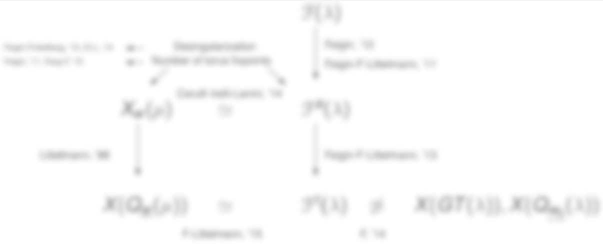

PBW and toric degenerations of flag varieties

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Let us introduce the main objects in this talk:

Let $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \mathfrak{b} \oplus \mathfrak{n}^-$

$U(\mathfrak{g}) = U(\mathfrak{n}^-)\, U(\mathfrak{h})\, U(\mathfrak{n}^+)$

$P^+ \Leftrightarrow$ simple f.d. $\mathfrak{sl}_n$-modules

$\lambda \in P^+ \Leftrightarrow V(\lambda)$

$V(\lambda) = U(\mathfrak{n}^-).v_\lambda,\ v_\lambda \in P^+$

Let $w \in W$, the Weylgroup,

Demazure module:

$$V_w(\lambda) := U(b).v_{w(\lambda)} \subset V(\lambda)$$

Let SL_n, B, N^- be the corresponding algebraic groups

$$\mathcal{F} = \{\, \underline{U} \in \prod \mathrm{Gr}(i,n) \mid \text{ where } \underline{U} \text{ is } \{0\} = U_0 \subset U_1 \subset \ldots \subset U_{n-1} \subset U_n = \mathbb{C}^n,\ \dim U_i = i \}$$

$\mathcal{F}(\lambda) := \overline{N^-.[v_\lambda]} \subseteq \mathbb{P}(V(\lambda))$

For regular $\lambda \in P^+$: $\mathcal{F}(\lambda) \cong \mathcal{F}$

Schubert variety:

$$X_w(\lambda) := \overline{B.[v_{w(\lambda)}]} \subset \mathbb{P}(V(\lambda))$$

We will degenerate the simple module and the flag variety in several ways. Here is the first one:

$$U(\mathfrak{n}^-)_s := \langle x_{i_1} \cdots x_{i_\ell} \mid x_{i_j} \in \mathfrak{n}^- , \ell \leq s \rangle_{\mathbb{C}},$$

so

$$U(\mathfrak{n}^-)_s = U(\mathfrak{n}^-)_{s-1} + \mathfrak{n}^- U(\mathfrak{n}^-)_{s-1}.$$

Then by the PBW theorem $\operatorname{gr} U(\mathfrak{n}^-) \cong S(\mathfrak{n}^-) = U(\mathfrak{n}^{-,a})$.

Because of the adjoint action

$$n^+ . \left(U(\mathfrak{n}^-)_s \, U(\mathfrak{b}) \, / \, U(\mathfrak{b}) \right) \subseteq \left(U(\mathfrak{n}^-)_s \, U(\mathfrak{b}) \, / \, U(\mathfrak{b}) \right),$$

there is an "adjoint" action of $\mathfrak{n}^+$ on $S(\mathfrak{n}^-)$ and $\mathfrak{n}^{-,a}$. We set, using this action,

$$\mathfrak{g}^a := \mathfrak{b} \oplus \mathfrak{n}^{-,a}.$$

We denote further

$$V_s(\lambda) = U(\mathfrak{n}^-)_s . v_\lambda,$$

then $V^a(\lambda) := \operatorname{gr} V(\lambda)$ is a cyclic $\mathfrak{g}^a$-module.

The group corresponding to the abelian Lie algebra $\mathfrak{n}^{-,a}$ is $\mathbb{G}_a^N$, where $N = \dim \mathfrak{n}^-$. We fix $\lambda \in P^+$ and define

$$\mathcal{F}^a(\lambda) := \overline{\mathbb{G}_a^N.[v_\lambda]} \subset \mathbb{P}(V^a(\lambda)).$$

Let us consider the first example, $\lambda = \omega_i$, $V(\lambda) = \Lambda^i \mathbb{C}^n$.

Here we can use:

the nilpotent radical is abelian $\Leftrightarrow$ the highest weight is rectangular

This implies

$$\mathcal{F}^a(\omega_i) \cong \mathcal{F}(\omega_i) \cong \mathrm{Gr}(i,n),$$

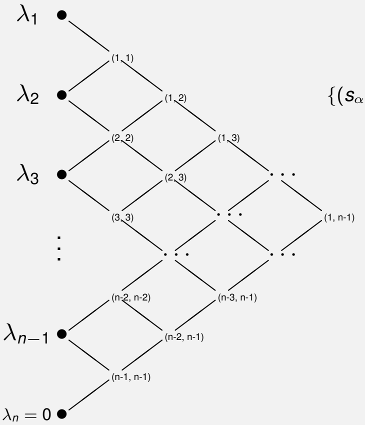
and we should not expect such an isomorphism in general.
In fact, there is the following alternative description:

Theorem (Feigin, '12)

For regular λ, the degenerated flag variety can be desribed as:

$$\{\underline{U} \in \prod_{i=0}^{n} \mathrm{Gr}(i,n) \mid \dim U_i = i\,;\ \mathrm{pr}_{i+1} U_i \hookrightarrow U_{i+1}\}$$

Going back to $V^a(\lambda)$. There is a monomial basis of $V^a(\lambda)$ as follows: let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0) \in P^+$ and



Define $P(\lambda) \subset \mathbb{R}^N_{\geq 0}$ as:

$$\{(s_\alpha) \in \mathbb{R}^N_{\geq 0} \mid \sum_{\alpha \in \mathbf{p}} s_\alpha \leq \lambda_i - \lambda_j \,,\ \forall \text{ paths } \mathbf{p} : \lambda_i \to \lambda_j\}$$

Ardila-Bliem-Salazar called this the *marked chain polytope* associated to this poset.

Denote the lattice points $S(\lambda) := P(\lambda) \cap \mathbb{Z}^N$.

Example: $\mathbb{C}^n$

$S(\omega_1) = \{0, a_{1,1} = 1, a_{1,2} = 1, \cdots, a_{1,n-1} = 1\}$

We have studied this polytope quite a bit...

Lemma (Feigin-F-Littelmann, '11)

For all $\lambda, \mu \in P^+$, $P(\lambda)$ is a normal polytope and $P(\lambda + \mu) = P(\lambda) + P(\mu)$.

We can identify monomials in $S(\mathfrak{n}^-)$ with lattice points in $\mathbb{Z}_{\geq 0}^{|R^+|=N}$:
for $\mathbf{s} = (s_\alpha) \in \mathbb{Z}_{\geq 0}^N$, we denote $f^{\mathbf{s}} = \prod_\alpha f_\alpha^{s_\alpha} \in S(\mathfrak{n}^-)$.

Theorem (Feigin-F-Littelmann, '11)

The set

$$\{f^{\mathbf{s}}.v_\lambda \in V^a(\lambda) \mid \mathbf{s} \in S(\lambda)\}$$

is a basis of $V^a(\lambda)$. The annihilating ideal is generated by

$$\{U(\mathfrak{n}^+).f_\alpha^{\lambda(h_\alpha)+1} \mid \alpha > 0\}.$$

This ideal is not monomial, for example

$$\left(f_{\alpha_1+\alpha_2} f_{\alpha_2+\alpha_3} - f_{\alpha_2} f_{\alpha_1+\alpha_2+\alpha_3}\right).e_1 \wedge e_2 = 0 \in (\Lambda^2 \mathbb{C}^n)^a.$$

Adjustment of the grading (Fang-F-Reineke): the annihilating ideal is monomial.

Moreover, a homogeneous total order on monomials in $S(\mathfrak{n}^-)$ which is a refinement of the PBW order is provided, such that $S(\lambda)$ parametrizes the basis in the associated graded module $V^t(\lambda)$ (which is again a $S(\mathfrak{n}^-)$-module).

Now, having this more degenerated $\mathfrak{n}^{-,a}$-module $V^t(\lambda)$, we can define:

$$\mathcal{F}^t(\lambda) := \overline{\mathbb{G}_a^N.[v_\lambda]} \subset \mathbb{P}(V^t(\lambda)).$$

We can relate this to the toric variety $X(P(\lambda))$ associated to the normal polytope $P(\lambda)$:
For $\mathbf{z} = (z_1, \ldots, z_N) \in (\mathbb{C}^*)^N$, $\mathbf{s} = (s_1, \ldots, s_N) \in S(\lambda)$, denote

$$\mathbf{z}^{\mathbf{s}} = \prod z_i^{s_i}.$$

Let $S(\lambda) = \{\mathbf{s}^1, \ldots, \mathbf{s}^K\}$, then $X(P(\lambda))$ is defined to be the closure of

$$\{(\mathbf{z}^{\mathbf{s}^1} : \cdots : \mathbf{z}^{\mathbf{s}^K}) \mid \mathbf{z} \in (\mathbb{C}^*)^N\} \subset \mathbb{P}(\mathbb{C}^K).$$

Proposition (Feigin-F-Littelmann, '13)

For $\lambda \in P^+$:
$\mathcal{F}^t(\lambda)$ is a toric variety and isomorphic to $X(P(\lambda))$.

$\mathcal{F}(\lambda)$

$\downarrow$ Feigin, '12; Feigin-F-Littelmann, '11

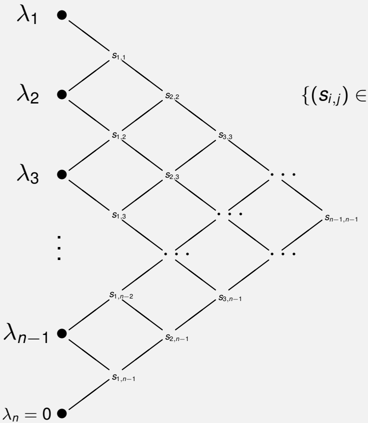
flat degenerations $\to$ $\mathcal{F}^a(\lambda)$

$\downarrow$ Feigin-F-Littelmann, '13

$\mathcal{F}^t(\lambda)$

Newton-Okounkov body!

Is this actually new??
What about Gelfand-Tsetlin polytopes and degenerations for example?



Define $GT(\lambda) \subset \mathbb{R}^N_{\geq 0}$ as:

$$\{(s_{i,j}) \in \mathbb{R}^N \mid s_{i,j} \geq s_{i+1,j+1} \geq s_{i,j+1} \text{ and } \lambda_j \geq s_{1,j} \geq \lambda_{j+1}\}$$

This is also called the *marked order polytope*.

Example: $\mathbb{C}^n$, then the lattice points are

$\{0, e_{1,1}, e_{1,1} + e_{2,2}, \cdots, e_{1,1} + \ldots + e_{n-1,n-1}\}$.

This provides also a toric degeneration

$X(GT(\lambda))$ of $\mathcal{F}(\lambda)$, Gonciulea-Lakshmibai, '96.

Let P be a finite poset, $A \subset P$ containing at least all extremal elements and $\lambda \in \mathbb{Z}_{\geq 0}^{|A|}$ be a marking.
Similar to the examples one defines the marked chain polytope $\mathcal{C}(P, A, \lambda)$ and the marked order polytope $\mathcal{O}(P, A, \lambda)$. A first result is:

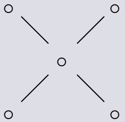
Theorem

1. *$\mathcal{C}(P, A, \lambda)$ and $\mathcal{O}(P, A, \lambda)$ have the same number of lattice points (Ardila-Bliem-Salazar, '11).*
2. *Closed formula for the number of facets in any marked poset polytope (F, '15).*

But we are more interested whether the two polytopes are isomorphic:

Theorem (F, '15)

$\mathcal{C}(P, A, \lambda)$ and $\mathcal{O}(P, A, \lambda)$ are unimodular equivalent if and only if the poset P does not contain a star subposet



Hence, the toric varieties $X(GT(\lambda))$ and $\mathcal{F}^t(\lambda)$ are isomorphic if and only if $\lambda_3 = 0$ or $\lambda_1 = \lambda_{n-2}$ or $\lambda_2 = \lambda_{n-1}$.

$$\mathcal{F}(\lambda)$$

$\downarrow$ Feigin, '12; Feigin-F-Littelmann, '11

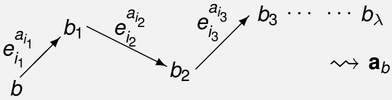
$$\mathcal{F}^a(\lambda)$$

$\downarrow$ Feigin-F-Littelmann, '13

$$\mathcal{F}^t(\lambda) \quad \not\cong \quad X(GT(\lambda))$$

F, '15

Let $B(\lambda)$ be the crystal graph, $b \in B(\lambda)$, $\underline{w_0} = s_{i_1} \cdots s_{i_N}$ a reduced decomposition.



$$\rightsquigarrow \mathbf{a}_b = (a_{i_1}, a_{i_2}, \cdots) \in \mathbb{Z}^N_{\geq 0}$$

Theorem (Littelmann '98, Berenstein-Zelevinsky '00)

$\exists$ *a normal polytope* $Q_{\underline{w_0}}(\lambda)$*, called the string polytope, whose lattice points are precisely* $\{\mathbf{a}_b \mid b \in B(\lambda)\}$.

The Gelfand-Tsetlin polytope corresponds to $w_0 = s_1\, s_2 s_1\, s_3 s_2 s_1 \,\cdots\, s_{n-1} \cdots s_1$.
There are many reduced decompositions,

$$\text{Stanley, '84} \; : \; \binom{n}{2}!/1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3),$$

and hence many polytopes and hence many toric varieties. But:

Lemma

In general, $\mathcal{F}^t(\lambda)$ *is not isomorphic to* $X(Q_{\underline{w_0}}(\lambda))$ *for any reduced decomposition.*

Unfortunately, the result is less detailed than for GT-polytopes, but work in progress...

$$\mathcal{F}(\lambda)$$

$\downarrow$ Feigin, '12; Feigin-F-Littelmann, '11

$$\mathcal{F}^a(\lambda)$$

$\downarrow$ Feigin-F-Littelmann, '13

$$\mathcal{F}^t(\lambda) \quad \not\cong \quad X(GT(\lambda)), X(Q_{\underline{w_0}}(\lambda))$$

F, '15

That could be the end of the story, but here comes a little piece of magic:

$$\begin{pmatrix} \ddots & & & \\ & \ddots & & \\ & & \ddots & \\ \mathfrak{n}^{-,a} & & & \ddots \end{pmatrix} \hookrightarrow \begin{pmatrix} \ddots & & & & \vdots & \ddots & & & \\ & \ddots & \mathfrak{b}_2 & & \vdots & & \ddots & & \mathfrak{b}_3 \\ & & \ddots & & \vdots & & & \ddots & \\ & & & \ddots & \vdots & \mathfrak{b}_1 & & & \ddots \\ & & & & \ddots & \cdots & \cdots & \cdots & \cdots \\ & & & & & \ddots & & & \\ & & & & & & \ddots & \mathfrak{b}_2 & \\ & & & & & & & \ddots & \\ & & & & & & & & \ddots \end{pmatrix}$$

Lemma ((Cerulli Irelli)-Lanini-Littelmann)

Via the maps $\mathfrak{n}^{-,a} \hookrightarrow \mathfrak{b}_1$ *and* $\mathfrak{b} \hookrightarrow \mathfrak{b}_2$, *there is an embedding of Lie algebras*

$$\mathfrak{g}^a \hookrightarrow \tilde{\mathfrak{b}}/\mathfrak{b}_3 \twoheadleftarrow \tilde{\mathfrak{b}} \subset \mathfrak{sl}_{2n}.$$

Moreover, if $w = (s_n s_{n+1} \cdots s_{2n-2}) \, \cdots \, (s_3 s_4) \, (s_2)$, then

$$\mathfrak{b}_1 = \langle e_\alpha \mid w^{-1}(\alpha) < 0 \rangle_{\mathbb{C}}.$$

For any $\mu \in P_{2n}^+$, the Demazure module is defined as

$$V(\mu) \supset V_w(\mu) := U(\tilde{\mathfrak{b}}).v_{w(\mu)} = U(\mathfrak{b}_1).v_{w(\mu)}.$$

Using this identification, there is an action of $\mathfrak{g}^a$ on any Demazure submodule $V_w(\mu)$ of a simple $\mathfrak{sl}_{2n}$-module $V(\mu)$.

Theorem (CL-L-L)

For any $\lambda \in P^+$, $\exists \, \mu \in P_{2n}^+$ *such that* $V^a(\lambda) \cong V_w(\mu)$ *as* $\mathfrak{g}^a$*-modules. Moreover,* $\mathcal{F}^a(\lambda) \cong X_w(\mu)$, *the Schubert variety associated with* w *in the (partial) flag variety* $\mathcal{F}(\mu)$.

Cohomology ring?

Cl-L, '14 ← Desingularization?

Fang-F, '15 ← Number of torus fixpoints?

$$
\begin{tikzcd}
 & & \mathcal{F}(\lambda) \arrow[d, "\text{Feigin, '12}" right, "\text{Feigin-F-Littelmann, '11}" right] & & \\
X_w(\mu) \arrow[d, "\text{F, '14}" left] & \overset{\text{Cerulli Irelli-Lanini, '14}}{\simeq} & \mathcal{F}^a(\lambda) \arrow[d, "\text{Feigin-F-Littelmann, '13}" right] & & \\
X(P_w(\mu)) & \underset{\text{F, '14}}{\simeq} & \mathcal{F}^t(\lambda) & \underset{\text{F, '15}}{\not\simeq} & X(GT(\lambda)), X(Q_{\underline{w_0}}(\lambda))
\end{tikzcd}
$$

Let us use this isomorphism also in the toric case:

The construction of string polytopes (via a reduced decomposition and the crystal graph) works for Demazure modules as well.
We fix

$$w = (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_3 s_4)(s_2).$$

and $\mu \in P^+_{2n}$:

Lemma (Littelmann, '98)

There exists a normal polytope (called the string polytope)

$$Q_{\underline{w}}(\mu)$$

whose set of lattice points parametrizes a monomial basis of the Demazure module $V_w(\mu)$.

The polytope is described recursively and hence certain properties such as number of facets can not be read off immediately. We can still consider the corresponding toric variety $X(Q_{\underline{w}}(\mu))$.

$\mathcal{F}(\lambda)$

Feigin, '12

Feigin-F-Littelmann, '11

Feigin-Finkelberg, '13, CI-L, '14 ← Desingularization

Feigin, '11, Fang-F, '15 ← Number of torus fixpoints

Cerulli Irelli-Lanini, '14

$X_w(\mu) \simeq \mathcal{F}^a(\lambda)$

Littelmann, '98

Kaveh, '11

Alexeev-Brion,'04

Feigin-F-Littelmann, '13

$X(Q_{\underline{w}}(\mu)) \simeq \mathcal{F}^t(\lambda) \not\simeq X(GT(\lambda)), X(Q_{\underline{w_0}}(\lambda))$

F-Littelmann, '15

F, '14

Newton-Okounkov body!

Desingularization? Crystal graph? Littlewood-Richardson rule? Other types?

End.