Quantum PBW filtration and monomial ideals

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Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a simple complex Lie algebra.We set

$$\deg(x) = 1 \,\forall\, x \in \mathfrak{n}^-$$

and consider the induced filtration on $U(\mathfrak{n}^-)$:

$$U(\mathfrak{n}^-)_s := \langle x_{i_1} \cdots x_{i_\ell} \mid x_{i_j} \in \mathfrak{n}^-, \ell \leq s \rangle_{\mathbb{C}}.$$

Now, since $xy - yx - [x, y] = 0$, the associated graded is isomorphic to $S(\mathfrak{n}^-) = \mathbb{C}[n^-]$.

This filtration is stable for the left $\mathfrak{n}^+$-action, in fact $\mathfrak{n}^+$ acts by differential operators. We have a degeneration:

$$\mathfrak{g} \rightsquigarrow \mathfrak{g}^a = \mathfrak{b} \oplus \mathfrak{n}^{-,a}.$$

The corresponding algebraic group is

$$G \rightsquigarrow G^a = B^- \ltimes \mathbb{G}_a^{\dim n^-}.$$

Let us turn to cyclic, highest weight $\mathfrak{g}$-modules:

Let $M = U(\mathfrak{n}^-).v_m$ and consider the induced filtration

$$\cdots U(\mathfrak{n}^-)_{s-1}.v_m \subset U(\mathfrak{n}^-)_s.v_m \subset U(\mathfrak{n}^-)_{s+1}.v_m \subset M.$$

The associated graded module is a $\mathbb{C}[n^-]$-module, we denote this module M^a.
Moreover, M^a is a $\mathfrak{b} \oplus \mathfrak{n}^{-,a}$-module and hence a $B^- \ltimes \mathbb{G}_a^{\dim n^-}$-module.

We are for now mainly interested in $V^a(\lambda)$, the associated graded module of the simple, finite-dimensional $\mathfrak{g}$-module $V(\lambda)$.

More general here: Replace $\mathfrak{n}^-$ by any nilpotent Lie algebra and M a $\mathfrak{n}^-$-module with generators $\{m_i \mid i \in I\}$. Especially interesting: Demazure module.

Let us consider $\mathfrak{g} = \mathfrak{sl}_n$ and $M = \bigwedge^k \mathbb{C}^n$. Consider

$$v = v_{i_1} \wedge \ldots \wedge v_{i_k}, \text{ with } i_1 < \ldots < i_\ell \leq k < i_{\ell+1} < \ldots < i_k$$

and denote $\{j_1 < \ldots < j_{k-\ell}\} = \{1, \ldots, k\} \backslash \{i_1, \ldots, i_\ell\}$.
The PBW degree of v is $k - \ell$ and

$$\left(f_{\alpha_{j_{\sigma(1)}} + \ldots + \alpha_{i_k}} \cdots f_{\alpha_{j_{\sigma(k-\ell)}} + \ldots + \alpha_{i_{\ell+1}}} \right) .v_1 \wedge \ldots \wedge v_k = v \text{ for any } \sigma \in S_{k-\ell}.$$

So if we want to describe a monomial basis, we have to make a choice:

1. The choice $\sigma = \mathrm{id}$ was made by Feigin-F-Littelmann.
2. The choice σ as the longest element in $S_{k-\ell}$ was made by Backhaus-Desczyk to uniform the following construction for all cominuscule weights of simple Lie algebras.

We will make the first choice, $\sigma = \mathrm{id}$ and stay for the rest of the talk in the $\mathfrak{sl}_n$-case.

A Dyck path is a sequence of positive roots $\mathbf{p} = \beta(0), \ldots, \beta(s)$ such that $\beta(0), \beta(s)$ are simple and

$$\beta(p) = \alpha_i + \ldots + \alpha_j \Rightarrow \beta(p+1) \in \{\alpha_{i+1} + \ldots + \alpha_j, \alpha_i + \ldots + \alpha_{j+1}\}.$$

We denote the set of all Dyck paths starting in α_i and ending in α_j by $\mathbb{D}_{i,j}$. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq 0\}$ and define (following Vinberg)

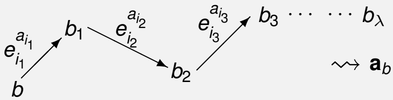
$$P(\lambda) = \left\{ (s_\alpha) \in \mathbb{R}^N_{\geq 0} \mid \sum_{\alpha \in \mathbf{p}} s_\alpha \leq \lambda_i - \lambda_j \,, \forall\, \mathbf{p} \in \mathbb{D}_{i,j}, \forall\, i \leq j \right\}$$

Theorem (Feigin-F-Littelmann ’11)

For any dominant, integral λ:

1. *The annihilating ideal of $v_\lambda \in V^a(\lambda)$ is generated by $\{U(\mathfrak{n}^+).f_\alpha^{\lambda(h_\alpha)+1} \mid \alpha > 0\}$.*
2. *The set $\{f^{\mathbf{s}}.v_\lambda \in V^a(\lambda) \mid \mathbf{s} \in S(\lambda) = P(\lambda) \cap \mathbb{Z}^N\}$ is a basis of $V^a(\lambda)$.*
3. *$P(\lambda)$ is normal and $P(\lambda) + P(\mu) = P(\lambda + \mu)$ for any dominant integral μ.*

Let $B(\lambda)$ be the crystal graph, $b \in B(\lambda)$, $\underline{w_0} = s_{i_1} \cdots s_{i_N}$ a reduced decomposition.



$\rightsquigarrow \mathbf{a}_b = (a_{i_1}, a_{i_2}, \cdots) \in \mathbb{Z}_{\geq 0}^N$

Theorem (Littelmann '98, Berenstein-Zelevinsky '00, Alexseev-Brion '04, Kaveh '11)

$\exists$ *a normal polytope $Q_{\underline{w_0}}(\lambda)$, called the string polytope, whose lattice points are precisely $\{\mathbf{a}_b \mid b \in B(\lambda)\}$. The associated toric variety $X(Q_{\underline{w_0}}(\lambda))$ is a flat degeneration of $\mathcal{F}(\lambda)$. $Q_{\underline{w_0}}(\lambda)$ is the Newton-Okounkov Body of $\mathcal{F}(\lambda)$.*

The Gelfand-Tsetlin polytope corresponds to $w_0 = s_1\ s_2 s_1\ s_3 s_2 s_1\ \cdots\ s_{n-1} \cdots s_1$.
There are many reduced decompositions,

$$\text{Stanley, '84} \ : \ \binom{n}{2}!/1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3),$$

and hence many polytopes and hence many toric varieties. But:

Lemma

There exists λ such that for every reduced decomposition of w_0, the polytope $Q_{\underline{w_0}}(\lambda)$ is not isomorphic to $P(\lambda)$.

In this sense, the polytope $P(\lambda)$ is new.

Let us consider a geometric interpretation: we define

$$\mathcal{F}^a(\lambda) := \overline{\mathbb{G}_a^N.[v_\lambda]} \subset \mathbb{P}(V^a(\lambda)) \ , \ \mathcal{F}^t(\lambda) := \overline{\mathbb{G}_a^N.[v_\lambda]} \subset \mathbb{P}(V^t(\lambda))$$

(here: $V^t(\lambda) = \operatorname{gr}^t V(\lambda)$ for an appropriate homogeneous total $\mathbb{N}^N$-order).

Theorem (Feigin '12, Feigin-F-Littelmann '13)

For any dominant, integral weight λ: $\mathcal{F}^a(\lambda)$ is a flat degeneration of $\mathcal{F}(\lambda)$ and $\mathcal{F}^t(\lambda)$ is a flat degeneration of both.

Feigin's proof contains a description of the degenerated Plücker relations and even more a description in terms of subspaces

$$\mathcal{F}^a(\lambda) = \mathcal{F}^a := \{\underline{U} \in \prod_{i=1}^{n} \operatorname{Gr}(i,n) \mid \dim(U_i) = i \text{ and } \operatorname{pr}_{i+1} U_i \subset U_{i+1}\},$$

here

$$\operatorname{pr}_{i+1} : \mathbb{C}^n \longrightarrow \mathbb{C}^n \ : \ \sum_j a_j e_j \mapsto \sum_{j \neq i+1} a_j e_j .$$

Two more interesting identifications of this degenerated flag variety:

Theorem

Let λ be dominant integral then

1. *The degenerated flag variety $\mathcal{F}^a(\lambda)$ is a Schubert variety $X_{w,\mu}$ inside a partial flag variety for SL_{2n} (Cerulli Irelli-Lanini '14).*
2. $H^0(X_w, \mathcal{L}_\mu) \cong_{\mathfrak{g}^a} V^a(\lambda)$ *(Cerulli Irelli-Lanini-Littelmann '15).*
3. $P(\lambda) \cong Q_{\underline{w}}(\mu)$ *and hence* $\mathcal{F}^t(\lambda) \cong X(Q_{\underline{w}}(\mu))$ *(F-Littelmann '15).*

The other one is in terms of quiver Grassmannian and due to Cerulli-Irelle-Feigin-Reineke and you will see more in this direction in the next talk:

Theorem (Cerulli Irelli-Feigin-Reineke)

The degenerated flag variety $\mathcal{F}^a$ is isomorphic to the quiver Grassmannian $\mathrm{Gr}_{\dim A}(A \oplus A^*)$, where A is the path algebra of the equioriented Dynkin quiver of type A.

So we have

$$\mathcal{F}^a \cong \mathcal{F}^a(\lambda) \cong X_{w,\mu} \cong \mathrm{Gr}_{\dim A}(A \oplus A^*).$$

Our goal was to define/study a PBW filtration for quantum groups $U_q(\mathfrak{g})$:

$$\to \mathbb{N}\text{-filtration with } \operatorname{gr} U_q(\mathfrak{n}^-) \cong \mathbb{C}_q[\mathfrak{n}^-]$$

Let $E_i, F_i, K_i^{\pm 1}$ be the generators subject to the usual relations and T_i Lusztig's automorphism

$$T_i(E_i) = -F_i K_i,\ T_i(F_i) = -K^{-1} E_i,\ T_i(K_j) = K_j K_i^{-c_{ij}}$$

and

$$T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r q_i^{-r} E_i^{(s)} E_j E_i^{(r)}, T_i(E_j) = \sum_{r+s=-c_{ij}} (-1)^r q_i^r F_i^{(r)} F_j F_i^{(s)}.$$

We fix a reduced decomposition of $w_0 = s_{i_1} \cdots s_{i_N}$ and define for $\beta = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t})$ the PBW root vector

$$F_\beta = T_{i_1} T_{i_2} \cdots T_{i_{t-1}}(F_{i_t}) \in U_q(\mathfrak{n}^-).$$

Ordered monomials in the F_β form a basis of $U_q(\mathfrak{n}^-)$.

For $\lambda \in P^+$, we denote $V_q(\lambda)$ the simple $U_q(\mathfrak{g})$-module of highest weighht λ and type 1, with highest weight vector v_λ.

Setting $\deg F_\alpha = 1$ for all $\alpha > 0$ is not working out for us:

1. Let $\mathfrak{g} = \mathfrak{sl}_4$ and fix the reduced expression $w_0 = s_1 s_2 s_1 s_3 s_2 s_1$. The following relation holds in $U_q(\mathfrak{n}^-)$:
$$F_{\alpha_2+\alpha_3} F_{\alpha_1+\alpha_2} = F_{\alpha_1+\alpha_2} F_{\alpha_2+\alpha_3} - (q - q^{-1}) F_{\alpha_2} F_{\alpha_1+\alpha_2+\alpha_3},$$
which specializes to $f_{\alpha_2+\alpha_3} f_{\alpha_1+\alpha_2} = f_{\alpha_1+\alpha_2} f_{\alpha_2+\alpha_3}$ in $U(\mathfrak{n}^-)$.
2. Let $\mathfrak{g}$ be of type G_2 and fix the reduced expression $w_0 = s_1 s_2 s_1 s_2 s_1 s_2$. We have in $U_q(\mathfrak{n}^-)$:
$$F_{3\alpha_1+2\alpha_2} F_{3\alpha_1+\alpha_2} = q^{-3} F_{3\alpha_1+\alpha_2} F_{3\alpha_1+2\alpha_2} + (1 - q^{-2} - q^{-4} + q^{-6}) F^{(3)}_{2\alpha_1+\alpha_2},$$
which specializes to $f_{3\alpha_1+2\alpha_2} f_{3\alpha_1+\alpha_2} = f_{3\alpha_1+\alpha_2} f_{3\alpha_1+2\alpha_2}$ in $U(\mathfrak{n}^-)$.

To find an appropriate grading we use Ringel's identification of $U_q(\mathfrak{n}^-)$ with the Hall algebra $H(Q)$:

Let Q be the equioriented Dynkin quiver of type A, D, E and for every positive root let U_α be the indecomposable representation of dimension vector α.

$$\{\text{isomorphism classes } [M]\} \leftrightarrow \{\text{functions } R^+ \longrightarrow \mathbb{N}\,,\ \beta \mapsto \mathbf{m}(\beta)\},$$

the same parametrization of a PBW basis of $U_q(\mathfrak{n}^-)$. We denote this set $\mathcal{B}$.

If we fix a reduced decomposition of $w_0 = s_{i_1} \cdots s_{i_N}$, then the isomorphism $U_q(\mathfrak{n}^-) \longrightarrow H(Q)$ is induced by the assignment

$$F^{\mathbf{m}} \mapsto F_{[M]} := q^{\dim \mathrm{End}(M) - \dim M} u_{[M]} = u_{[U_{\beta_1}]}^{\mathbf{m}(\beta_1)} \cdots u_{[U_{\beta_N}]}^{\mathbf{m}(\beta_N)}$$

To construct a filtration on $U_q(\mathfrak{n}^-)$ we shouldl consider possible degree functions on $\mathcal{B}$. Remember: the associated graded should be isomorphic to $\mathbb{C}_q[\mathfrak{n}^-]$.

Let us consider possible degree functions $w : \mathcal{B} \longrightarrow \mathbb{N}$ on isomorphism classes $[M] \in \mathcal{B}$. We call w (strongly) admissible if

- $w([M]) = 0 \Leftrightarrow [M] = 0$,
- $w(X) \leq w(M) + w(N)$ for every short exact sequence $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$,
- (and $<$ iff only if the exact sequence is non-split).

Lemma (Fang-F-Reineke, '15)

This function induces a filtration on $U_q(\mathfrak{n}^-)$, where $\mathcal{F}_n$ is spanned by $F_{[M]}$ with $w([M]) \leq n$. Moreover, the associated graded algebra is isomorphic to $\mathbb{C}_q[\mathfrak{n}^-]$.

Here is the main result that we are using from Hall algebras:

Theorem (Fang-F-Reineke)

w is admissible iff $w(M) = \mathrm{dimHom}(V, M)$ for some Q representation V.
w is strongly admissible iff V contains at least one direct summand of all simple and all non-projective indecomposable U_α.

Example: Type A_n and we consider the *canonical* choice, $V = \bigoplus_{\alpha \in R^+} U_\alpha$, one copy of each indecomposable. Then

$$\deg F_{\alpha_i + \ldots + \alpha_j} = (j - i + 1)(n - j + 1).$$

Let us apply this new grading in the non-quantum case. Then

$$\deg f_{i,j} = (j-i+1)(n-i+1) \quad \text{instead of } 1,$$

and among all the $\sigma \in S_{k-\ell}$ there is a unique one, such that

$$f_{\alpha_{j_{\sigma(1)}}+\ldots+\alpha_{i_{\ell+1}}} \cdots f_{\alpha_{j_{\sigma(k-\ell)}}+\ldots+\alpha_{i_{\ell+k-\ell}}}$$

has minimal degree! This is precisely $\sigma = \mathrm{id}$.
If we denote $V^{\mathcal{F}}(\lambda)$ the associated graded module (of the simple module $V(\lambda)$), then

Theorem (Fang-F-Reineke, '15)

$S(\lambda)$ parametrizes a basis of $V^{\mathcal{F}}(\lambda)$ and the defining ideal is monomial.

Remark

This is special about this filtration: The monomial basis is uniquely determined by forcing the grading to be strongly admissible. This implies that the polytope is somehow canonical. In this sense, this might be a special Newton-Okounkov body of $\mathcal{F}(\lambda)$.

Here is the quantum version of the previous theorem:

Theorem (Fang-F-Reineke, '15)

The set

$$\{F^{\mathbf{p}}.v_\lambda \mid \mathbf{p} \in S(\lambda)\} \text{ forms a basis of } V_q^{\mathcal{F}}(\lambda)\},$$

and the annihilating ideal is monomial.

Remark

- *For other simply-laced Lie algebras, the ideal is not monomial in general: Consider $\mathfrak{so}_8$ and $V(\omega_1+\omega_3)_{-\omega_4}$, then there are two monomials of the same weight and degree, mapping to this weight space.*
- *Try the non-simply-laced case: $\mathfrak{sp}_n$. A good polytope is known but so far there is no admissible grading that provides exactly this monomial basis.*
- *The grading can be obtained by a specific reduced decomposition. Does any reduced decomposition leads to the associated graded algebra $\mathbb{C}_q[n^-]$? The polytope depends on the reduced decompositon.*