PBW and toric degenerations of flag varieties

Ghislain Fourier

University of Glasgow - Universität Bonn
Let us introduce the main objects in this talk:

Let $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \mathfrak{b} \oplus \mathfrak{n}^-$

Let $U(\mathfrak{g}) = U(\mathfrak{n}^-) U(\mathfrak{h}) U(\mathfrak{n}^+)$

$P^+ \iff$ simple f.d. $\mathfrak{sl}_n$-modules

$\lambda \in P^+ \iff V(\lambda)$

$V(\lambda) = U(\mathfrak{n}^-).v_\lambda, v_\lambda \in P^+$

Let $w \in W$, the Weyl group,

Demazure module:

$V_w(\lambda) := U(\mathfrak{b}).v_{w(\lambda)} \subset V(\lambda)$

Let $SL_n, B, N^-$ be the corresponding algebraic groups

$F = \{ U \in \prod \text{Gr}(i, n) \mid \text{where } U \text{ is} \}$

$\{0\} = U_0 \subset U_1 \subset \ldots \subset U_{n-1} \subset U_n = \mathbb{C}^n, \dim U_i = i \}$

$F(\lambda) := \overline{N^-.[v_\lambda]} \subset \mathbb{P}(V(\lambda))$

For regular $\lambda \in P^+: F(\lambda) \cong F$

Schubert variety:

$X_w(\lambda) := \overline{B.[v_{w(\lambda)}]} \subset \mathbb{P}(V(\lambda))$
We will degenerate the simple module and the flag variety in several ways. Here is the first one:

\[ U(n^-)_s := \langle x_{i_1} \cdots x_{i_\ell} \mid x_{i_j} \in n^-, \ell \leq s \rangle_C, \]

so

\[ U(n^-)_s = U(n^-)_{s-1} + n^- U(n^-)_{s-1}. \]

Then by the PBW theorem \( \text{gr } U(n^-) \cong S(n^-) = U(n^-, a). \)

Because of the adjoint action

\[ n^+ \cdot (U(n^-)_s U(b) / U(b)) \subseteq (U(n^-)_s U(b) / U(b)), \]

there is an "adjoint" action of \( n^+ \) on \( S(n^-) \) and \( n^-, a \). We set, using this action,

\[ g^a := b \oplus n^-, a. \]

We denote further

\[ V_s(\lambda) = U(n^-)_s V_\lambda, \]

then \( V^a(\lambda) := \text{gr } V(\lambda) \) is a cyclic \( g^a \)-module.
The group corresponding to the abelian Lie algebra $n^-, a$ is $G^N_a$, where $N = \dim n^-$. We fix $\lambda \in P^+$ and define

$$\mathcal{F}^a(\lambda) := \overline{G^N_a \cdot [v_\lambda]} \subset \mathbb{P}(V^a(\lambda)).$$

Let us consider the first example, $\lambda = \omega_i, V(\lambda) = \Lambda^i \mathbb{C}^n$.

Here we can use:

the nilpotent radical is abelian $\iff$ the highest weight is rectangular

This implies

$$\mathcal{F}^a(\omega_i) \cong \mathcal{F}(\omega_i) \cong \text{Gr}(i, n),$$

and we should not expect such an isomorphism in general. In fact, there is the following alternative description:

**Theorem (Feigin, ’12)**

For regular $\lambda$, the degenerated flag variety can be described as:

$$\{ U \in \prod_{i=0}^{n} \text{Gr}(i, n) \mid \dim U_i = i ; \text{pr}_{i+1} U_i \hookrightarrow U_{i+1} \}$$
Going back to $V^a(\lambda)$. There is a monomial basis of $V^a(\lambda)$ as follows: let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0) \in P^+$ and define $P(\lambda) \subset \mathbb{R}^N_{\geq 0}$ as:

$$\{(s_{\alpha}) \in \mathbb{R}^N_{\geq 0} | \sum_{\alpha \in p} s_{\alpha} \leq \lambda_i - \lambda_j, \forall \text{ paths } p : \lambda_i \rightarrow \lambda_j\}$$

Ardila-Bliem-Salazar called this the marked chain polytope associated to this poset. Denote the lattice points $S(\lambda) := P(\lambda) \cap \mathbb{Z}^N$.

Example: $\mathbb{C}^n$

$$S(\omega_1) = \{0, a_{1,1} = 1, a_{1,2} = 1, \cdots, a_{1,n-1} = 1\}$$

We have studied this polytope quite a bit...
Lemma (Feigin-F-Littelmann, ’11)

For all $\lambda, \mu \in P^+$, $P(\lambda)$ is a normal polytope and $P(\lambda + \mu) = P(\lambda) + P(\mu)$.

We can identify monomials in $S(n^-)$ with lattice points in $\mathbb{Z}^{|R^+|=N}$: for $s = (s_\alpha) \in \mathbb{Z}^N_{\geq 0}$, we denote $f^s = \prod_\alpha f^{s_\alpha}_\alpha \in S(n^-)$.

Theorem (Feigin-F-Littelmann, ’11)

The set

$$\{ f^s . v_\lambda \in V^a(\lambda) \mid s \in S(\lambda) \}$$

is a basis of $V^a(\lambda)$. The annihilating ideal is generated by

$$\{ U(n^+) . f^{\lambda(h_\alpha)+1}_\alpha \mid \alpha > 0 \}.$$  

This ideal is not monomial, for example

$$\left( f_{\alpha + \alpha_2} f_{\alpha_2 + \alpha_3} - f_{\alpha_2} f_{\alpha_1 + \alpha_2 + \alpha_3} \right) . e_1 \wedge e_2 = 0 \in (\Lambda^2 \mathbb{C}^n)^a.$$  

Adjustment of the grading (Fang-F-Reineke): the annihilating ideal is monomial.

Moreover, a homogeneous total order on monomials in $S(n^-)$ which is a refinement of the PBW order is provided, such that $S(\lambda)$ parametrizes the basis in the associated graded module $V^t(\lambda)$ (which is again a $S(n^-)$-module).
Now, having this more degenerated $\mathfrak{n}^{-,a}$-module $V^t(\lambda)$, we can define:

$$\mathcal{F}^t(\lambda) := \mathbb{G}_a^N[\nu_\lambda] \subset \mathbb{P}(V^t(\lambda)).$$

We can relate this to the toric variety $X(P(\lambda))$ associated to the normal polytope $P(\lambda)$:

For $\mathbf{z} = (z_1, \ldots, z_N) \in (\mathbb{C}^*)^N$, $\mathbf{s} = (s_1, \ldots, s_N) \in S(\lambda)$, denote

$$\mathbf{z}^\mathbf{s} = \prod z_i^{s_i}.$$

Let $S(\lambda) = \{\mathbf{s}^1, \ldots, \mathbf{s}^K\}$, then $X(P(\lambda))$ is defined to be the closure of

$$\{(\mathbf{z}^{s_1} : \cdots : \mathbf{z}^{s^K}) \mid \mathbf{z} \in (\mathbb{C}^*)^N\} \subset \mathbb{P}(\mathbb{C}^K).$$

**Proposition (Feigin-F-Littelmann, ’13)**

*For $\lambda \in P^+$:*

$\mathcal{F}^t(\lambda)$ is a toric variety and isomorphic to $X(P(\lambda))$. 
Toric degeneration

\[
\mathcal{F}(\lambda)
\]

Feigin, '12
Feigin-F-Littelmann, '11

\[
\mathcal{F}^a(\lambda)
\]

Feigin-F-Littelmann, '13

\[
\mathcal{F}^t(\lambda)
\]

Newton-Okounkov body!
Is this actually new??
What about Gelfand-Tsetlin polytopes and degenerations for example?

Define $GT(\lambda) \subset \mathbb{R}^N_{\geq 0}$ as:

$$\{(s_{i,j}) \in \mathbb{R}^N \mid s_{i,j} \geq s_{i+1,j+1} \geq s_{i,j+1} \text{ and } \lambda_j \geq s_{1,j} \geq \lambda_{j+1}\}$$

This is also called the marked order polytope.

Example: $\mathbb{C}^n$, then the lattice points are

$$\{0, e_{1,1}, e_{1,1} + e_{2,2}, \ldots, e_{1,1} + \ldots + e_{n-1,n-1}\}.$$  

This provides also a toric degeneration $X(GT(\lambda))$ of $F(\lambda)$, Gonciulea-Lakshmi, ’96.
Let $P$ be a finite poset, $A \subset P$ containing at least all extremal elements and $\lambda \in \mathbb{Z}_{\geq 0}^{\lvert A \rvert}$ be a marking.

Similar to the examples one defines the marked chain polytope $\mathcal{C}(P, A, \lambda)$ and the marked order polytope $\mathcal{O}(P, A, \lambda)$. A first result is:

**Theorem**

1. $\mathcal{C}(P, A, \lambda)$ and $\mathcal{O}(P, A, \lambda)$ have the same number of lattice points (Ardila-Bliem-Salazar, ’11).
2. Closed formula for the number of facets in any marked poset polytope (F, ’15).

But we are more interested whether the two polytopes are isomorphic:

**Theorem (F, ’15)**

$\mathcal{C}(P, A, \lambda)$ and $\mathcal{O}(P, A, \lambda)$ are unimodular equivalent if and only if the poset $P$ does not contain a star subposet

$$
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
$$

Hence, the toric varieties $X(\text{GT}(\lambda))$ and $\mathcal{F}^t(\lambda)$ are isomorphic if and only if $\lambda_3 = 0$ or $\lambda_1 = \lambda_{n-2}$ or $\lambda_2 = \lambda_{n-1}$. 
\[ \mathcal{F}(\lambda) \]

Feigin, '12

Feigin-F-Littelmann, '11

\[ \mathcal{F}^a(\lambda) \]

Feigin-F-Littelmann, '13

\[ \mathcal{F}^t(\lambda) \not\cong X(GT(\lambda)) \]

F, '15
Let $B(\lambda)$ be the crystal graph, $b \in B(\lambda)$, $w_0 = s_{i_1} \cdots s_{i_N}$ a reduced decomposition.

\[
\begin{array}{cccc}
\quad & e_{i_1} & b_1 & e_{i_2} & b_2 & e_{i_3} & b_3 & \cdots & \cdots & b_{\lambda} \\
\vdots & b & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

\[\sim \Rightarrow a_b = (a_{i_1}, a_{i_2}, \cdots) \in \mathbb{Z}_\geq 0^N\]

Theorem (Littelmann '98, Berenstein-Zelevinsky '00)

\[\exists \text{ a normal polytope } Q_{w_0}(\lambda), \text{ called the string polytope, whose lattice points are precisely } \{a_b \mid b \in B(\lambda)\}.\]

The Gelfand-Tsetlin polytope corresponds to $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \cdots s_{n-1} \cdots s_1$.

There are many reduced decompositions,

Stanley, '84 : \[\binom{n}{2}! / 1^{n-1} 3^{n-2} 5^{n-3} \cdots (2n-3),\]

and hence many polytopes and hence many toric varieties. But:

Lemma

In general, $\mathcal{F}^t(\lambda)$ is not isomorphic to $X(Q_{w_0}(\lambda))$ for any reduced decomposition.

Unfortunately, the result is less detailed than for GT-polytopes, but work in progress...
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String polytopes

\[ \mathcal{F}(\lambda) \]

Feigin, '12
Feigin-F-Littelmann, '11

\[ \mathcal{F}^a(\lambda) \]

Feigin-F-Littelmann, '13

\[ \mathcal{F}^t(\lambda) \not\cong X(GT(\lambda)), X(Q_{w_0}(\lambda)) \]

F, '15
That could be the end of the story, but here comes a little piece of magic:
Lemma ((Cerulli Irelli)-Lanini-Littelmann)

Via the maps \( n^{-,a} \hookrightarrow b_1 \) and \( b \hookrightarrow b_2 \), there is an embedding of Lie algebras

\[
g^a \hookrightarrow \tilde{b}/b_3 \hookrightarrow \tilde{b} \subset \mathfrak{sl}_{2n}.
\]

Moreover, if \( w = (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_3 s_4) \cdots (s_2) \), then

\[
b_1 = \langle e_\alpha \mid w^{-1}(\alpha) < 0 \rangle_\mathbb{C}.
\]

For any \( \mu \in P_{2n}^+ \), the Demazure module is defined as

\[
V(\mu) \supset V_w(\mu) := U(\tilde{b}).v_{w(\mu)} = U(b_1).v_{w(\mu)}.
\]

Using this identification, there is an action of \( g^a \) on any Demazure submodule \( V_w(\mu) \) of a simple \( \mathfrak{sl}_{2n} \)-module \( V(\mu) \).

Theorem (CL-L-L)

For any \( \lambda \in P^+ \), \( \exists \mu \in P_{2n}^+ \) such that \( V^a(\lambda) \cong V_w(\mu) \) as \( g^a \)-modules. Moreover, \( \mathcal{F}^a(\lambda) \cong X_w(\mu) \), the Schubert variety associated with \( w \) in the (partial) flag variety \( \mathcal{F}(\mu) \).
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A surprising isomorphism

\[ \mathcal{F}(\lambda) \]

\[ \mathcal{F}^a(\lambda) \]

\[ X(P_w(\mu)) \cong \]

\[ X(G_T(\lambda)), X(Q_{w_0}(\lambda)) \]

\[ X_w(\mu) \cong \]

\[ \text{Cerulli Irelli-Lanini, '14} \]

\[ \text{Fang-F, '15} \]

\[ \text{Feigin-F-Littelmann, '11} \]

\[ \text{Feigin-F-Littelmann, '13} \]

\[ \text{Feigin, '12} \]

\[ \text{Desingularization?} \]

\[ \text{Number of torus fixpoints?} \]

\[ \text{Cohomology ring?} \]

\[ \text{CI-L, '14} \]

\[ \text{F, '14} \]

\[ \text{F, '14} \]

\[ \text{F, '15} \]
Let us use this isomorphism also in the toric case:

The construction of string polytopes (via a reduced decomposition and the crystal graph) works for Demazure modules as well. We fix

\[ w = (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_3 s_4) (s_2). \]

and \( \mu \in P^+_{2n} \):

**Lemma (Littelmann, ’98)**

*There exists a normal polytope (called the string polytope)*

\[ Q_w(\mu) \]

*whose set of lattice points parametrizes a monomial basis of the Demazure module \( V_w(\mu) \).*

The polytope is described recursively and hence certain properties such as number of facets can not be read off immediately. We can still consider the corresponding toric variety \( X(Q_w(\mu)) \).
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Finally: A nice diagram

\[ \mathcal{F}(\lambda) \]

\[ \mathcal{F}^a(\lambda) \]

\[ X_w(\mu) \rightleftharpoons \text{Desingularization} \leftarrow \text{Number of torus fixpoints} \]

\[ X(Q_w(\mu)) \rightleftharpoons \text{Cerulli Irelli-Lanini, '14} \]

\[ X(GT(\lambda)), X(Q_{w_0}(\lambda)) \]

Feigin, '12
Feigin-Finkelberg, '13
CI-L, '14
Feigin, '11, Fang-F, '15
Feigin-F-Littelmann, '11
Feigin-F-Littelmann, '13
Cerulli Irelli-Lanini, '14
X_w(\mu) \rightleftharpoons \text{Desingularization} \leftarrow \text{Number of torus fixpoints} \]

\[ \mathcal{F}^t(\lambda) \not\cong \]

\[ X(GT(\lambda)), X(Q_{w_0}(\lambda)) \]

F, '14
Kaveh, '11
Alexeev-Brion, '04
Littelmann, '98

Desingularization? Crystal graph? Littlewood-Richardson rule? Other types?

End.