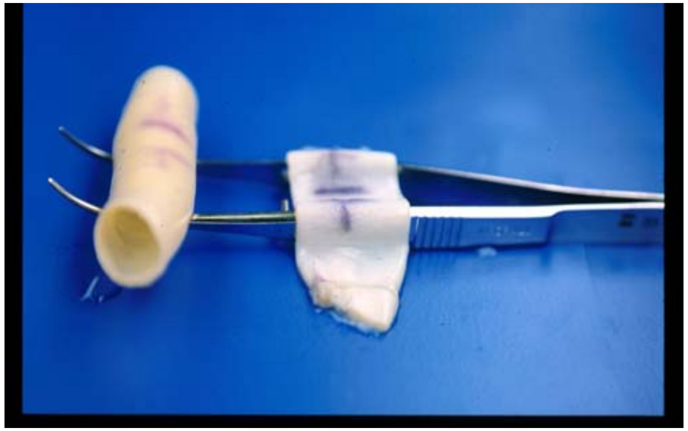
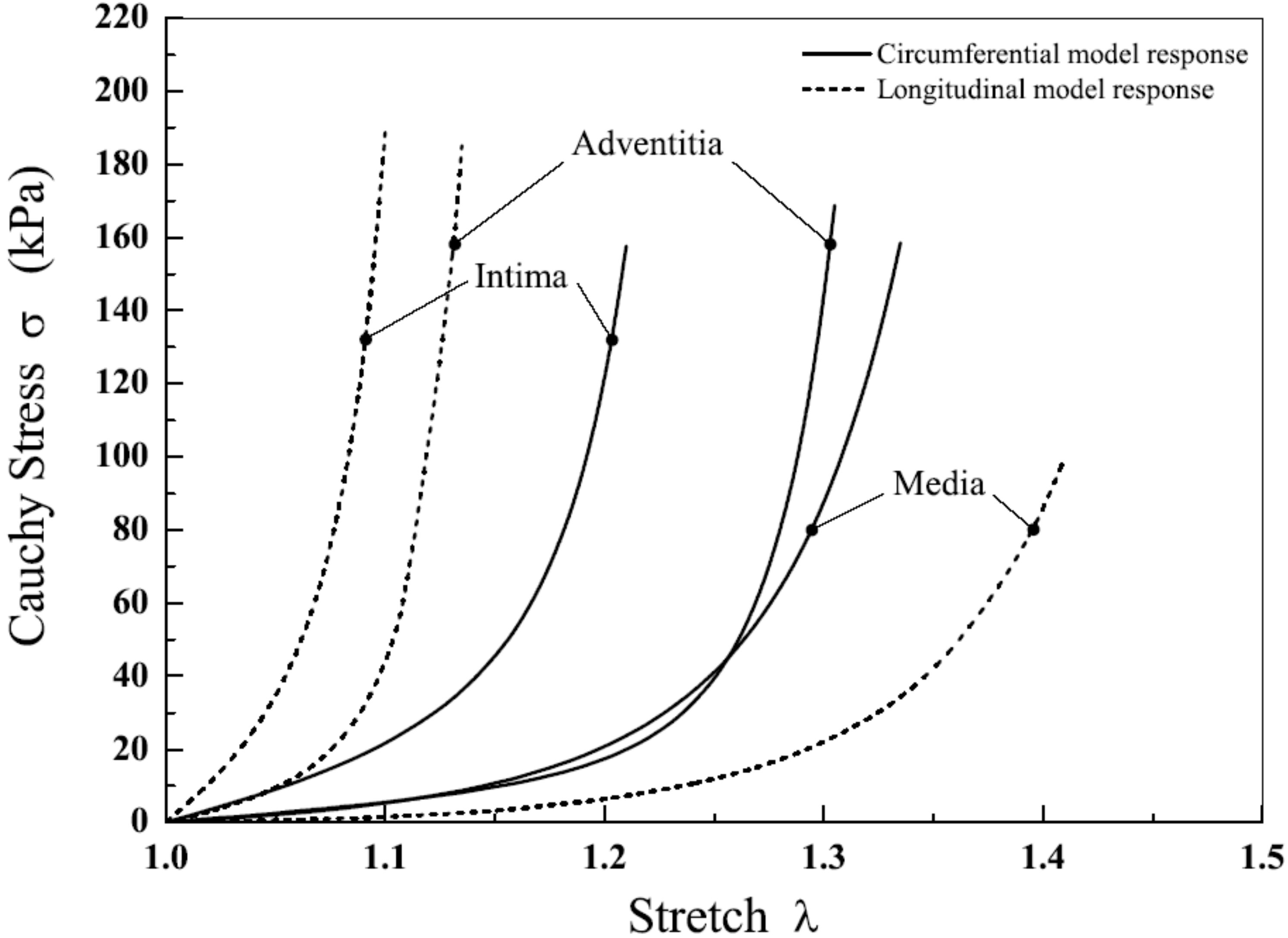
However!

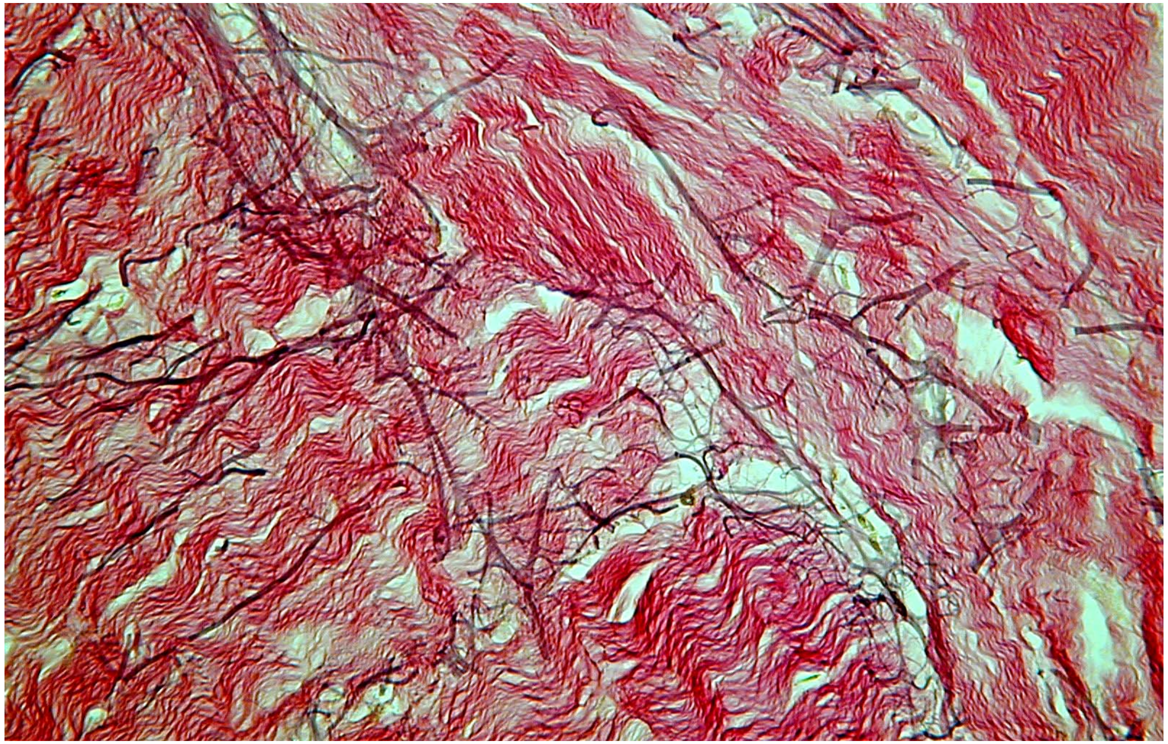
The behaviours of the separate layers
are very different

Stiffness of Media and Adventitia Compared



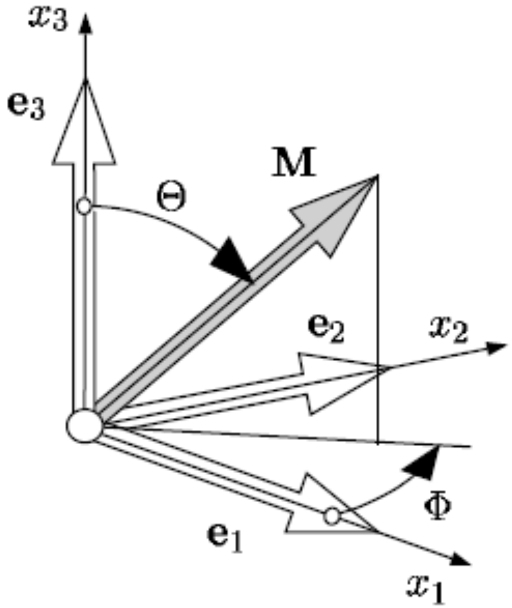
From Holzapfel, Sommer, Regitnig 2004 – mean data for aged coronary artery layers





Collagen fibres in an iliac artery (adventitia)

Description of distributed fibre orientations



$$\mathbf{M} = \sin\Theta\cos\Phi\,\mathbf{e}_1 + \sin\Theta\sin\Phi\,\mathbf{e}_2 + \cos\Theta\,\mathbf{e}_3$$

Orientation density distribution $\rho(\mathbf{M})$ $\quad \rho(-\mathbf{M}) = \rho(\mathbf{M})$

Normalized $$\frac{1}{4\pi}\int_\omega \rho(\mathbf{M})\mathrm{d}\omega = 1$$

Generalized structure tensor

$$\mathbf{H} = \frac{1}{4\pi}\int_{\omega} \rho(\mathbf{M})\mathbf{M} \otimes \mathbf{M} \mathrm{d}\omega \quad \longrightarrow \quad \alpha_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Transversely isotropic distribution $\rho(\mathbf{M}) \longrightarrow \rho(\Theta)$

$$\boxed{\mathbf{H} = \kappa\mathbf{I} + (1 - 3\kappa)\mathbf{e}_3 \otimes \mathbf{e}_3}$$

$$\kappa = \frac{1}{4}\int_0^{\pi} \rho(\Theta)\sin^3\Theta \mathrm{d}\Theta$$

mean fibre direction

Parameter calculated from given $\rho(\Theta)$

or treated as a phenomenological parameter

$\kappa = 1/3 \longrightarrow$ isotropy

$\kappa = 0 \longrightarrow$ transverse isotropy
no fibre dispersion

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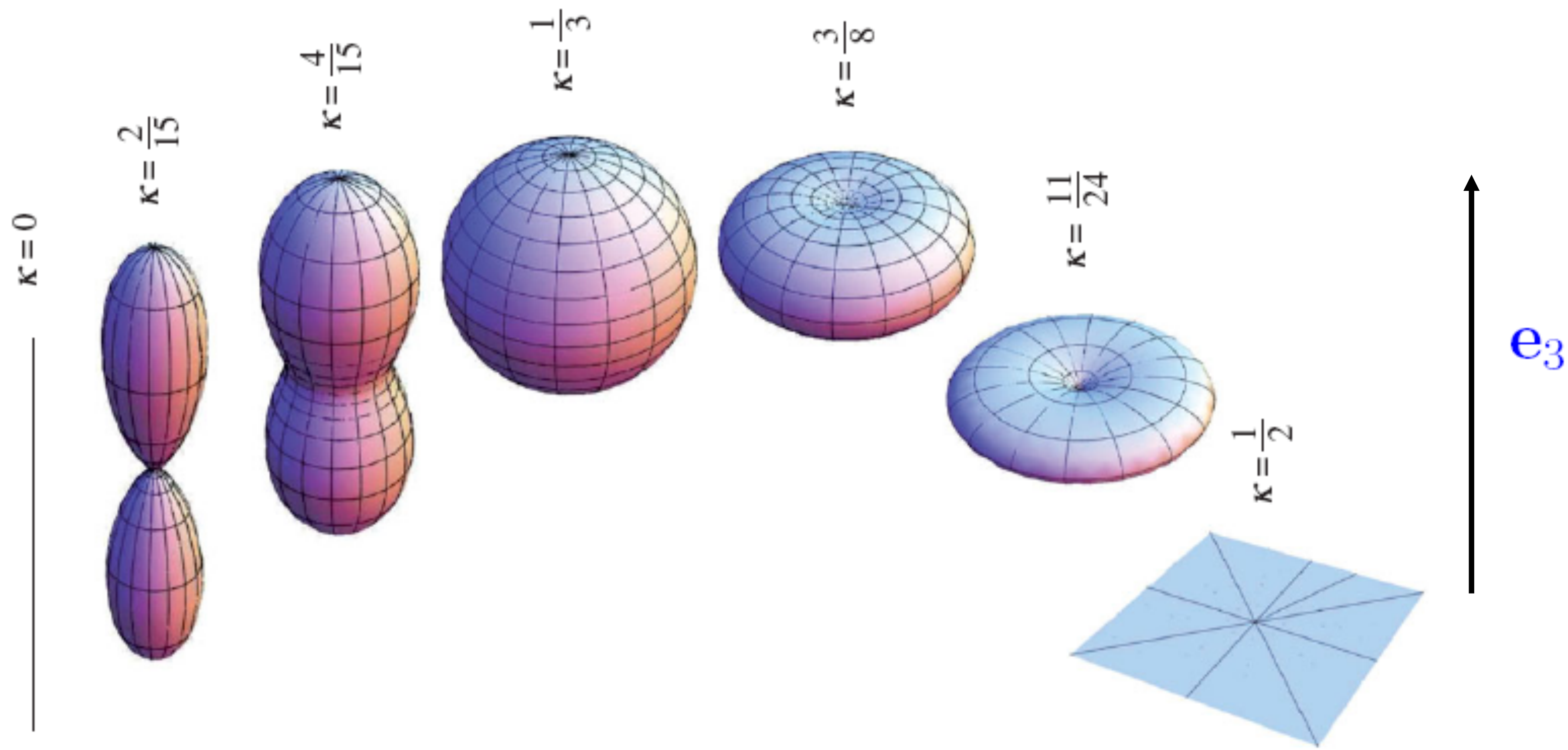
Fibre orientation distribution – illustration



Plot of $\rho(\mathbf{M})\mathbf{M}$

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Deformation invariant based on

now the mean direction

$$\mathbf{H} = \kappa\,\mathbf{I} + (1 - 3\kappa)\mathbf{M} \otimes \mathbf{M}$$

$$K \equiv \mathrm{tr}(\mathbf{HC}) = \kappa\,\underbrace{\mathrm{tr}\,\mathbf{C}}_{I_1} + (1 - 3\kappa)\underbrace{\mathbf{M} \cdot (\mathbf{CM})}_{I_4}$$

$$K' = \kappa\,\mathrm{tr}\,\mathbf{C} + (1 - 3\kappa)\mathbf{M}' \cdot (\mathbf{CM}')$$

$I_4 \longrightarrow K$ $I_6 \longrightarrow K'$

$$W = \frac{1}{2}\mu_1(I_1 - 3) + \frac{\mu_2}{2\mu_3}\{\exp[\mu_3(K - 1)^2] + \exp[\mu_3(K' - 1)^2] - 2\}$$

material constants

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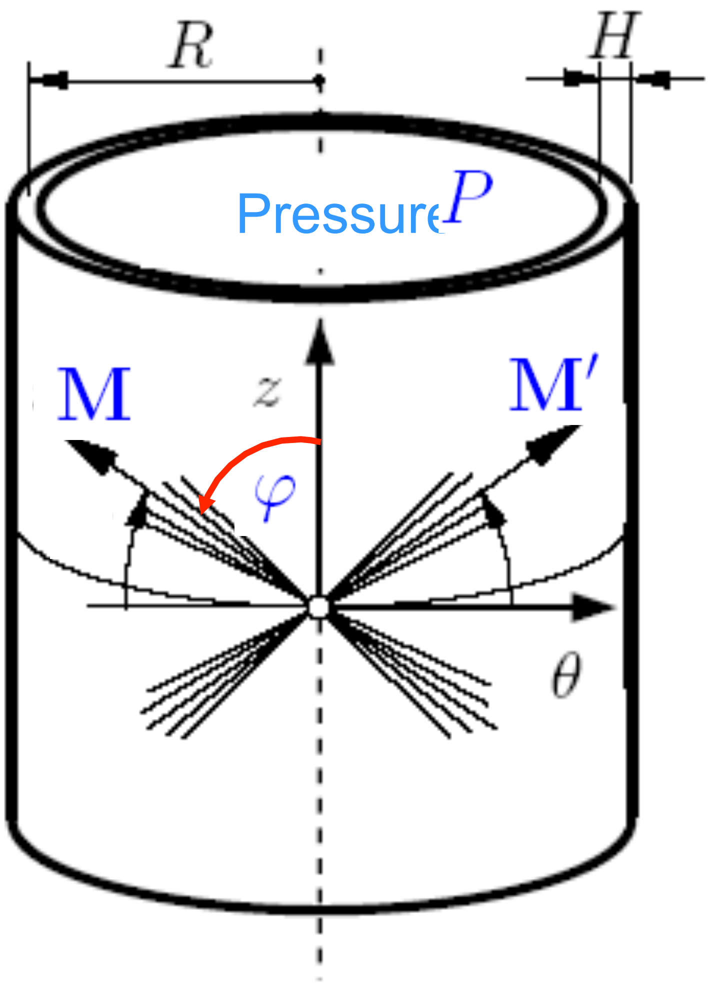
Application to a **thin**-walled tube



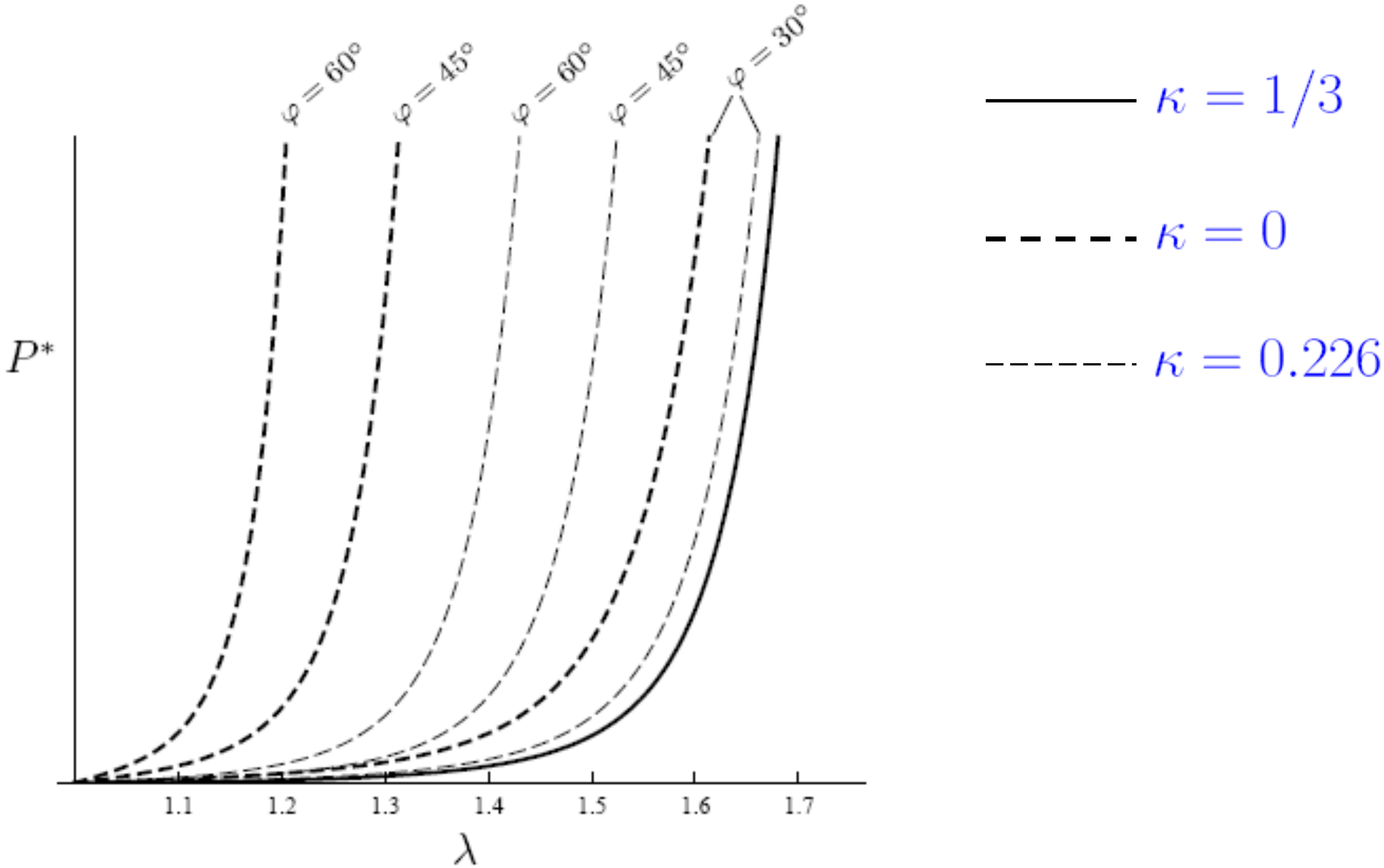
$$W \longrightarrow \hat{W}(\lambda, \lambda_z)$$

$$P^* \equiv \frac{PR}{H} = \lambda^{-1}\lambda_z^{-1}\frac{\partial \hat{W}}{\partial \lambda}$$

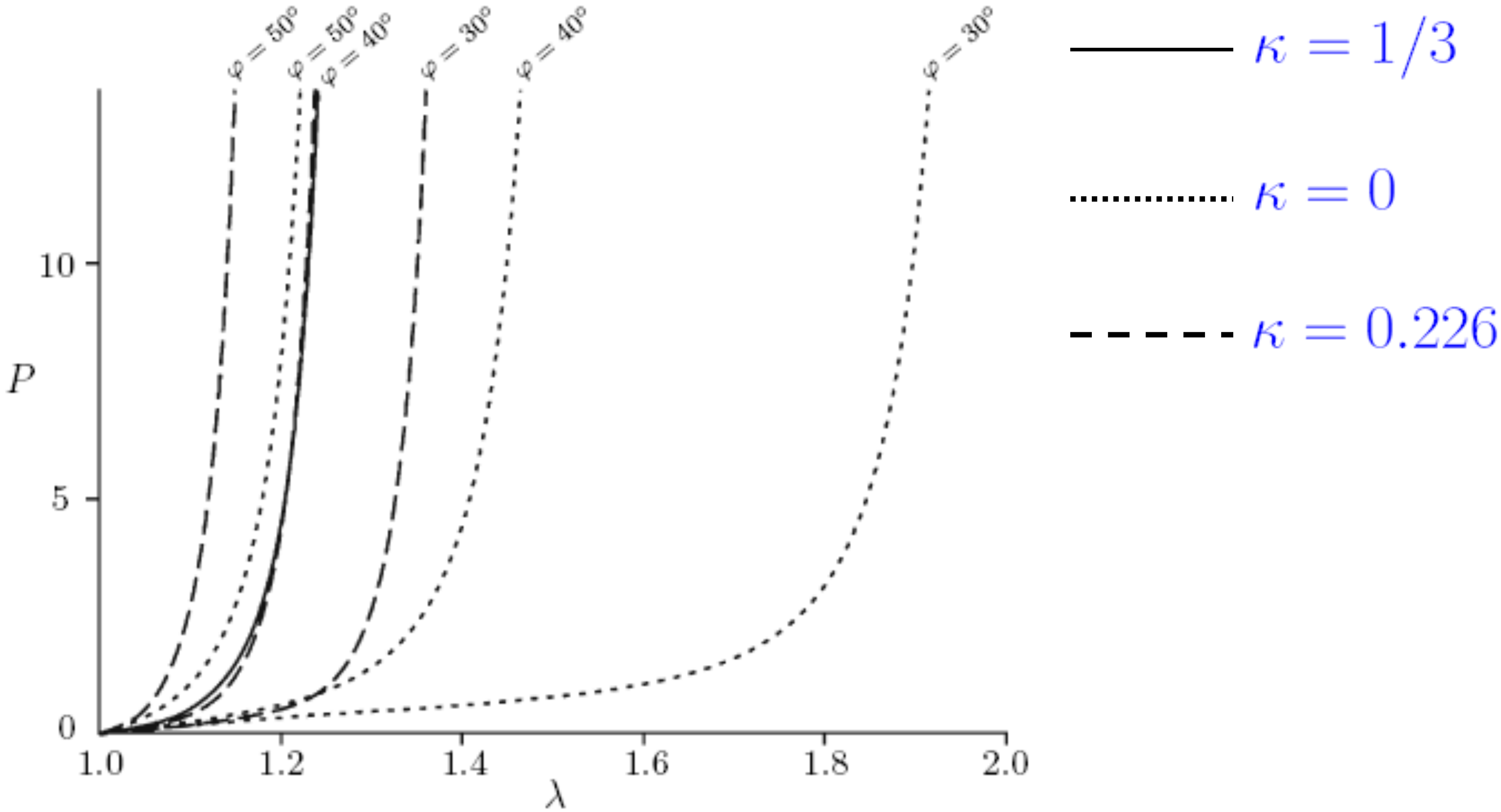
reduced axial load

$$F^* \equiv \frac{F}{2\pi RH} = \frac{\partial \hat{W}}{\partial \lambda_z} - \frac{1}{2}\lambda^2 P^*$$

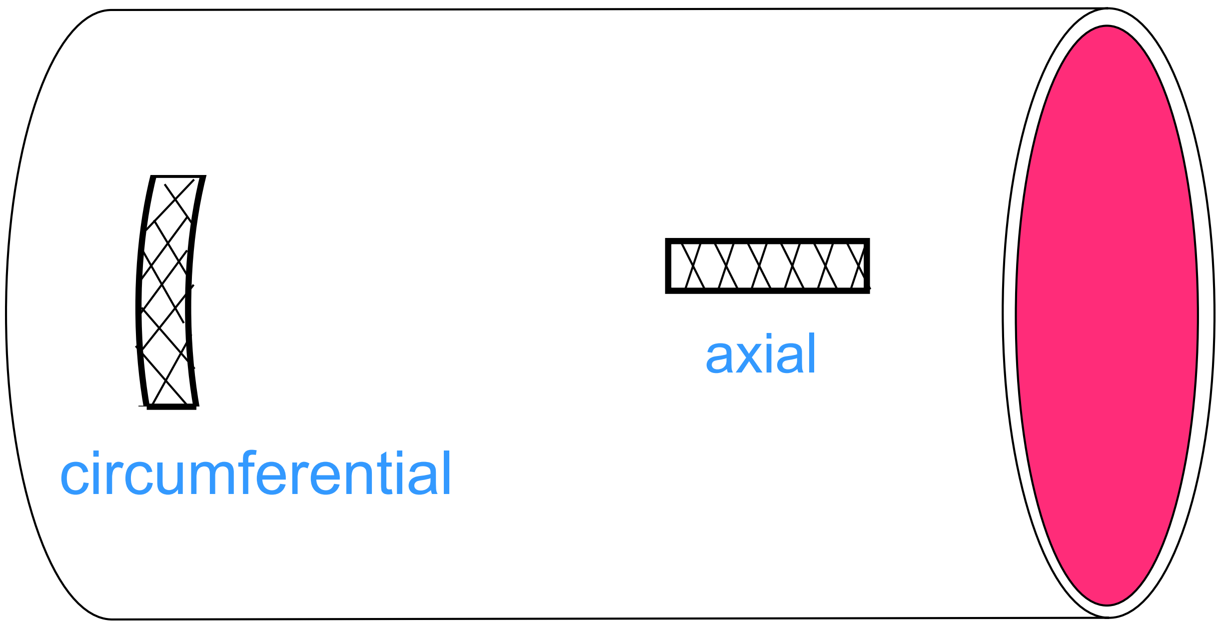
Pressure vs circumferential stretch for a tube with $\lambda_z = 1$



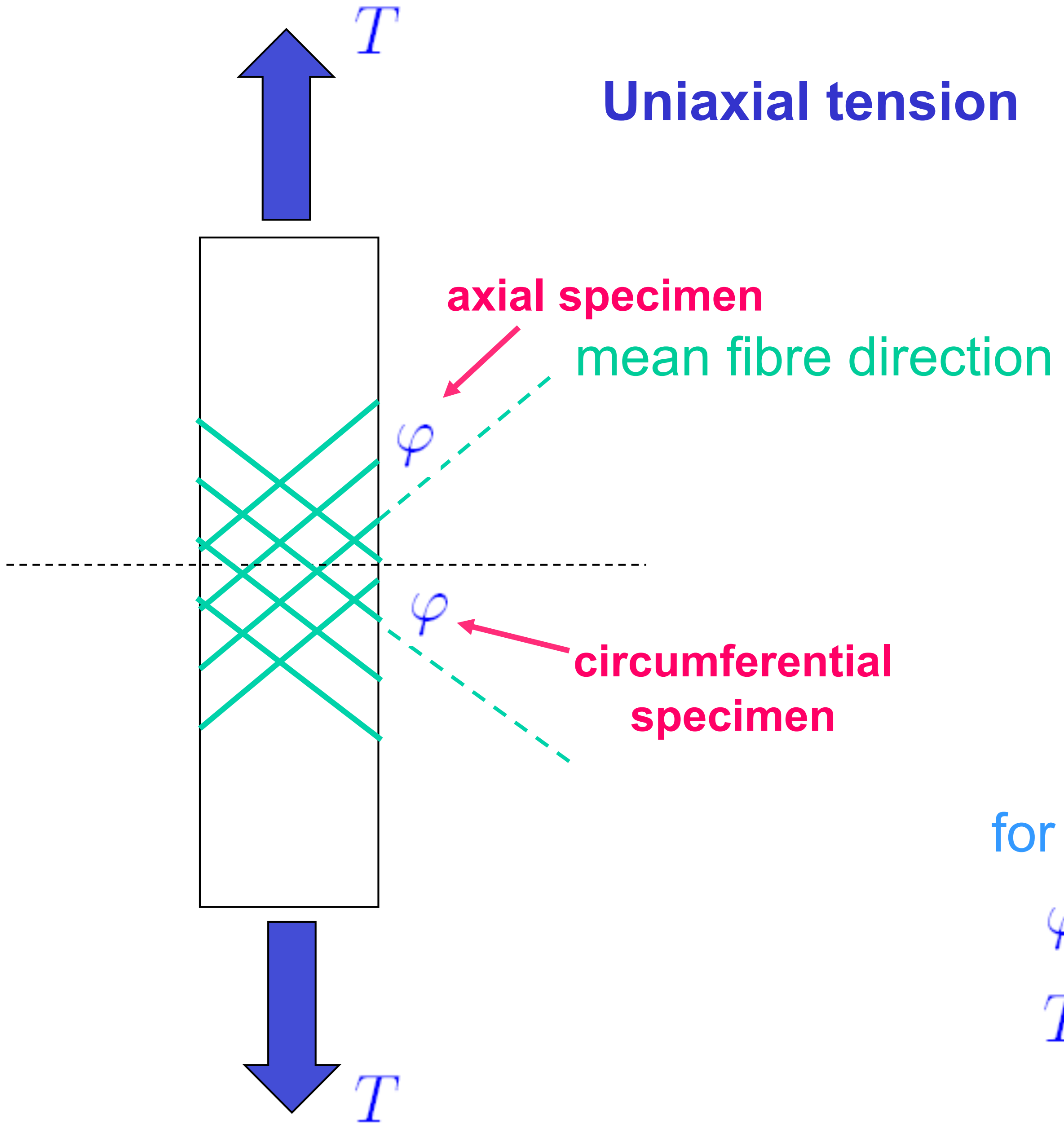
Pressure vs circumferential stretch for a tube with $F = 0$ (P in kPa)



Application to uniaxial tension of axial and circumferential strips



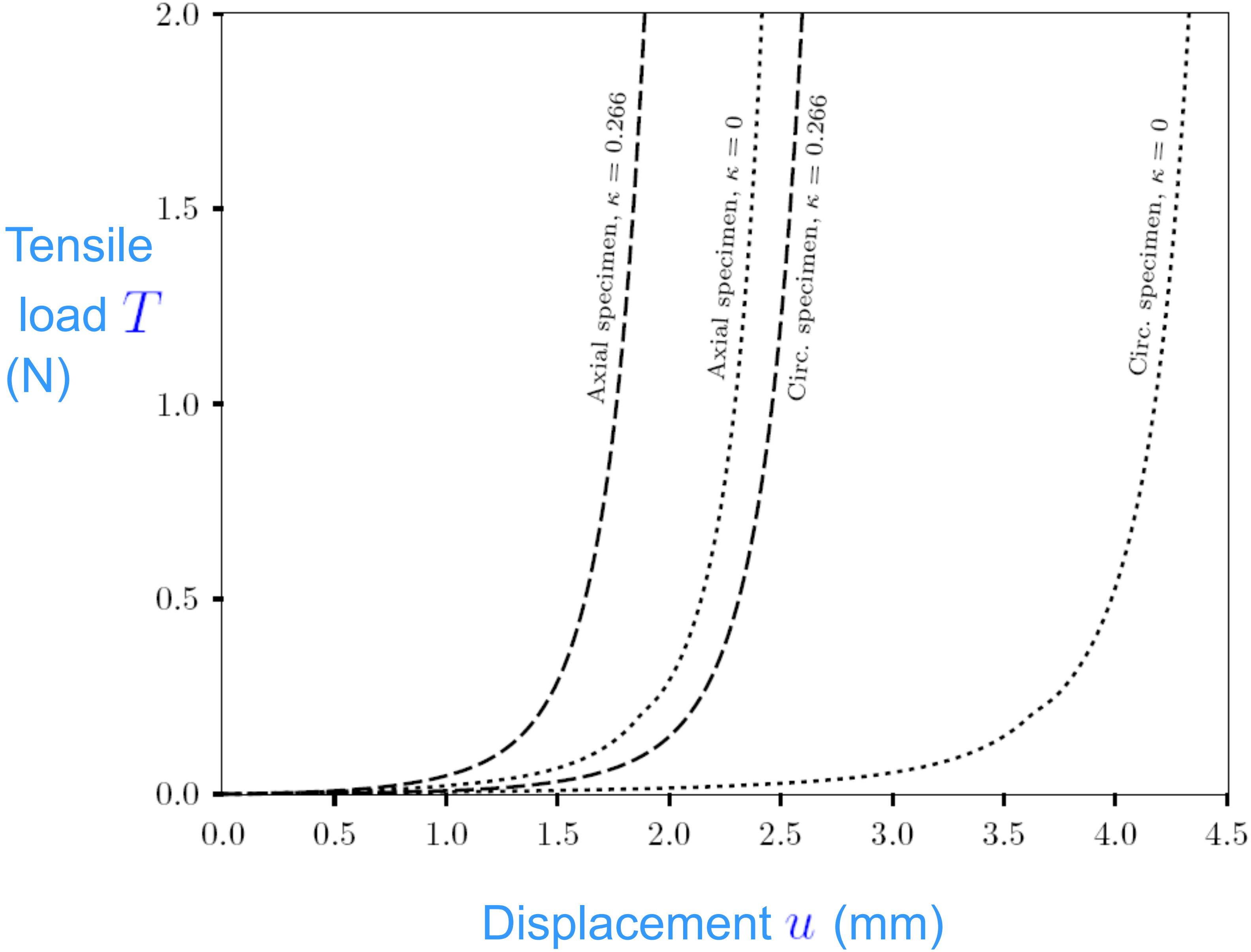
Uniaxial tension



for simulations

$$\varphi = 40^\circ$$

$$T = 1\,\mathrm{N}$$



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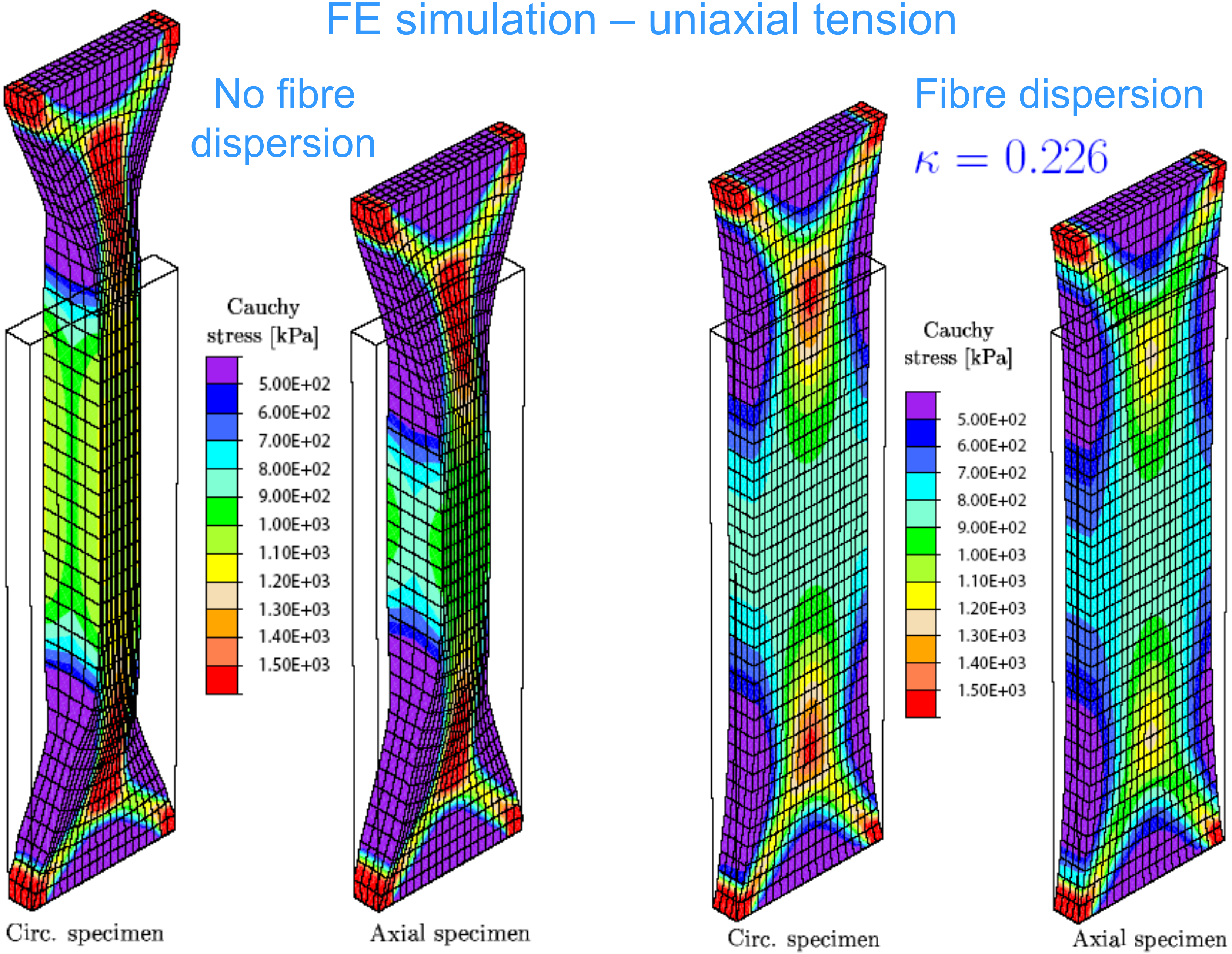
FE simulation – uniaxial tension



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Conclusion

In modelling the mechanics of soft tissue
it is essential to account for the dispersion of
collagen fibre directions

– it has a substantial effect

Reference

Gasser, Ogden, Holzapfel J. R. Soc. Interface (2006)

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Structure and Modelling of the Myocardium

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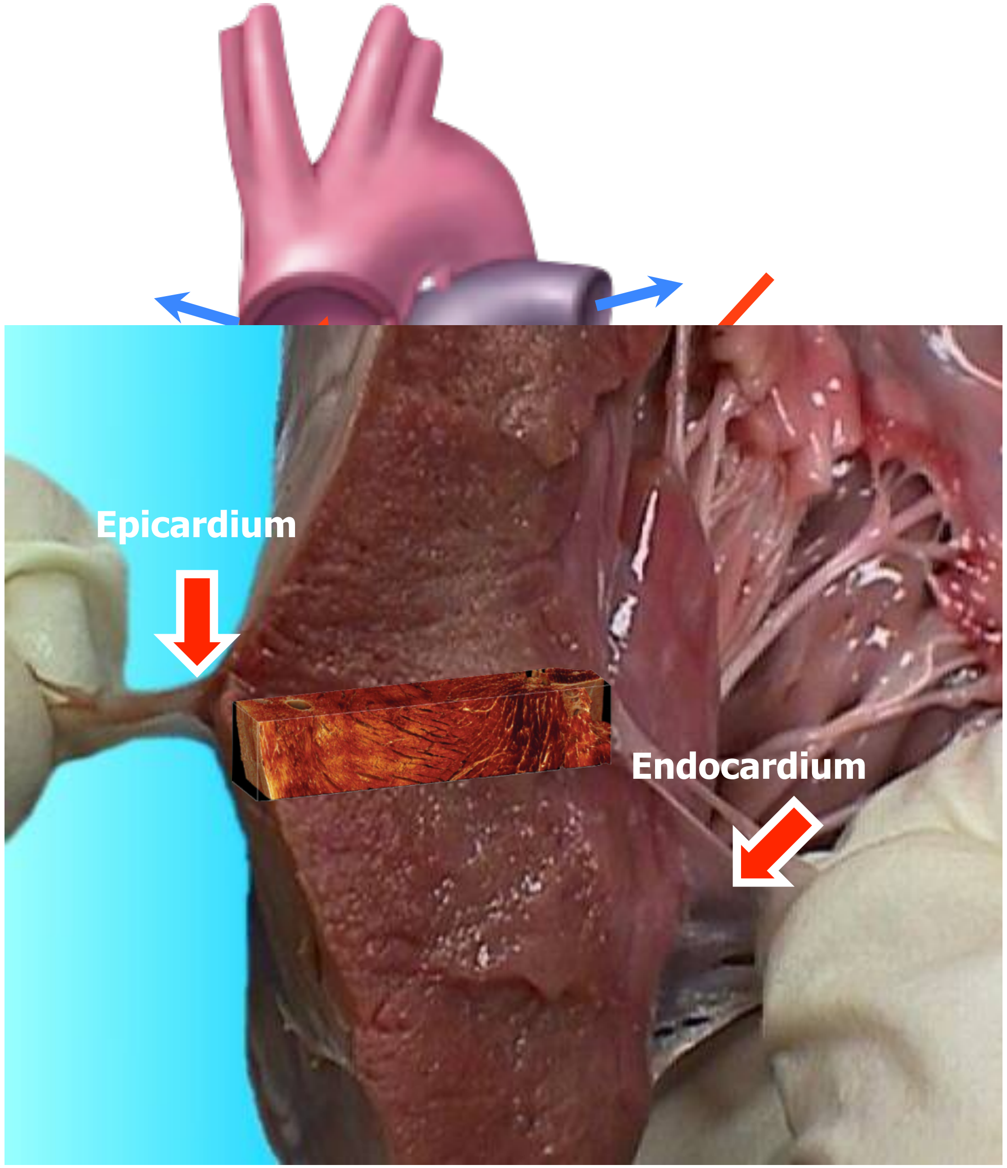
Lectures, Xi'an, April 2011

Anatomy of the Heart



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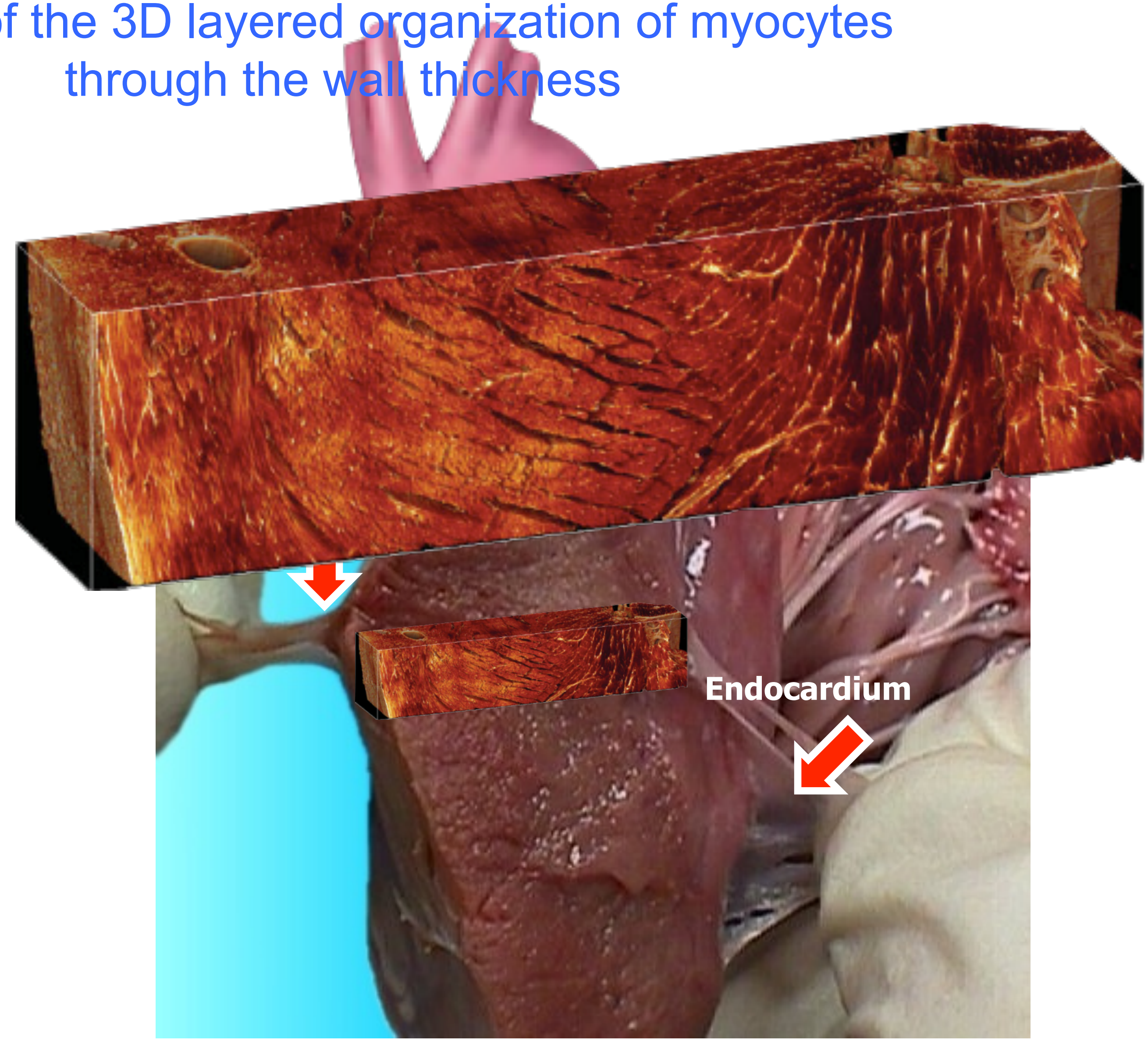
Anatomy of the Heart

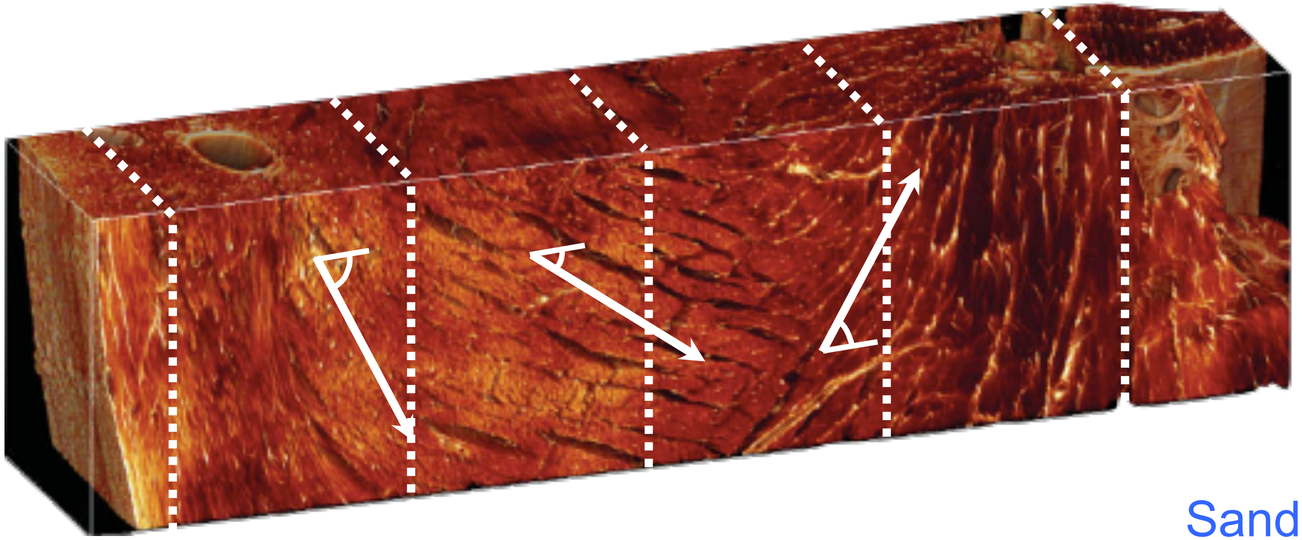
Change of the 3D layered organization of myocytes through the wall thickness



Structure of the Left Ventricle Wall

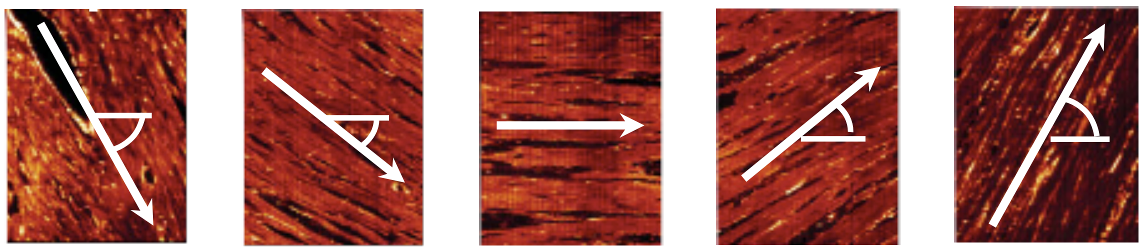
Change of the 3D layered organization of myocytes through the wall thickness

Epicardium (external)



Endocardium (internal)

Sands et. al. (2005)



Locally: three mutually orthogonal directions can be identified forming planes with distinct material responses

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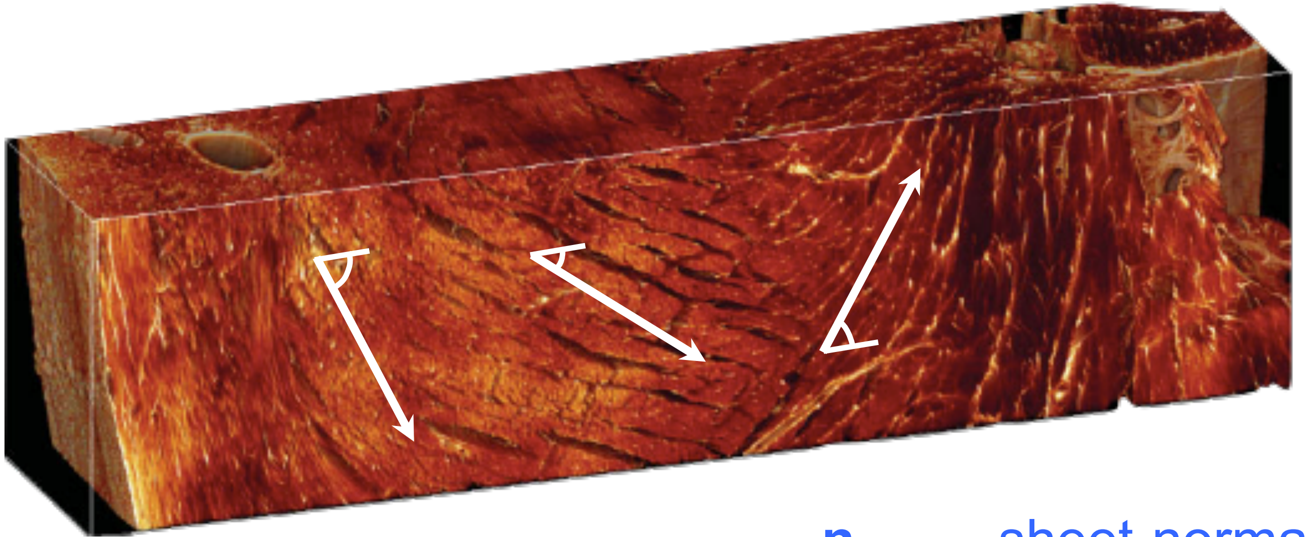
Structure of the Left Ventricle Wall

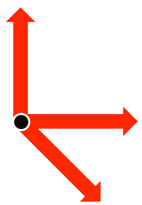
Change of the 3D layered organization of myocytes through the wall thickness

Epicardium (external)

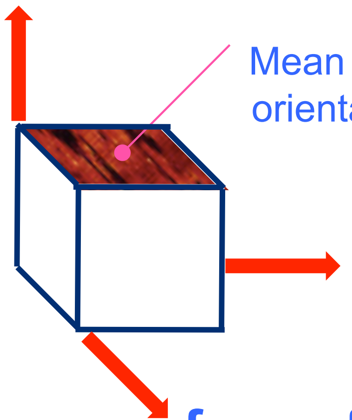


Endocardium (internal)

Basis vectors



$\mathbf{n}_0$ ... sheet-normal



Mean fibre orientation

$\mathbf{s}_0$ ... sheet axis

$\mathbf{f}_0$ ... fibre axis

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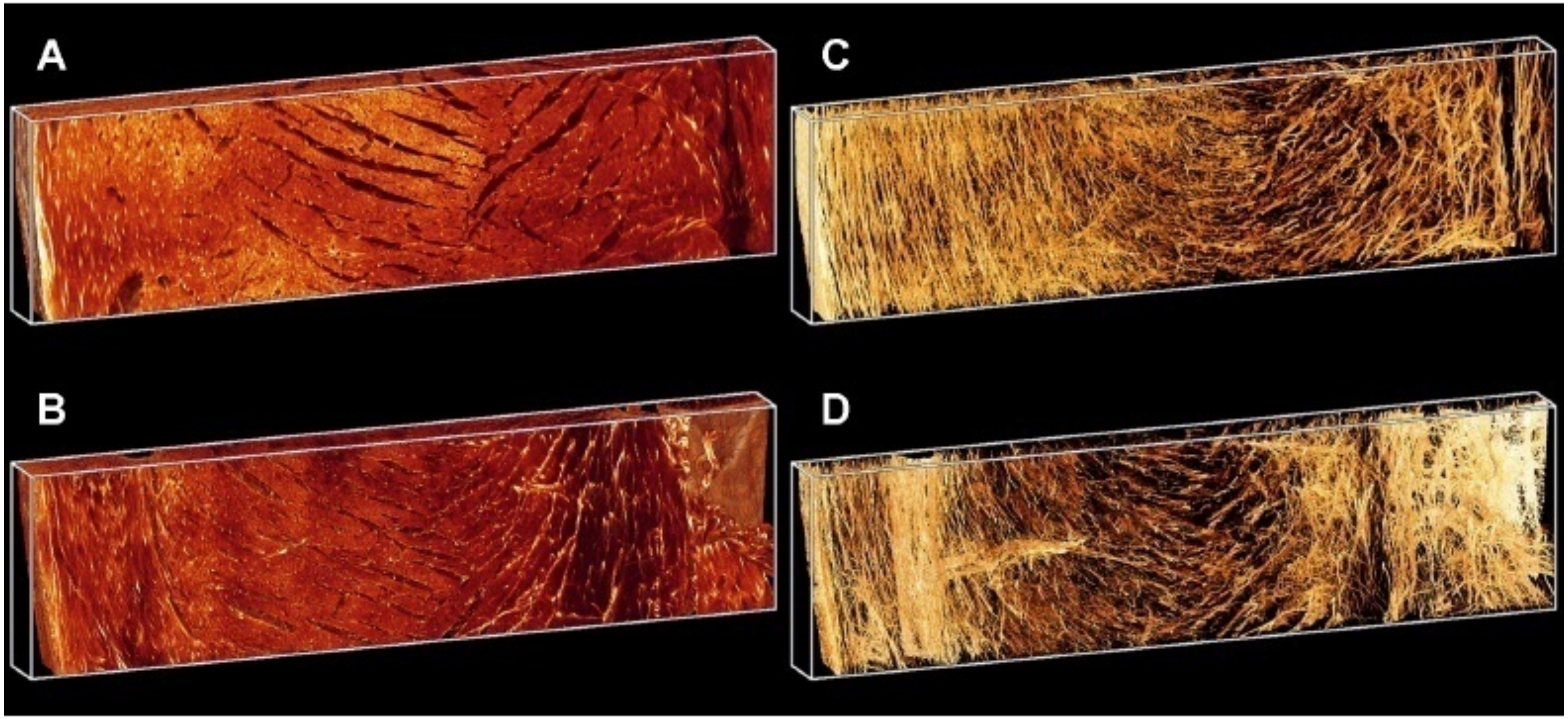
An alternative view
from Pope et al. (2008)



collagen fibres exposed

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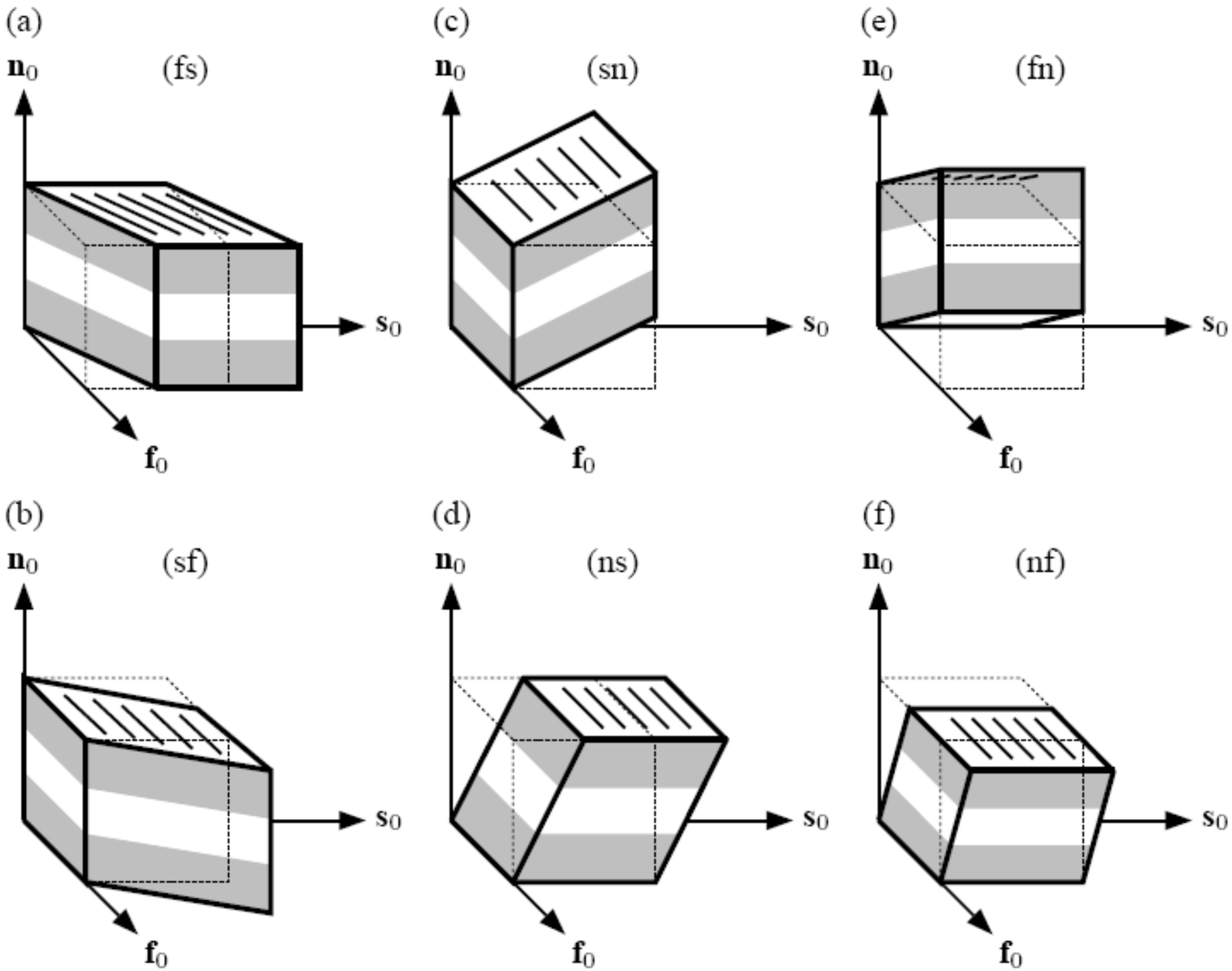
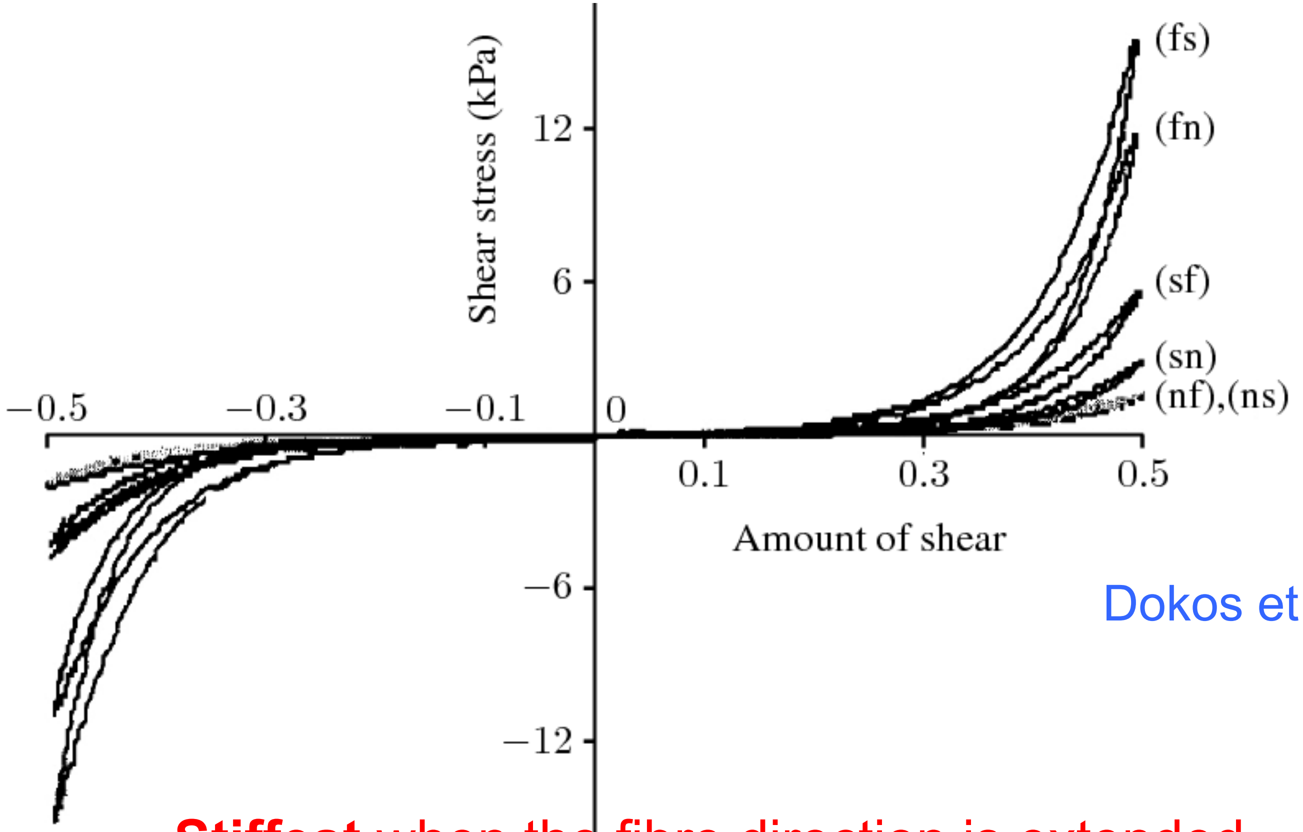
Simple Shear of a Cube



6 modes of simple shear

Mechanics of the Myocardium

Simple shear tests on a cube of a typical myocardial specimen in the fs, fn and sn planes



Dokos et al. (2002)

Stiffest when the fibre direction is extended
Least stiff for normal direction
Intermediate stiffness for sheet direction

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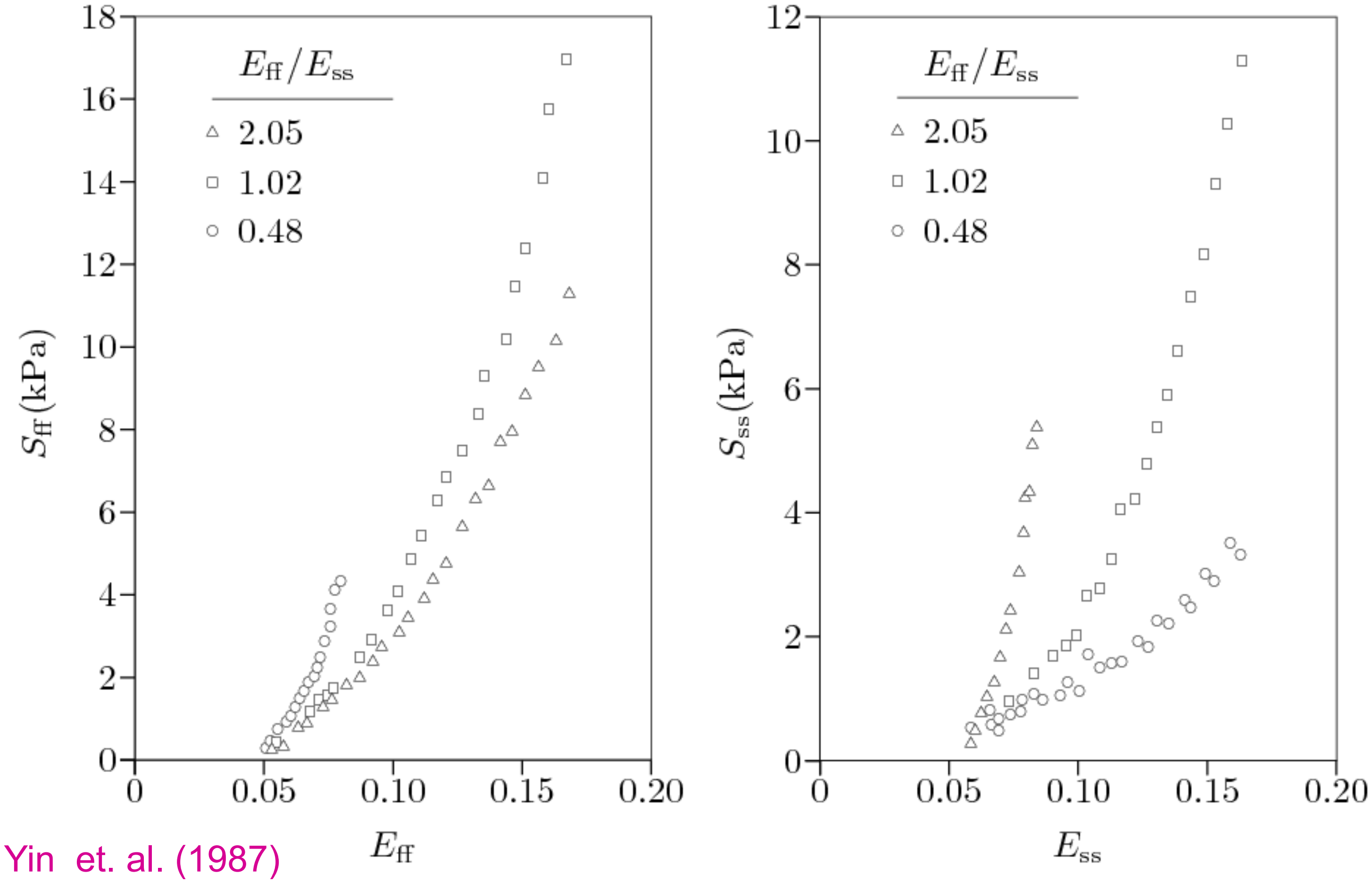
C o n s e q u e n c e

Within the context of (incompressible, nonlinear) elasticity theory, myocardium should be modelled as a non-homogenous, thick-walled, orthotropic, material

Mechanics of the Myocardium

Biaxial loading in the fs plane of canine left ventricle



Yin et. al. (1987)

The only biaxial data available!

Limitations: e.g. no data in the low-strain region (0 – 0.05)

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Structurally Based Model

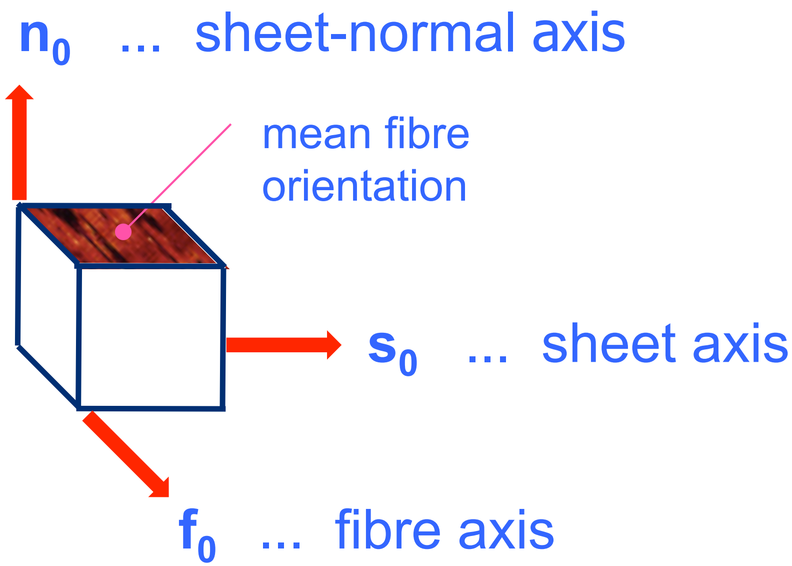
Define

$$I_{4\,\mathrm{f}} = \mathbf{f}_0 \cdot (\mathbf{C}\mathbf{f}_0)$$

$$I_{4\,\mathrm{s}} = \mathbf{s}_0 \cdot (\mathbf{C}\mathbf{s}_0)$$

$$I_{4\,\mathrm{n}} = \mathbf{n}_0 \cdot (\mathbf{C}\mathbf{n}_0)$$

$$\sum_{i=\mathrm{f,s,n}} I_{4\,i} = I_1$$



$$I_{8\,\mathrm{fs}} = I_{8\,\mathrm{sf}} = \mathbf{f}_0 \cdot (\mathbf{C}\mathbf{s}_0)$$

$$I_{8\,\mathrm{fn}} = I_{8\,\mathrm{nf}} = \mathbf{f}_0 \cdot (\mathbf{C}\mathbf{n}_0)$$

$$I_{8\,\mathrm{sn}} = I_{8\,\mathrm{ns}} = \mathbf{s}_0 \cdot (\mathbf{C}\mathbf{n}_0)$$

direction coupling invariants

$I_{5\,\mathrm{f}}, I_{5\,\mathrm{s}}, I_{5\,\mathrm{n}}$ expressible in terms of the other invariants

Structurally Based Model

General framework →

compressible material: 7 independent invariants

$$I_1 \quad I_2 \quad I_{4\,\mathrm{f}} \quad I_{4\,\mathrm{s}} \quad I_{8\,\mathrm{fs}} \quad I_{8\,\mathrm{fn}} \quad I_{8\,\mathrm{ns}}$$

incompressible material: 6 independent invariants

$$I_1 \quad \not{I_2} \quad I_{4\,\mathrm{f}} \quad I_{4\,\mathrm{s}} \quad I_{8\,\mathrm{fs}} \quad I_{8\,\mathrm{fn}}$$

Cauchy stress tensor

isotropic contribution — anisotropic contribution

$$\boldsymbol{\sigma} = \underbrace{2\psi_1\mathbf{B} - p\mathbf{I}}_{\text{isotropic}} + \underbrace{2\psi_{4\,\mathrm{f}}\mathbf{f}\otimes\mathbf{f} + 2\psi_{4\,\mathrm{s}}\mathbf{s}\otimes\mathbf{s} + \psi_{8\,\mathrm{fs}}(\mathbf{f}\otimes\mathbf{s} + \mathbf{s}\otimes\mathbf{f})}_{\text{anisotropic}}$$

$$+\psi_{8\,\mathrm{fn}}(\mathbf{f}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{f})$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$$

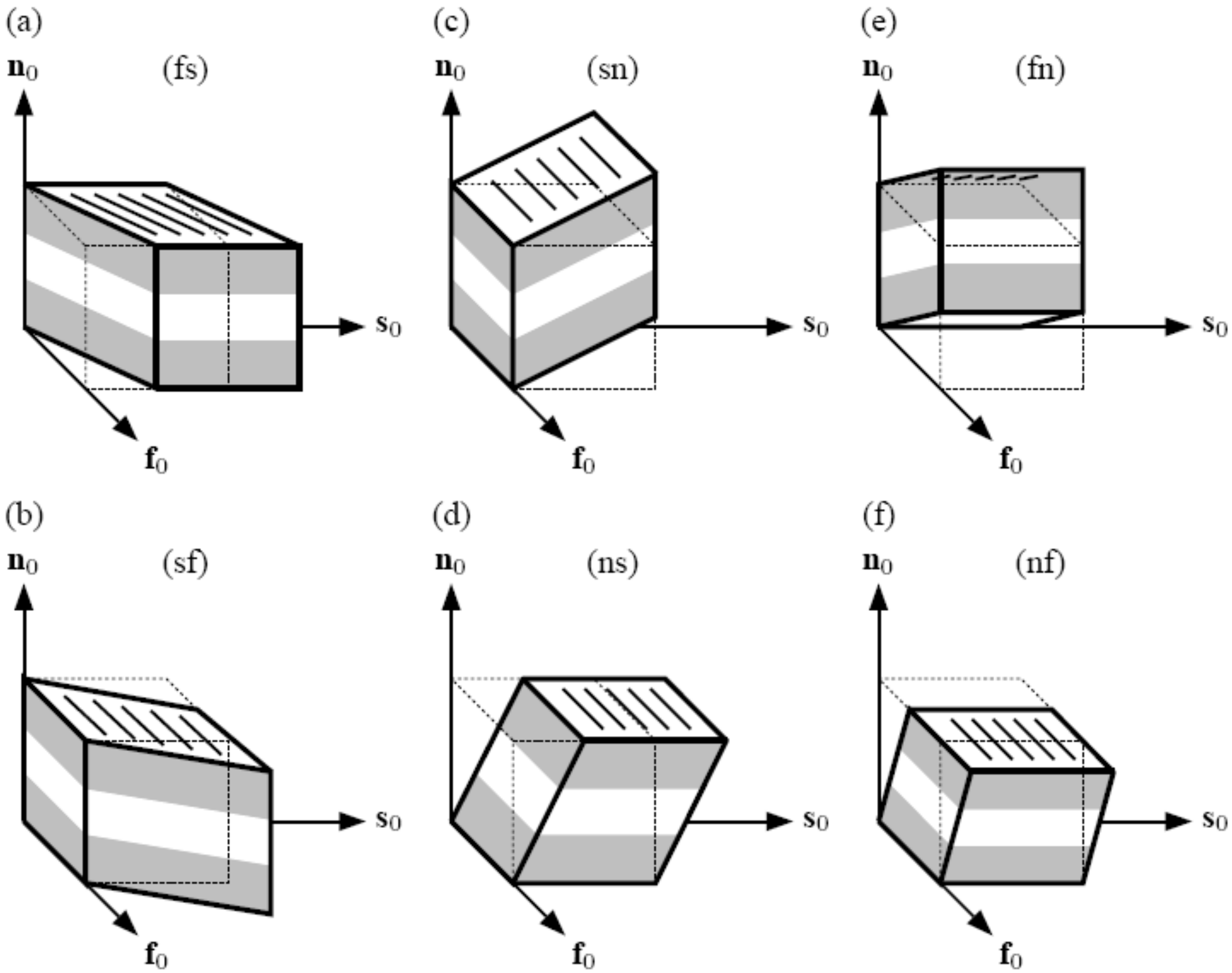
left Cauchy-Green tensor

$$\mathbf{f} = \mathbf{F}\mathbf{f}_0$$

$$\mathbf{s} = \mathbf{F}\mathbf{s}_0$$

$$\psi_{4\,\mathrm{i}} = \partial\Psi/\partial I_{4\,\mathrm{i}}, \; i = \mathrm{f}, \mathrm{s} \qquad \Psi \equiv W$$

Simple Shear of a Cube



6 modes of simple shear

Simple Shear of a Cube

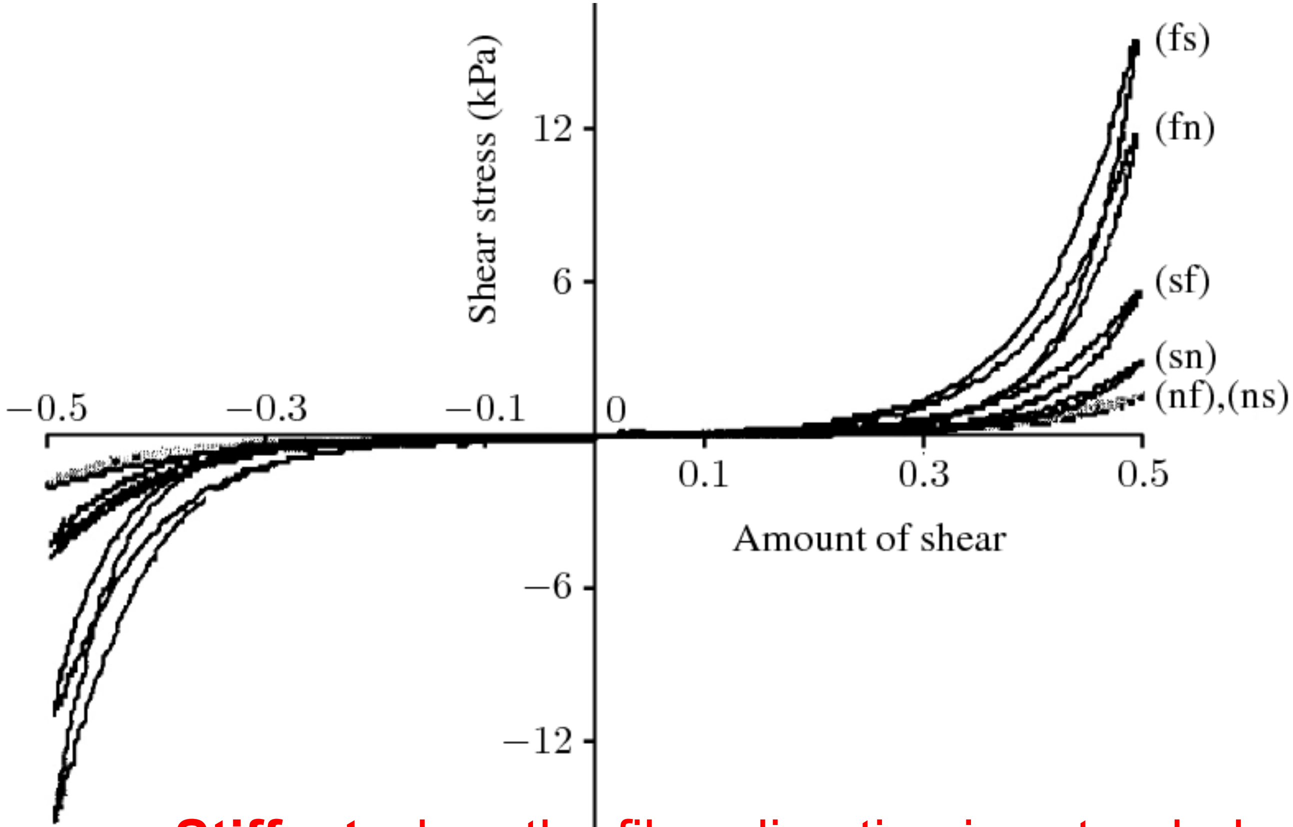
Shear stress versus amount of shear γ for the 6 modes:

$$
\begin{aligned}
\text{(fs)}: \quad \sigma_{\text{fs}} &= 2(\psi_1 + \psi_2 + \psi_{4\,\text{f}})\gamma + \psi_{8\,\text{fs}} \\
\text{(fn)}: \quad \sigma_{\text{fn}} &= 2(\psi_1 + \psi_2 + \psi_{4\,\text{f}})\gamma + \psi_{8\,\text{fn}} \\
\text{(sf)}: \quad \sigma_{\text{fs}} &= 2(\psi_1 + \psi_2 + \psi_{4\,\text{s}})\gamma + \psi_{8\,\text{fs}} \\
\text{(sn)}: \quad \sigma_{\text{sn}} &= 2(\psi_1 + \psi_2 + \psi_{4\,\text{s}})\gamma \\
\text{(nf)}: \quad \sigma_{\text{fn}} &= 2(\psi_1 + \psi_2)\gamma + \psi_{8\,\text{fn}} \\
\text{(ns)}: \quad \sigma_{\text{sn}} &= 2(\psi_1 + \psi_2)\gamma
\end{aligned}
$$

The modes in which the fibres are stretched are (fs) and (fn)

Dokos et al. (2002) data

Simple shear tests on a cube of a typical myocardial specimen in the fs, fn and sn planes



Stiffest when the fibre direction is extended
Least stiff for normal direction
Intermediate stiffness for sheet direction

Simple Shear of a Cube

Shear stress versus amount of shear for the 6 modes:

$$\text{(fs)}: \quad \sigma_{\text{fs}} = 2(\psi_1 + \psi_2 + \psi_{4\,\text{f}})\gamma + \psi_{8\,\text{fs}}$$

$$\text{(fn)}: \quad \sigma_{\text{fn}} = 2(\psi_1 + \psi_2 + \psi_{4\,\text{f}})\gamma + \cancel{\psi_{8\,\text{fn}}}$$

$$\text{(sf)}: \quad \sigma_{\text{fs}} = 2(\psi_1 + \psi_2 + \psi_{4\,\text{s}})\gamma + \psi_{8\,\text{fs}}$$

$$\text{(sn)}: \quad \sigma_{\text{sn}} = 2(\psi_1 + \psi_2 + \psi_{4\,\text{s}})\gamma$$

$$\text{(nf)}: \quad \sigma_{\text{fn}} = 2(\psi_1 + \psi_2)\gamma + \cancel{\psi_{8\,\text{fn}}}$$

$$\text{(ns)}: \quad \sigma_{\text{sn}} = 2(\psi_1 + \psi_2)\gamma$$

The modes in which the fibres are stretched are (fs) and (fn)

A Specific Strain-energy Function

$$\Psi(I_1, I_{4\,\mathrm{f}}, I_{4\,\mathrm{s}}, I_{8\,\mathrm{fs}}) \longrightarrow$$

$$\Psi = \frac{a}{2b}\exp[b(I_1 - 3)]$$

isotropic term

$$+ \sum_{i=\mathrm{f,s}} \frac{a_i}{2b_i}\{\exp[b_i(I_{4\,i} - 1)^2] - 1\}$$

transversely isotropic terms

$$I_{4\,\mathrm{f}} > 1 \quad I_{4\,\mathrm{s}} > 1$$

$$+ \frac{a_{\mathrm{fs}}}{2b_{\mathrm{fs}}}[\exp(b_{\mathrm{fs}} I_{8\,\mathrm{fs}}^2) - 1]$$

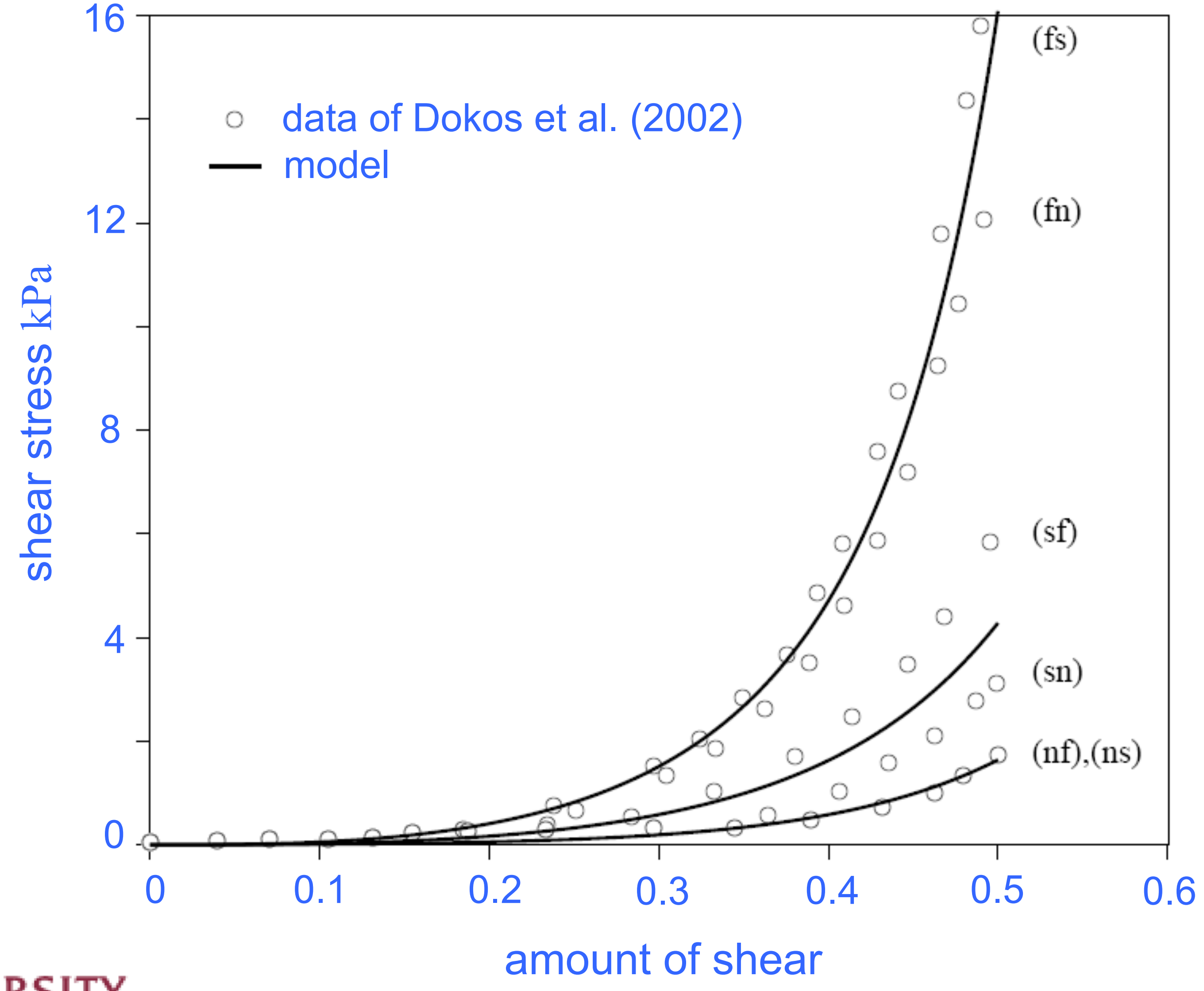
orthotropic term

discriminates shear behaviour

8 constants $a \quad b \quad a_{\mathrm{f}} \quad a_{\mathrm{s}} \quad b_{\mathrm{f}} \quad b_{\mathrm{s}} \quad a_{\mathrm{fs}} \quad b_{\mathrm{fs}}$

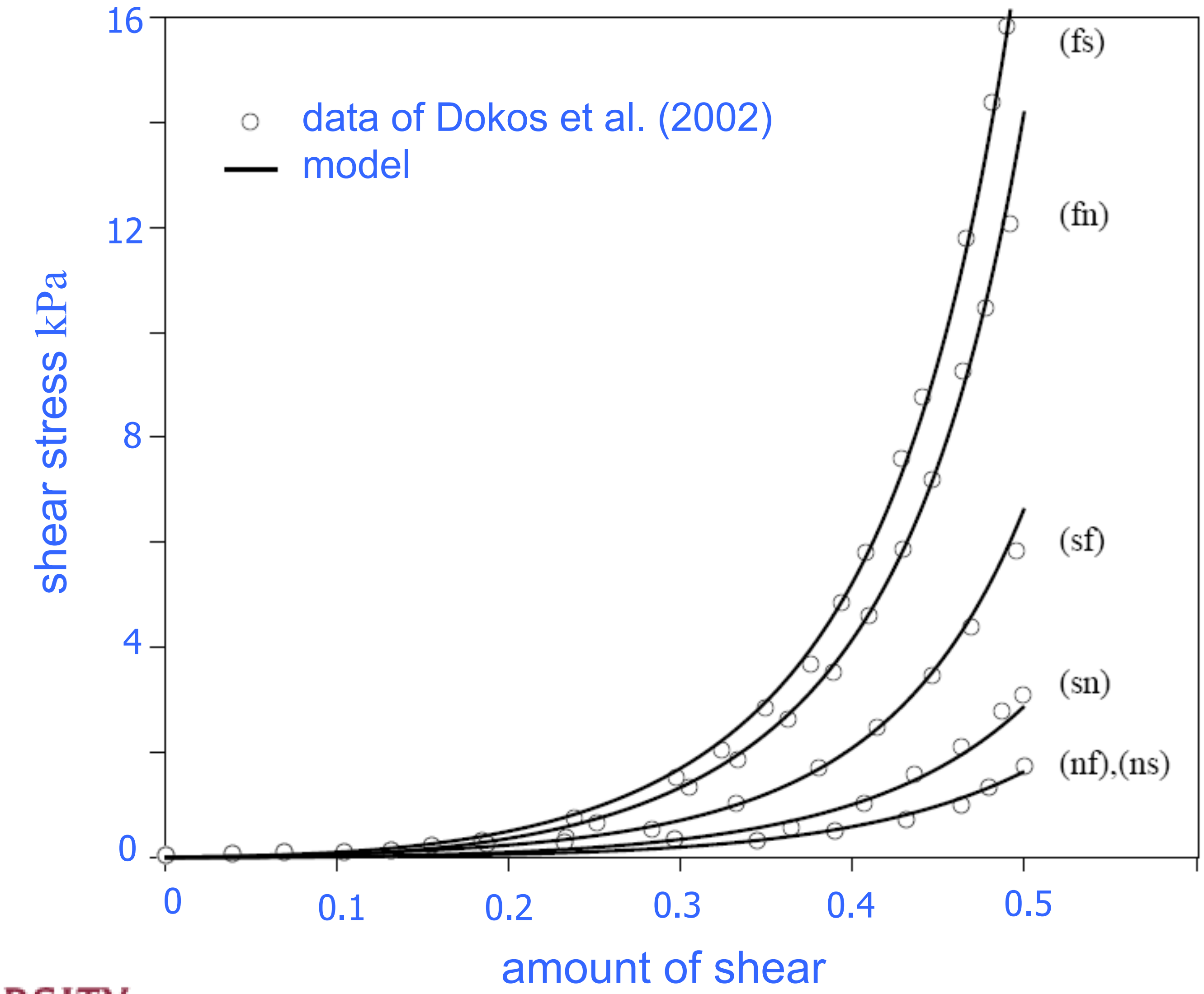
Simple Shear of a Cube

Fit without $I_{8\,\mathrm{fs}}$ term



Simple Shear of a Cube

Fit with $I_{8\,\mathrm{fs}}$ term



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Reference

Holzapfel, Ogden Phil. Trans. R. Soc. Lond. A (2009) **myocardium**

Health warning

Much more data needed

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