New invariants for groups

D. A. Cruickshank   S. J. Pride

1 March 2001

Contents

1 Introduction

If C is a commutative ring and M is a finitely generated C-module, then there is a descending chain of ideals in C, the *Fitting ideals*, $\mathrm{Fit}_\lambda(M)$ ($\lambda \in \mathbb{Z}$), defined as follows. Take an exact sequence

$$F' \xrightarrow{\partial} F \xrightarrow{\varepsilon} M \to 0, \tag{1.1}$$

with F, F' free C-modules, and F of finite rank r. Let D be the matrix of ∂ with respect to bases for F' and F. Then

$$\mathrm{Fit}_\lambda(M) = \begin{cases} C, & \lambda \geq r, \\ \text{the ideal generated} & \\ \text{by all } (r-\lambda)\times(r-\lambda) & \\ \text{subdeterminants of } D, & \lambda < r \end{cases}$$

(with the convention that if there are no $(r-\lambda)\times(r-\lambda)$ submatrices of D, then $\mathrm{Fit}_\lambda(M) = 0$). This chain is independent of the choice of exact sequence (1.1) and of the choice of bases for F and F'. The chain is thus an invariant of M; see [40, pp. 145–147].

On the other hand, if G is a finitely generated group, then there is the chain of *Alexander ideals*, $\mathrm{A}_\lambda(G)$ ($\lambda \in \mathbb{Z}$), defined as follows: take a presentation $\mathcal{P} = \langle \boldsymbol{x}; \boldsymbol{r}\rangle$ for G with $\boldsymbol{x}$ finite (so, the group $G(\mathcal{P})$ defined by $\mathcal{P}$ is isomorphic to G); compute the *Alexander matrix*,

$$D = \left[\overline{\frac{\partial R}{\partial x}}\right]_{\substack{R\in\boldsymbol{r}\\ x\in\boldsymbol{x}}},$$

the entries of which lie in the integral group ring $\mathbb{Z}G(\mathcal{P})$ (here, $\overline{}$ denotes the image in $\mathbb{Z}G(\mathcal{P})$); apply the abelianising map $\mathbb{Z}G(\mathcal{P}) \to \mathbb{Z}G(\mathcal{P})^{\mathrm{ab}}$, to the entries (here, $G(\mathcal{P})^{\mathrm{ab}}$ is the quotient of $G(\mathcal{P})$ by its derived subgroup), to obtain a matrix D^{ab}; then,

$$\mathrm{A}_\lambda(\mathcal{P}) = \begin{cases} \mathbb{Z}G(\mathcal{P})^{\mathrm{ab}}, & \lambda \geq |\boldsymbol{x}|, \\ \text{the ideal generated} & \\ \text{by all } (|\boldsymbol{x}|-\lambda)\times(|\boldsymbol{x}|-\lambda) & \\ \text{subdeterminants of } D^{\mathrm{ab}}, & \lambda < |\boldsymbol{x}|. \end{cases}$$

These ideals lie in $\mathbb{Z}G(\mathcal{P})^{\mathrm{ab}}$. They are invariants of G in the sense that if $\mathcal{P}'$ is another presentation of G, so that there is a group isomorphism $\phi : G(\mathcal{P}) \to G(\mathcal{P}')$, then the induced ring isomorphism of $\mathbb{Z}G(\mathcal{P})^{\mathrm{ab}}$ and $\mathbb{Z}G(\mathcal{P}')^{\mathrm{ab}}$ carries $\mathrm{A}_\lambda(\mathcal{P})$ to $\mathrm{A}_\lambda(\mathcal{P}')$ for all λ. This is proved by considering the effect on the Alexander matrix of Tietze transformations of presentations; see [24], [31].

Now, if F and F' are free $\mathbb{Z}G(\mathcal{P})$-modules with bases $\boldsymbol{x}$ and $\boldsymbol{r}$ respectively, then the Alexander matrix determines a map $\partial : F' \to F$, and it turns out that we have an exact sequence

$$F' \xrightarrow{\partial} F \xrightarrow{\varepsilon} \mathrm{I}G(\mathcal{P}) \to 0, \tag{1.2}$$

where $\mathrm{I}G(\mathcal{P})$ is the augmentation ideal of $G(\mathcal{P})$. Consequently, the Alexander ideals of G are akin to some sort of Fitting ideals for the module $\mathrm{I}G(\mathcal{P})$. Note, however, that we start in a non-commutative setting, and then pass to a commutative setting, essentially by applying $\mathbb{Z}G(\mathcal{P})^{\mathrm{ab}} \otimes_{\mathbb{Z}G(\mathcal{P})} -$ to (1.2).

There are, in fact, higher-dimensional Fitting ideals [62]. Let M be a module of type FP_n over the commutative ring C. Thus, there is a partial free resolution

$$\mathcal{F}: \quad F_{n+1} \xrightarrow{\partial_{n+1}} F_n \to \cdots \to F_1 \to F_0 \to M \to 0,$$

with $r_i = \mathrm{rk}_C\, F_i$ finite for $i = 0, \ldots, n$. Then, setting $\overrightarrow{\chi}_n(\mathcal{F}) = r_n - r_{n-1} + \cdots + (-1)^n r_0$, and letting D_n be the matrix of ∂_{n+1} with respect to a choice of bases for F_{n+1} and F_n,

$$\mathrm{Fit}^n_\lambda(M) = \begin{cases} C, & \lambda \geq \overrightarrow{\chi}_n(\mathcal{F}), \\ \text{the ideal generated} & \\ \text{by all } (\overrightarrow{\chi}_n(\mathcal{F}) - \lambda) \times (\overrightarrow{\chi}_n(\mathcal{F}) - \lambda) & \\ \text{subdeterminants of } D_n, & \lambda < \overrightarrow{\chi}_n(\mathcal{F}). \end{cases}$$

Again, these are invariants of M.

Our aim in this paper is to begin the development of a theory of higher-dimensional Alexander ideals for groups. This theory can also be extended to monoids, and we will say a little about this at the end of the paper.

We begin in Section 3 by discussing chains of ideals for (non-negative) free chain complexes. If $\mathcal{F}$ is such a complex of type FP_n over a (not necessarily commutative) ring R, then, for any representation $\rho : R \to \mathrm{Mat}_k(C)$ with C commutative, we have a chain $\mathrm{E}^\rho_n(\mathcal{F})$ of ideals in C. We introduce the important notion of Tietze transformations of chain complexes, and show that, if $\mathcal{F}'$ is Tietze equivalent to $\mathcal{F}$, then $\mathrm{E}^\rho_n(\mathcal{F}) = \mathrm{E}^\rho_n(\mathcal{F}')$. Other basic properties of these chains are discussed. We also introduce the important concepts of E-triviality and E-linkage. A complex $\mathcal{F}$ is $\mathrm{E}^\rho[m,n]$-trivial if the chains $\mathrm{E}^\rho_i(\mathcal{F})$ $(m \leq i \leq n)$ are trivial in a particular sense. The $\mathrm{E}^\rho[m,n]$-linked condition is a weakening of $\mathrm{E}^\rho[m,n]$-triviality.

In Section 4, we define for any R-module M of type FP_n the chain $\mathrm{E}^\rho_n(M)$ by considering free resolutions of M of type FP_n. Since any two such resolutions are Tietze equivalent, these chains are well defined. Properties of these chains follow from properties established for arbitrary chain complexes in Section 3. We also have the notions of $\mathrm{E}^\rho[m,n]$-trivial and $\mathrm{E}^\rho[m,n]$-linked modules. These properties behave very well with respect to short exact sequences. We discuss projective modules, and prove that any finitely generated projective R-module is $\mathrm{E}^\rho[0,\infty]$-trivial for all representations $\rho : R \to \mathrm{Mat}_k(C)$ with C indecomposable. This result is intimately connected with work of Hattori [35] and Stallings [60] and others on ranks of projective modules (the Hattori–Stallings rank).

In Section 5, we come to our main focus, the definition and theory of higher-dimensional Alexander ideals for groups. Let K be a commutative ring. For any group G, we then have the group ring KG. Regarding K as a KG-module ${}_GK$ with trivial G-action, we say that G is of type FP_n over K if the module

${}_GK$ is. In this case, for any $\rho : KG \to \mathrm{Mat}_k(C)$, we have a well-defined chain of ideals $\mathrm{E}^\rho_n(G) = \mathrm{E}^\rho_n({}_GK)$. This chain is an invariant of the group G. However, from a group-theoretic point of view we really want chains which are invariants of the *isomorphism type* of G. To obtain such chains, we must restrict ourselves to representations ρ which are 'canonical' in some sense. For simplicity, we concentrate on one-dimensional representations τ^T, arising from what we call an abelianising function ^{T}. An example of such a representation is $\tau^{\mathrm{ab}}_G : KG \to KG^{\mathrm{ab}}$. For any abelianising function, we define $\mathrm{E}^T_n(G, K)$ to be $\mathrm{E}^{\tau^T_G}_n(G)$, for G of type FP_n over K. If $\phi : H \to G$ is a group isomorphism, then there will be a group isomorphism $\phi^T : H^T \to G^T$, and the extension of this to a ring isomorphism $KH^T \to KG^T$ will carry the chain $\mathrm{E}^T_n(H, K)$ to the chain $\mathrm{E}^T_n(G, K)$. In this sense then, $\mathrm{E}^T_n(-, K)$ is a group invariant.

The most important coefficient ring is $K = \mathbb{Z}$, thanks to a 'universal coefficient lemma' in Section 5.1. In particular, $\mathrm{E}^{\mathrm{ab}}_1(G, \mathbb{Z})$ is the chain of classical Alexander ideals, and, in general, the chains $\mathrm{E}^{\mathrm{ab}}_n(G, \mathbb{Z})$ are the higher-dimension analogues of these. The chain $\mathrm{E}^{\mathrm{ab}}_2(G, \mathbb{Z})$ is particularly amenable to computation, using the theory of spherical pictures over group presentations (Section 5.2).

We give some examples of E-ideals in Sections 5.2– 5.6.

We say that a group G is $\mathrm{E}^T[m, n]$-trivial over K if ${}_GK$ is $\mathrm{E}^{\tau^T_G}[m, n]$-trivial (in other words, the chains $\mathrm{E}^T_i(G, K)$ $(m \le i \le n)$ satisfy the appropriate triviality conditions), and we define $\mathrm{E}^T[m, n]$-linked groups similarly. These classes of groups are rather interesting, and some results concerning them are given in Sections 5.5, 5.6 and 5.8. There is a connection between E-triviality and Serre's question of whether groups of type FP are of type FL. This is discussed in Section 5.7.

It turns out that, for a module M over a ring R, the elements in the centre of R which annihilate M have an influence on the chains $\mathrm{E}^\rho_n(M)$ (Section 4.1). In the context of groups, this translates to considering finite conjugacy classes of elements of G, and leads to group-theoretic results about the behaviour of such conjugacy classes. These results have a similar flavour to Gottlieb's theorem that a group G of type FL with $\chi(G) \neq 0$ has trivial centre.

As well as the classical Alexander ideals, there are also the classical Alexander *polynomials*, which can be derived from the Alexander ideals [1], [24]. We can similarly obtain higher-dimensional Alexander polynomials, which we consider in Section 5.10.

Although we have concentrated here on one-dimensional representations of groups, the ideals $\mathrm{E}^\rho_n(G)$ for general representations $\rho : KG \to \mathrm{Mat}_k(C)$ are also of use. We intend to discuss this in more detail in future work. In the present paper, however, we content ourselves with showing an intimate connection with ideas of R. Swan, M. Lustig and others on minimality of resolutions in Section 5.11.

We finish the paper, in Section 6, by giving a brief outline of how the theory of E-ideals can be extended from groups to monoids. Further work on this direction will appear in [25].

We finish this introduction with a 'philosophical' remark, namely, that the theory of E-ideals is somehow 'transverse' to homology theory. To compute the nth chain $\mathrm{E}_n^\rho(M)$ for an R-module M and a representation $\rho : R \mapsto \mathrm{Mat}_k(C)$, we in essence use the section

$$F_{n+1} \xrightarrow{\partial_{n+1}} F_n$$

of a free resolution $\mathcal{F}$ of M, and then apply the tensor $C^k \otimes_R -$. On the other hand, to compute the nth Tor-group $\mathrm{Tor}_n^R(C^k, M)$ we consider the section

$$\xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n}$$

of $\mathcal{F}$ and apply $C^k \otimes_R -$. When C is a principal ideal domain (and M is of type FP_n) the two theories give the same amount of information (but presented in different ways); see Theorem 3.5 in Section 3.2. However, in general, the two theories give complementary information.

We mention that in the context of groups, the well-known fact that the Alexander polynomial of the fundamental group of a knot complement ('knot group') evaluated at 1 is ± 1, is just a manifestation of the relationship mentioned above between homology and E-ideals over a p.i.d. In this case, the p.i.d. is $\mathbb{Z}$ and the chains $\mathrm{E}_0^{\mathrm{triv}}(G, \mathbb{Z})$, $\mathrm{E}_1^{\mathrm{triv}}(G, \mathbb{Z})$ (where $\mathrm{triv} : \mathbb{Z}G \to \mathbb{Z}; g \mapsto 1$ is the trivial abelianising function) carry the same information as the integral homology groups $\mathrm{H}_0(G)$, $\mathrm{H}_1(G)$ (see the remark at the end of Section 5.3).

Finally, we emphasise that the new invariants are, in general, quite distinct from the higher-dimensional Fitting ideals. The manner in which they differ is elucidated in remarks at the beginning of Section 4.1 and at the end of Section 5.1.

2 Notation

All rings will have an identity, 1, and all ring homomorphisms will preserve the identity. Modules will be left modules, unless otherwise stated.

Let R be a ring.

The *centre* $\{x : xy = yx \text{ for all } y \in R\}$ of R will be denoted by $Z(R)$. For an R-module B, the *annihilator* $\{x : xb = 0 \text{ for all } b \in B\}$ will be denoted by $\mathrm{Ann}(B)$.

A *non-negative chain complex* (over R) is a collection $\mathcal{B} = (B_i, \partial_i)_{i \geq 0}$ of R-modules B_i and module homomorphisms $\partial_i : B_i \to B_{i-1}$ (we take ∂_0 to be the zero homomorphism $B_0 \to 0$) such that $\partial_i \partial_{i+1} = 0$ for all i. If all the modules are free, then we call $\mathcal{B}$ a non-negative *free* chain complex and, if each B_i is projective, we call $\mathcal{B}$ a non-negative *projective* chain complex. If, for some l, $B_i = 0$ for all $i > l$, then we say that $\mathcal{B}$ is *finite*, and write it as $(B_i, \partial_i)_{i=0}^l$. Since all chain complexes here will be non-negative, we will quite often omit the adjective 'non-negative'. Also, we will sometimes omit reference to the boundary maps ∂_i and just write $\mathcal{B} = (B_i)_{i \geq 0}$.

The ith homology, $\mathrm{H}_i(\mathcal{B})$, of $\mathcal{B}$ is $\ker\partial_i/\mathrm{im}\,\partial_{i+1}$; $\mathcal{B}$ is *exact* in dimension i if, and only if, $\mathrm{H}_i(\mathcal{B}) = 0$.

For $m \geq 0$, we define the mth *shift*, $\mathcal{B}^{[m]}$, of $\mathcal{B}$ to be the chain complex with $B_i^{[m]} = B_{i+m}$ $(i \geq 0)$, $\partial_i^{[m]} = \partial_{i+m}$ $(i > 0)$ (and, of course, $\partial_0^{[m]} = 0$).

A *free resolution* of a module M is a free chain complex $\mathcal{F} = (F_i, \partial_i)_{i\geq 0}$ which is exact in positive dimensions, and such that there is a surjective homomorphism $\varepsilon : F_0 \to M$ with $\mathrm{im}\,\partial_1 = \ker\varepsilon$. We write $\mathcal{F} \xrightarrow{\varepsilon} M \to 0$. Similarly, we can have a *projective* resolution of M.

Let C be a commutative ring.

By a *representation* of R (over C) we mean a ring homomorphism ρ from R to the matrix ring $\mathrm{Mat}_k(C)$ of all $k \times k$ matrices over C. We emphasise that ρ must send the identity of R to the identity matrix $I_k \in \mathrm{Mat}_k(C)$. The *dimension*, $\dim(\rho)$, of ρ is k.

A *non-zero-divisor* in C is an element u which is not a zero-divisor (that is, $uv = 0$, for $v \in C$ implies that $v = 0$).

An *ascending chain of ideals* in C is a family $\{J_\kappa\}_{\kappa\in\mathbb{Z}}$ of ideals of C such that $J_\kappa \subseteq J_{\kappa+1}$ $(\kappa \in \mathbb{Z})$. Similarly, $\{J_\kappa\}_{\kappa\in\mathbb{Z}}$ is a *descending chain of ideals* if $J_{\kappa+1} \subseteq J_\kappa$ $(\kappa \in \mathbb{Z})$. If $J = \{J_\lambda\}_{\lambda\in\mathbb{Z}}$, $J' = \{J'_\lambda\}_{\lambda\in\mathbb{Z}}$ are two chains (either both ascending or both descending), then for $m \in \mathbb{Z}$ the *convolution of J and J', suspended by m*, is the chain whose λth ideal is

$$\sum_{\kappa\in\mathbb{Z}} J_{m+\kappa} J'_{\lambda-\kappa}$$

We denote this chain by $J *^{[m]} J'$. If $m = 0$, then we omit the superscript $^{[m]}$. More generally, we can consider the convolution product

$$\mathop{*}_{u\in\boldsymbol{u}}{}^{[m]} J_u$$

of a family of chains J_u $(u \in \boldsymbol{u})$.

For an $l \times m$ matrix X over C we have the descending chain $J(X) = \{J_\lambda(X)\}_{\lambda\in\mathbb{Z}}$ of *elementary ideals*, where

$$J_\lambda(X) = \begin{cases} C, & \lambda \leq 0, \\ \text{the ideal generated by all} & \\ \lambda \times \lambda \text{ subdeterminants of } X, & 0 < \lambda \leq \min\{l, m\}, \\ 0, & \lambda > \min\{l, m\}. \end{cases}$$

We allow l, but not m, to be ∞.

If X' is another $l \times m$ matrix over C and if each row (respectively, column) of X' is a linear combination of rows (respectively, columns) of X, then $J_\lambda(X') \subseteq J_\lambda(X)$ for all λ. If X has the form

$$\begin{bmatrix} Y & 0 \\ Z & Y' \end{bmatrix},$$

then

$$J(X) \supseteq J(Y) * J(Y'),$$

with equality if $Z = 0$. If Y' is in fact an identity matrix, say $Y' = I_k$, then

$$J_\lambda(X) = J_{\lambda-k}(Y)$$

for all $\lambda \in \mathbb{Z}$.

For further discussion on elementary ideals see [24].

3 Free chain complexes

3.1 Ideals and their basic properties

Let R be a ring with the invariance of rank property (that is, the rank of a free R-module is well defined). Consider a non-negative free chain complex $\mathcal{F} = (F_i, \partial_i)_{i\geq 0}$, where F_i is a free R-module of rank r_i. The chain complex is said to be of *type* FP_n $(n \geq 0)$ if $r_i < \infty$ for $i = 0, 1, \ldots, n$. If this is the case, we define the *directed partial Euler characteristic*, $\overrightarrow{\chi} = \overrightarrow{\chi}_n(\mathcal{F})$, of $\mathcal{F}$ to be $r_n - r_{n-1} + \cdots + (-1)^n r_0$. The complex is said to be of *type* FP_∞ if $r_i < \infty$ for all i.

Fix a representation $\rho : R \to \mathrm{Mat}_k(C)$, and suppose that $\mathcal{F}$ is of type FP_n. We then obtain an ascending chain

$$\mathrm{E}^\rho_n(\mathcal{F}) = \left(\mathrm{E}^\rho_{n,\lambda}(\mathcal{F})\right)_{\lambda\in\mathbb{Z}}$$

of ideals in C as follows. Choose ordered bases $\boldsymbol{z}_{n+1}, \boldsymbol{z}_n$ for F_{n+1}, F_n respectively, and consider the $r_{n+1} \times r_n$ matrix D of ∂_{n+1} with respect to these bases. Let D^ρ be the $kr_{n+1} \times kr_n$ matrix over C obtained by applying the map ρ to each entry of D. Then

$$\mathrm{E}^\rho_{n,\lambda}(\mathcal{F}) = J_{k\overrightarrow{\chi}_n - \lambda}(D^\rho).$$

It follows from a remark in Section 2 that $\mathrm{E}^\rho_{n,\lambda}(\mathcal{F})$ is independent of the choice of ordered bases $\boldsymbol{z}_{n+1}, \boldsymbol{z}_n$.

Note that, if we regard C^k as a (C, R)-bimodule with R acting via ρ, then $C^k \otimes_R \mathcal{F}$ is a chain complex of free C-modules and

$$\mathrm{E}^\rho_n(\mathcal{F}) = \mathrm{E}^{1_C}_n(C^k \otimes_R \mathcal{F}), \tag{3.1}$$

where $1_C : C \to C$ is the identity homomorphism.

Also, if $\alpha : C \to C'$ is a ring homomorphism (with C' commutative), then we have the induced homomorphism (also denoted α) from $\mathrm{Mat}_k(C)$ to $\mathrm{Mat}_k(C')$, and thus the composition

$$\alpha\rho : R \to \mathrm{Mat}_k(C').$$

It is easy to see that, for all $\lambda \in \mathbb{Z}$,

$$\mathrm{E}^{\alpha\rho}_{n,\lambda}(\mathcal{F}) = \left(\alpha\, \mathrm{E}^\rho_{n,\lambda}(\mathcal{F})\right), \tag{3.2}$$

where $(\cdot)$ denotes the 'ideal generated by' (these brackets are, of course, superfluous when α is surjective). We abbreviate this to

$$\mathrm{E}_n^{\alpha\rho}(\mathcal{F}) = \alpha\,\mathrm{E}_n^{\rho}(\mathcal{F}).$$

If a complex $\mathcal{F}$ is of type FP_{m-1}, then, for $n \geq m$, $\mathcal{F}$ is of type FP_n if, and only if, the shift $\mathcal{F}^{[m]}$ is of type FP_{n-m}, and we have the dimension shifting formula

$$\mathrm{E}^{\rho}_{n-m,\lambda}(\mathcal{F}^{[m]}) = \mathrm{E}^{\rho}_{n,\lambda+(-1)^{n-m-1}k\overrightarrow{\chi}_{m-1}(\mathcal{F})}(\mathcal{F}) \qquad (\lambda \in \mathbb{Z}). \tag{3.3}$$

There are three important properties of these chains of ideals. The first is invariance under what we call a Tietze transformation, the second is a relationship between successive chains $\mathrm{E}^{\rho}_{n-1}(\mathcal{F})$, $\mathrm{E}^{\rho}_{n}(\mathcal{F})$, and the third is a relationship between successive terms $\mathrm{E}^{\rho}_{n,\lambda-1}(\mathcal{F})$, $\mathrm{E}^{\rho}_{n,\lambda}(\mathcal{F})$ of a given chain $\mathrm{E}^{\rho}_{n}(\mathcal{F})$.

For $j \geq 0$, we define a *Tietze transformation of rank j* on a free chain complex $\mathcal{F}$ to be an operation as follows. Let F be a free R-module, and let $\phi : F \to F_j$ be an R-homomorphism. Then replace the part

$$\cdots \xrightarrow{\partial_{j+2}} F_{j+1} \xrightarrow{\partial_{j+1}} F_j \xrightarrow{\partial_j} \cdots$$

of $\mathcal{F}$ by

$$\cdots \xrightarrow{\partial'_{j+2}} F_{j+1} \oplus F \xrightarrow{\partial'_{j+1}} F_j \oplus F \xrightarrow{\partial'_j} \cdots,$$

where, for $f_i \in F_i$, $f \in F$,

$$\begin{aligned}
\partial'_j(f_j, f) &= \partial_j(f_j + \phi(f)),\\
\partial'_{j+1}(f_{j+1}, f) &= (\partial_{j+1}(f_{j+1}) - \phi(f), f),\\
\partial'_{j+2}(f_{j+2}) &= (\partial_{j+2}(f_{j+2}), 0).
\end{aligned}$$

This gives a new free chain complex $\mathcal{F}'$. The Tietze transformation is said to be *finitary* if F is of finite rank. It is easily shown that $\mathcal{F}'$ has the same homology as $\mathcal{F}$.

Remark. We can clearly define Tietze transformations on non-negative *projective* chain complexes in an analogous way. These will be needed in Section 4.3.

Theorem 3.1. *Let $\mathcal{F}$ be of type FP_n. If $\mathcal{F}'$ is obtained from $\mathcal{F}$ by a Tietze transformation of rank j (finitary if $j \leq n$), then $\mathcal{F}$ is of type FP_n and* $\mathrm{E}^{\rho}_{n}(\mathcal{F}') = \mathrm{E}^{\rho}_{n}(\mathcal{F})$.

Proof. Notice that if D_{j-1}, D_j, D_{j+1} are the matrices for $\partial_j, \partial_{j+1}, \partial_{j+2}$ with respect to chosen bases, and if we choose a basis for the free module F, then the matrices D'_{j-1}, D'_j, D'_{j+1} for $\partial'_j, \partial'_{j+1}, \partial'_{j+2}$ with respect to the induced bases will be of the following forms:

$$D'_{j-1} = \begin{bmatrix} D_{j-1} \\ Y \end{bmatrix}, \qquad D'_j = \begin{bmatrix} D_j & 0 \\ Z & I_r \end{bmatrix}, \qquad D'_{j+1} = \begin{bmatrix} D_{j+1} & 0 \end{bmatrix}.$$

Here, $r = \operatorname{rank} F$, Y is an $r \times r_{j-1}$ matrix whose rows are linear combinations of the rows of D_{j-1}, and Z is some $r \times r_j$ matrix. The matrices for the remaining boundary maps are unchanged.

If $n < j-1$ or if $n > j+1$ and r is finite, then $\overrightarrow{\chi}_n(\mathcal{F}') = \overrightarrow{\chi}_n(\mathcal{F})$ and $\partial'_{n+1} = \partial_{n+1}$.

If $n = j-1$, then $\overrightarrow{\chi}_n(\mathcal{F}') = \overrightarrow{\chi}_n(\mathcal{F})$ and, from the form of D'_{j-1}, we deduce that $J_\kappa(D'^{\rho}_{j-1}) = J_\kappa(D^{\rho}_{j-1})$ $(\kappa \in \mathbb{Z})$.

If $n = j$, then (assuming that r is finite) $\overrightarrow{\chi}_n(\mathcal{F}') = r + \overrightarrow{\chi}_n(\mathcal{F})$ and, from the form of D'_j, we deduce that $J_{\kappa+rk}(D'^{\rho}_j) = J_\kappa(D^{\rho}_j)$ $(\kappa \in \mathbb{Z})$.

Finally, if $n = j+1$, then (for r finite) $\overrightarrow{\chi}_n(\mathcal{F}') = \overrightarrow{\chi}_n(\mathcal{F})$ and, from the form of D'_{j+1}, $J_\kappa(D'^{\rho}_{j+1}) = J_\kappa(D^{\rho}_{j+1})$ $(\kappa \in \mathbb{Z})$.

We can thus conclude that, in all cases, $\mathrm{E}^{\rho}_{n,\lambda}(\mathcal{F}') = \mathrm{E}^{\rho}_{n,\lambda}(\mathcal{F})$ $(\lambda \in \mathbb{Z})$, as required.

Theorem 3.2. *Let $\mathcal{F} = (F_i, \partial_i)_{i \geq 0}$ be of type FP_n $(n > 0)$. Then, for $\kappa + \lambda < 0$,*

$$\mathrm{E}^{\rho}_{n,\kappa}(\mathcal{F})\, \mathrm{E}^{\rho}_{n-1,\lambda}(\mathcal{F}) = 0.$$

Proof. Consider the non-negative free chain complex $\widehat{\mathcal{F}} = (\widehat{F}_i, \widehat{\partial}_i)_{i \geq 0}$, where $\widehat{F}_0 = F_0$, $\widehat{F}_i = F_i \oplus F_{i-1}$ $(i > 0)$, and

$$\begin{aligned} \widehat{\partial}_1(f_1, f_0) &= \partial_1(f_1) + f_0, \\ \widehat{\partial}_i(f_i, f_{i-1}) &= (\partial_i(f_i) + f_{i-1}, -\partial_{i-1}(f_{i-1})) \qquad (i > 1) \end{aligned}$$

(this is the mapping cylinder of the identity chain map $\mathcal{F} \to \mathcal{F}$). If D_n, D_{n-1} are the matrices for ∂_{n+1}, ∂_n with respect to chosen bases for F_{n+2}, F_{n+1}, F_n, then the matrix $\widehat{D}_n$ for $\widehat{\partial}_{n+1}$ with respect to the induced bases will have the form

$$\widehat{D}_n = \begin{bmatrix} D_n & 0 \\ I_{r_n} & -D_{n-1} \end{bmatrix}.$$

Thus, for all $p, q \in \mathbb{Z}$,

$$J_{p+q}\left(\widehat{D}^{\rho}_n\right) \supseteq J_p\left(D^{\rho}_n\right) J_q\left(D^{\rho}_{n-1}\right).$$

Hence, for all $\kappa, \lambda \in \mathbb{Z}$,

$$\begin{aligned} \mathrm{E}^{\rho}_{n,\kappa+\lambda}(\widehat{\mathcal{F}}) &= J_{k(\overrightarrow{\chi}_n + \overrightarrow{\chi}_{n-1}) - (\kappa+\lambda)}\left(\widehat{D}^{\rho}_n\right) \\ &\supseteq J_{k\overrightarrow{\chi}_n - \kappa}\left(D^{\rho}_n\right) J_{k\overrightarrow{\chi}_{n-1} - \lambda}\left(D^{\rho}_{n-1}\right) \\ &= \mathrm{E}^{\rho}_{n,\kappa}(\mathcal{F})\, \mathrm{E}^{\rho}_{n-1,\lambda}(\mathcal{F}). \end{aligned}$$

But $\mathcal{F}$ can be obtained from the zero chain complex 0 by a sequence of Tietze transformations (finitary in dimensions lower than n), so $\mathrm{E}^{\rho}_n(\widehat{\mathcal{F}}) = \mathrm{E}^{\rho}_n(0)$ by Theorem 3.1. Since

$$\mathrm{E}^{\rho}_{n,\lambda}(0) = \begin{cases} C, & \lambda \geq 0, \\ 0, & \lambda < 0, \end{cases}$$

the result follows.

Remark. In general, let $\mathcal{F}' = (F'_i, \partial'_i)_{i\geq 0}$, $\mathcal{F} = (F_i, \partial_i)_{i\geq 0}$ be free chain complexes, let $\theta : \mathcal{F}' \to \mathcal{F}$ be a chain map and let $\widehat{\mathcal{F}} = (\widehat{F}_i, \widehat{\partial}_i)_{i\geq 0}$ be the corresponding mapping cylinder. Thus, $\widehat{F}_0 = F_0$, $\widehat{F}_i = F_i \oplus F'_{i-1}$ $(i > 0)$, and

$$\begin{aligned}
\widehat{\partial}_1(f_1, f'_0) &= \partial_1(f_1) + \theta_0(f'_0), \\
\widehat{\partial}_i(f_i, f'_{i-1}) &= (\partial_i(f_i) + \theta_{i-1}(f'_{i-1}), -\partial_{i-1}(f'_{i-1})) \qquad (i > 1).
\end{aligned}$$

If D_n, D'_{n-1} are the matrices of ∂_{n+1}, ∂'_n with respect to chosen bases, then the matrix for $\widehat{\partial}_{n+1}$ will be of the form

$$\widehat{D}_n = \begin{bmatrix} D_n & 0 \\ \Theta_n & -D'_{n-1} \end{bmatrix},$$

where Θ_n is the matrix of the nth homomorphism θ_n of the chain map θ. An argument like that in the proof of Theorem 3.2 then gives the following proposition.

Proposition 3.1. *If $\mathcal{F}'$ is of type FP_{n-1} and $\mathcal{F}$ is of type FP_n, then $\mathrm{E}^\rho_n(\widehat{\mathcal{F}}) \supseteq \mathrm{E}^\rho_{n-1}(\mathcal{F}') * \mathrm{E}^\rho_n(\mathcal{F})$.*

For the third property, suppose that $\zeta \in Z(R)$. Then we have a chain map $\zeta \cdot 1_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$, whose ith homomorphism, $\zeta \cdot 1_{F_i} : F_i \to F_i$, is multiplication by ζ. This chain map is *homotopic to zero* if there are homomorphisms $\psi : F_{i-1} \to F_i$ $(i > 0)$ such that

$$\begin{aligned}
\zeta \cdot 1_{F_0} &= \partial_1 \psi_1, \\
\zeta \cdot 1_{F_i} &= \partial_{i+1}\psi_{i+1} + \psi_i \partial_i \qquad (i > 0).
\end{aligned}$$

Now, if $\rho : R \to \mathrm{Mat}_k(C)$ is *surjective*, then $\rho(\zeta) \in Z(\mathrm{Mat}_k(C))$, so $\rho(\zeta) = \overline{\zeta} I_k$ for some $\overline{\zeta} \in C$. Then

$$1_{C^k} \otimes \zeta \cdot 1_{\mathcal{F}} : C^k \otimes_R \mathcal{F} \to C^k \otimes_R \mathcal{F}$$

is equal to $\overline{\zeta} \cdot 1_{C^k} \otimes_R 1_{\mathcal{F}}$, and if $\zeta \cdot 1_{\mathcal{F}}$ is homotopic to zero, then so is $\overline{\zeta} \cdot 1_{C^k} \otimes_R 1_{\mathcal{F}}$. Applying (3.1), we deduce from [29, Theorem 1] the following result.

Theorem 3.3. *Suppose that $\mathcal{F}$ is of type FP_n. If $\zeta \cdot 1_{\mathcal{F}}$ is homotopic to zero and ρ is surjective, then, for $\lambda \in \mathbb{Z}$,*

$$\lambda \overline{\zeta}\, \mathrm{E}_{n,\lambda}(\mathcal{F}) \subseteq \mathrm{E}^\rho_{n,\lambda-1}(\mathcal{F}).$$

3.2 Derived invariants

For a free chain complex $\mathcal{F}$ of type FP_n, we define

$$\begin{aligned}
\nu^\rho_n = \nu^\rho_n(\mathcal{F}) &= \min\left\{\lambda \in \mathbb{Z} : \mathrm{E}^\rho_{n,\lambda}(\mathcal{F}) = C\right\}, \\
\delta^\rho_n = \delta^\rho_n(\mathcal{F}) &= \min\left\{\lambda \in \mathbb{Z} : \mathrm{E}^\rho_{n,\lambda}(\mathcal{F}) \neq 0\right\}.
\end{aligned}$$

By definition,

$$-k\overrightarrow{\chi}_{n\pm 1}(\mathcal{F}) \le \delta_n^\rho(\mathcal{F}) \le \nu_n^\rho(\mathcal{F}) \le k\overrightarrow{\chi}_n(\mathcal{F}).$$

If $\alpha : C \to C'$ is a homomorphism of commutative rings, we then have, using (3.2),

$$\delta_n^\rho(\mathcal{F}) \le \delta_n^{\alpha\rho}(\mathcal{F}) \le \nu_n^{\alpha\rho}(\mathcal{F}) \le \nu_n^\rho(\mathcal{F}). \tag{3.4}$$

Theorem 3.4. *Let $\mathcal{F}$ be of type FP_n $(n > 0)$.*

(i) *If one of the ideals $\mathrm{E}^\rho_{n-1,\delta^\rho_{n-1}}(\mathcal{F})$, $\mathrm{E}^\rho_{n,\delta^\rho_n}(\mathcal{F})$ contains a non-zero-divisor, then*

$$\delta^\rho_{n-1} + \delta^\rho_n \ge 0. \tag{3.5}$$

In particular, (3.5) *holds if C is an integral domain.*

(ii) *If C has a quotient which is an integral domain, then*

$$\nu^\rho_{n-1} + \nu^\rho_n \ge 0.$$

Proof. (i) If one of the ideals contains a non-zero-divisor, then the product of the two ideals cannot be zero, so (3.5) follows by Theorem 3.2.

(ii) If α is the quotient map, then we have

$$\begin{aligned} 0 &\le \delta^{\alpha\rho}_{n-1} + \delta^{\alpha\rho}_n && \text{(by (i))} \\ &\le \nu^\rho_{n-1} + \nu^\rho_n && \text{(by (3.4))}, \end{aligned}$$

as required.

It will be convenient to set

$$\begin{aligned} \mathrm{Q}^\rho_n(\mathcal{F}) &= \bigoplus_{\lambda=\delta^\rho_n}^{\nu^\rho_n - 1} \frac{\mathrm{E}^\rho_{n,\lambda+1}(\mathcal{F})}{\mathrm{E}^\rho_{n,\lambda}(\mathcal{F})} \left(= \bigoplus_{\lambda \ge \delta^\rho_n} \frac{\mathrm{E}^\rho_{n,\lambda+1}(\mathcal{F})}{\mathrm{E}^\rho_{n,\lambda}(\mathcal{F})} \right), \\ \mathrm{L}^\rho_n(\mathcal{F}) &= C^{\delta^\rho_{n-1}+\delta^\rho_n} \oplus \mathrm{Q}^\rho_n(\mathcal{F}) \qquad \text{(whenever (3.5) holds).} \end{aligned}$$

We take $\delta^\rho_{-1} = 0$, always.

Theorem 3.5. *If C is a principal ideal domain (p.i.d.), then*

$$\mathrm{L}^\rho_n(\mathcal{F}) \cong \mathrm{H}_n(C^k \otimes_R \mathcal{F}).$$

Proof. We can write $\mathrm{H}_n(C^k \otimes_R \mathcal{F})$ in the form

$$C^{q_n} \oplus \left(\bigoplus_{j=1}^{p_n} \frac{C}{(\eta_j)} \right)$$

for some uniquely determined $p_n, q_n \ge 0$ and some non-zero, non-units $\eta_j \in C$ (uniquely determined up to multiplication by units) such that $\eta_j | \eta_{j+1}$. Similarly, each $\mathrm{H}_i(C^k \otimes_R \mathcal{F})$ can be written in such a form; in particular, we have the numbers $q_i = \mathrm{rk}_C \mathrm{H}_n(C^k \otimes_R \mathcal{F})$ $(0 \leq i < n)$.

Set $\overline{F}_i = C^k \otimes_R F_i$, $\overline{\partial}_i = 1_{C^k} \otimes \partial_i$, $B_i = \mathrm{im}\,\overline{\partial}_{i+1}$ and $Z_i = \ker \overline{\partial}_i$. Since B_i and Z_i are submodules of the C-free module $\overline{F}_i$, they are both free. The short exact sequence

$$0 \to Z_i \to \overline{F}_i \xrightarrow{\overline{\partial}_i} B_{i-1} \to 0$$

then splits, so $\overline{F}_i = Z_i \oplus \widetilde{B}_{i-1}$, where $\widetilde{B}_{i-1}$ is a lift of B_{i-1} via $\overline{\partial}_i$ (take $\widetilde{B}_{-1} = 0$). We also have the short exact sequence

$$0 \to B_i \to Z_i \to \mathrm{H}_i(C^k \otimes_R \mathcal{F}) \to 0.$$

Considering the C-ranks of the modules in this sequence, we have

$$\mathrm{rk}_C Z_i = q_i + b_i,$$

where $b_i = \mathrm{rk}_C(B_i)$, and so $Z_i \cong B_i \oplus C^{q_i}$. Consequently,

$$\overline{F}_i \cong B_i \oplus C^{q_i} \oplus \widetilde{B}_{i-1}$$

and

$$\mathrm{rk}_C \overline{F}_i = b_i + q_i + b_{i-1}.$$

Choosing ordered bases for B_{n+1}, $\widetilde{B}_n$, B_n and $\widetilde{B}_{n-1}$ and for the free parts of $\mathrm{H}_i(C^k \otimes_R \mathcal{F})$ $(i = n, n+1)$ induces ordered bases for $\overline{F}_{n+1}$ and $\overline{F}_n$, with respect to which $\overline{\partial}_{n+1}$ has the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Delta_n & 0 & 0 \end{bmatrix} \tag{3.6}$$

(with one less column of zero matrices when $n = 0$). Here Δ_n is some $b_n \times b_n$ matrix over C. Because C is a p.i.d., we can choose bases for $\widetilde{B}_n$ and B_n such that Δ_n is a diagonal matrix. But this matrix is a presentation matrix for the torsion part of $\mathrm{H}_n(C^k \otimes \mathcal{F})$, we must have

$$\Delta_n = \mathrm{diag}(1, \ldots, 1, \eta_1, \ldots, \eta_{p_n}).$$

Now,

$$k\overrightarrow{\chi}_n(\mathcal{F}) = b_n + \delta,$$

where $\delta = q_n - q_{n-1} + \cdots + (-1)^n q_0$, and so

$$\begin{aligned} \mathrm{E}^{\rho}_{n,\lambda}(\mathcal{F}) &= J_{b_n+\delta-\lambda}(\Delta_n) \\ &= \begin{cases} C, & \lambda \geq p_n + \delta, \\ \left(\prod_{j=1}^{p_n+\delta-\lambda} \eta_j\right), & \delta \leq \lambda < p_n + \delta, \\ 0, & \lambda < \delta. \end{cases} \end{aligned}$$

Thus, since the η_j are non-zero and are not units, $\delta_n^\rho(\mathcal{F}) = \delta$ and $\nu_n^\rho(\mathcal{F}) = \delta + p_n$. Also, for $\delta \leq \lambda < \delta + p_n$,

$$\frac{\mathrm{E}^\rho_{n,\lambda+1}(\mathcal{F})}{\mathrm{E}^\rho_{n,\lambda}(\mathcal{F})} \cong \frac{C}{(\eta_{p_n+\delta-\lambda})},$$

so $\mathrm{Q}^\rho_n(\mathcal{F}) \cong \oplus_j C/(\eta_j)$ and the theorem follows.

3.3 E-triviality and E-linkage

Let $0 \leq m \leq n$. We will say that $\mathcal{F}$ is:

(i) $\mathrm{E}^\rho[m,n]$-*trivial* if $\nu_i^\rho = \delta_i^\rho$ $(m \leq i \leq n)$ and $\delta_i + \delta_{i+1} = 0$ $(m \leq i < n)$;

(ii) $\mathrm{E}^\rho[m,n]$-*linked* if $\mathrm{E}^\rho_{i,\delta_i^\rho}(\mathcal{F})$ contains a non-zero-divisor $(m \leq i \leq n)$ and $\delta_i + \delta_{i+1} = 0$ $(m \leq i < n)$.

We will say that $\mathcal{F}$ is $\mathrm{E}^\rho[m,\infty]$-*trivial* if it is $\mathrm{E}^\rho[m,n]$-trivial for all $n > m$, and that it is $\mathrm{E}^\rho[m,\infty]$-*linked* if it is $\mathrm{E}^\rho[m,n]$-linked for all $n > m$.

Note that, if $\mathcal{F}$ is $\mathrm{E}^\rho[m,n]$-trivial, then it is $\mathrm{E}^\rho[m,n]$-linked.

Clearly *a free chain complex* $\mathcal{F}$ *is* $\mathrm{E}^\rho[m,n]$*-trivial if, and only if,* $\mathrm{L}^\rho_m(\mathcal{F}) = \mathrm{Q}^\rho_m(\mathcal{F})$ *and* $\mathrm{L}^\rho_i(\mathcal{F}) = 0$ *for* $m < i \leq n$.

We obtain the next result using Theorem 3.5.

Proposition 3.2. *If C is a p.i.d., then*

(i) *$\mathrm{E}^\rho[m,n]$-triviality is the homological condition that $\mathrm{H}_m(C^k \otimes_R \mathcal{F})$ is torsion and $\mathrm{H}_i(C^k \otimes_R \mathcal{F}) = 0$ for $m < i \leq n$;*

(ii) *the $\mathrm{E}^\rho[m,n]$-linked property is the homological condition that $\mathrm{H}_i(C^k \otimes_R \mathcal{F})$ is torsion for $m < i \leq n$.*

4 Modules

4.1 Invariant ideals of modules

A module M is said to be *of type FP$_n$* if it has a free resolution which is of type FP$_n$. For an R-module M of type FP$_n$, we define $\mathrm{E}^\rho_n(M)$ to be $\mathrm{E}^\rho_n(\mathcal{F})$ for any free resolution $\mathcal{F} \xrightarrow{\varepsilon} M \to 0$ of M of type FP$_n$, and we define $\nu^\rho_n(M)$ to be $\nu^\rho_n(\mathcal{F})$ and $\delta^\rho_n(M)$ to be $\delta^\rho_n(\mathcal{F})$. These definitions are valid, since, if $\mathcal{F}' \xrightarrow{\varepsilon'} M \to 0$ is another free resolution of type FP$_n$ of M, then there are resolutions $\mathcal{F}^{(n+1)}$ and $\mathcal{F}'^{(n+1)}$ of type FP$_n$, which are obtained from $\mathcal{F}$ and $\mathcal{F}'$ respectively by a finite number of Tietze transformations (finitary in rank n and below), and which are identical in dimension $n+1$ and below. Theorem 3.1 then gives $\mathrm{E}^\rho_n(\mathcal{F}) = \mathrm{E}^\rho_n(\mathcal{F}^{(n+1)}) = \mathrm{E}^\rho_n(\mathcal{F}'^{(n+1)}) = \mathrm{E}^\rho_n(\mathcal{F}')$. For the first step in constructing these new resolutions, choose $\phi_0 : F'_0 \to F_0$ and $\psi_0 : F'_0 \to F_0$ such that $\varepsilon\phi_0 = \varepsilon'$ and $\varepsilon'\psi_0 = \varepsilon$. Then apply Tietze transformations of rank 0 to $\mathcal{F}$ and $\mathcal{F}'$ to obtain new resolutions

$$\mathcal{F}^{(1)}: \quad \cdots F_j \xrightarrow{\partial_j} F_{j-1} \to \cdots \xrightarrow{\partial_2^{(1)}} F_1 \oplus F_0' \xrightarrow{\partial_1^{(1)}} F_0 \oplus F_0' \xrightarrow{\varepsilon^{(1)}} M \to 0,$$
$$\mathcal{F}'^{(1)}: \quad \cdots F_j' \xrightarrow{\partial_j'} F_{j-1}' \to \cdots \xrightarrow{\partial_2'^{(1)}} F_0 \oplus F_1' \xrightarrow{\partial_1'^{(1)}} F_0 \oplus F_0' \xrightarrow{\varepsilon'^{(1)}} M \to 0,$$

with $\varepsilon^{(1)}(f_0, f_0') = \varepsilon(f_0 + \phi_0(f_0')) = \varepsilon(f_0) + \varepsilon'(f_0') = \varepsilon'^{(1)}(f_0, f_0')$. Repeat this step in dimension 1 by choosing $\phi_1 : F_0 \oplus F_1' \to F_1 \oplus F_0'$ and $\psi_1 : F_1 \oplus F_0' \to F_0 \oplus F_1'$ such that $\partial_1^{(1)}\phi_1 = \partial_1'^{(1)}$ and $\partial_1'^{(1)}\psi_1 = \partial_1^{(1)}$, and so on.

Remark. It should be noted that the chain $\mathrm{E}_n^{\rho}(M) = \mathrm{E}_n^{1_C}(C^k \otimes_R M)$ is not in general the same as the chain of n-dimensional Fitting ideals of $C^k \otimes_R M$. To calculate the latter, we take a free resolution of the C-module $C^k \otimes_R M$. However, to calculate the former we start with a free resolution $\mathcal{F}$ of the R-module M, and then pass to $C^k \otimes_R \mathcal{F}$. This is a chain complex of free C-modules, but is not in general a resolution of $C^k \otimes_R M$, since applying $C^k \otimes_R -$ will usually destroy exactness. See also the Remark at the end of Section 5.1.

We obtain the next result from Theorem 3.2.

Theorem 4.1. *For $\kappa + \lambda < 0$,*

$$\mathrm{E}_{n-1,\lambda}^{\rho}(M)\, \mathrm{E}_{n,\kappa}^{\rho}(M) = 0.$$

Remark. This was proved for commutative R in [62].

Suppose that $\zeta \in Z(R) \cap \mathrm{Ann}(M)$. Then, for any free resolution $\mathcal{F}$ of M, $\zeta \cdot 1_{\mathcal{F}}$ is homotopic to zero (see [20]), and so the next result follows from Theorem 3.3.

Theorem 4.2. *For $\zeta \in Z(R) \cap \mathrm{Ann}(M)$ and surjective ρ,*

$$\lambda\bar{\zeta}\, \mathrm{E}_{n,\lambda}^{\rho}(M) \subseteq \mathrm{E}_{n,\lambda-1}^{\rho}(M).$$

We define $\mathrm{Q}_n^{\rho}(M)$ to be $\mathrm{Q}_n^{\rho}(\mathcal{F})$ for some free resolution $\mathcal{F}$ of M of type FP_n, and we define $\mathrm{L}_n^{\rho}(M)$ to be $\mathrm{L}_n^{\rho}(\mathcal{F})$ (when defined). The next result is a corollary of Theorem 3.5.

Theorem 4.3. *If C is a p.i.d., then $\mathrm{L}_n^{\rho}(M) \cong \mathrm{Tor}_n^R(C^k, M)$.*

We will say that M is $\mathrm{E}^{\rho}[m,n]$*-linked* if M is of type FP_n and some (and hence any) free resolution of type FP_n of M is $\mathrm{E}^{\rho}[m,n]$-linked, and that M is $\mathrm{E}^{\rho}[m,n]$*-trivial* if some free resolution of M is. We will say that M is *eventually* E^{ρ}*-linked* if it is $\mathrm{E}^{\rho}[l,\infty]$-linked for some $l \geq 0$ and that it is *eventually* E^{ρ}*-trivial* if it is $\mathrm{E}^{\rho}[l,\infty]$-trivial for some $l \geq 0$. We then define $\delta^{\rho}(M)$ to be $(-1)^l \delta_l^{\rho}(M)$ (this is well defined, since $\delta_i^{\rho}(M) + \delta_{i+1}^{\rho}(M) = 0$ for $i \geq l$).

Proposition 4.1. *If F is a finitely generated, free R-module, then F is $\mathrm{E}^{\rho}[0,\infty]$-trivial for any ρ, and $\delta^{\rho}(F) = k\,\mathrm{rk}(F)$.*

More generally, recall that a module M is said to be of *type FL* if it has a *finite* free resolution $\mathcal{F} = (F_i)_{i=0}^{l}$ with each F_i finitely generated. The Euler characteristic $\chi(M)$ of M is then (well) defined [58], [40] to be $\operatorname{rk} F_0 - \operatorname{rk} F_1 + \cdots + (-1)^l \operatorname{rk} F_l (= (-1)^l \overrightarrow{\chi}_l(\mathcal{F}))$. It is easy to show by direct computation that M is $\mathrm{E}^\rho[l, \infty]$-trivial and $\delta_l^\rho(M) = k\overrightarrow{\chi}_l(\mathcal{F})$.

Theorem 4.4. *If M is of type FL, then M is eventually E^ρ-trivial and $\delta^\rho(M) = k\chi(M)$.*

We will discuss later the extension of this result to modules of *type FP* (see Theorem 4.11).

4.2 Short exact sequences of modules

We now consider the behaviour of the E^ρ-ideals with respect to short exact sequences. For this section, we fix a short exact sequence

$$0 \to M' \xrightarrow{\iota} M \to M'' \to 0$$

of R-modules.

Theorem 4.5. (i) *If M' is of type FP_{n-1} $(n > 0)$ and M is of type FP_n, then M'' is of type FP_n and*

$$\mathrm{E}_n^\rho(M'') \supseteq \mathrm{E}_{n-1}^\rho(M') * \mathrm{E}_n^\rho(M).$$

(ii) *If M' and M'' are of type FP_n $(n \geq 0)$, then so is M and*

$$\mathrm{E}_n^\rho(M) \supseteq \mathrm{E}_n^\rho(M') * \mathrm{E}_n^\rho(M'')$$

(with equality when the short exact sequence splits, that is, when $M \cong M' \oplus M''$).

(iii) *If M is of type FP_n $(n > 0)$ and M'' is of type FP_{n+1}, then M' is of type FP_n and*

$$\mathrm{E}_n^\rho(M') \supseteq \mathrm{E}_{n+1}^\rho(M'') * \mathrm{E}_n^\rho(M).$$

Proof. (i) If $\mathcal{F}'$ and $\mathcal{F}$ are free resolutions of M' and M respectively, then ι lifts to a chain map from $\mathcal{F}'$ to $\mathcal{F}$. The mapping cylinder of this chain map is then a free resolution of M'' [58]. The result follows from Proposition 3.1.

(ii) If $\mathcal{F}'$ and $\mathcal{F}''$ are free resolutions of M' and M'' respectively, then the horseshoe construction [57] gives a free resolution of M. The $(n+1)$th boundary map of this resolution will have a matrix of the form

$$\begin{bmatrix} D'_n & 0 \\ X_n & D''_n \end{bmatrix},$$

where D'_n and D''_n are matrices for the $(n+1)$th boundary maps of $\mathcal{F}'$ and $\mathcal{F}''$ respectively. An argument similar to that leading to Proposition 3.1 then gives the result. When the sequence splits, X_n will be 0, and we then obtain equality.

(iii) Let $\varepsilon : F \to M$ be a surjective homomorphism with F free of finite rank r. There are then free resolutions $\mathcal{F} \xrightarrow{\varepsilon} M \to 0$ and $\mathcal{F}'' \xrightarrow{\alpha\varepsilon} M'' \to 0$ of type FP_n and FP_{n+1} respectively, with $F_0 = F_0'' = F$. Then the shifts $\mathcal{F}^{[1]}$, $\mathcal{F}''^{[1]}$ are free resolutions of type FP_{n-1}, FP_n for $\ker\varepsilon$, $\ker\alpha\varepsilon$ respectively. Since we have a short exact sequence

$$0 \to \ker\varepsilon \to \ker\alpha\varepsilon \to M' \to 0,$$

we deduce from (i) and the dimension shifting formula (3.3), that for all $\lambda, \kappa \in \mathbb{Z}$

$$\begin{aligned} \mathrm{E}^{\rho}_{n,\lambda+\kappa}(M') &\supseteq \mathrm{E}^{\rho}_{n-1,\lambda}(\ker\varepsilon)\,\mathrm{E}^{\rho}_{n,\kappa}(\ker\alpha\varepsilon) \\ &= \mathrm{E}^{\rho}_{n,\lambda+(-1)^n r}(M)\,\mathrm{E}^{\rho}_{n+1,\kappa+(-1)^{n+1} r}(M''), \end{aligned}$$

and the result follows.

Remark. Part (ii) was proved for commutative R in [62].

We can generalise the parenthetical result of Theorem 4.5(ii) to arbitrary finite direct sums of modules.

Corollary 4.1. *If M_i, $i = 1, \ldots m$, are modules of type FP_n, then*

$$\mathrm{E}^{\rho}_n\left(\bigoplus_{i=1}^m M_i\right) = \mathop{*}_{i=1}^{m} \mathrm{E}^{\rho}_n(M_i).$$

We now use Theorem 4.5 to obtain some combination results. It will be convenient in what follows to denote $\mathrm{E}^{\rho}_{i,\lambda}(M')$ by $\mathrm{E}'_{i,\lambda}$, $\mathrm{E}^{\rho}_{i,\lambda}(M)$ by $\mathrm{E}_{i,\lambda}$, $\mathrm{E}^{\rho}_{i,\lambda}(M'')$ by $\mathrm{E}''_{i,\lambda}$, $\delta^{\rho}_i(M')$ by δ'_i, $\delta^{\rho}_i(M)$ by δ_i and $\delta^{\rho}_i(M'')$ by δ''_i (when defined).

Theorem 4.6. *Suppose that M' is $\mathrm{E}^{\rho}[m,n]$-linked $(0 \le m < n)$. Then, for $m < i \le n$, M is of type FP_i if, and only if, M'' is, and in this case*

$$\delta_i = \delta'_i + \delta''_i.$$

Furthermore, E_{i,δ_i} contains a non-zero-divisor if, and only if, $\mathrm{E}''_{i,\delta''_i}$ does.

If M' is in fact $\mathrm{E}^{\rho}[m,n]$-trivial, then

$$\mathrm{E}_{i,\lambda} = \mathrm{E}''_{i,\lambda-\delta'_i} \qquad (\lambda \in \mathbb{Z}).$$

Proof. By assumption, M' is of type FP_i (and hence of type FP_{i-1}). By Theorem 4.5(i), if M is FP_i, then so is M'' and, for all $\lambda \in \mathbb{Z}$,

$$\mathrm{E}''_{i,\lambda} \supseteq \mathrm{E}'_{i-1,\delta'_{i-1}}\,\mathrm{E}_{i,\lambda-\delta'_{i-1}}\,. \tag{4.1}$$

When $\lambda - \delta'_{i-1} \ge \delta_i$, the right-hand side cannot be 0 (and, furthermore, will contain a non-zero-divisor if E_{i,δ_i} $(\subseteq \mathrm{E}_{i,\lambda-\delta'_{i-1}})$ does). Thus, for $\lambda \ge \delta'_{i-1} + \delta_i$, $\mathrm{E}''_{i,\lambda} \ne 0$, so

$$\delta''_i \le \delta'_{i-1} + \delta_i. \tag{4.2}$$

On the other hand, by Theorem 4.5(ii), if M'' is FP_i, then so is M, and

$$\mathrm{E}_{i,\lambda} \supseteq \mathrm{E}'_{i,\delta'_i}\,\mathrm{E}''_{i,\lambda-\delta'_i}\,. \tag{4.3}$$

An argument like that above then gives

$$\delta_i \le \delta'_i + \delta''_i. \tag{4.4}$$

Since $\delta'_{i-1} = -\delta'_i$, we deduce from (4.2) and (4.4) that

$$\delta_i = \delta'_i + \delta''_i,$$

as required. If E_{i,δ_i} contains a non-zero-divisor, then, by the comment following (4.1), $\mathrm{E}''_{i,\delta''_i}$ also contains a non-zero-divisor. The reverse implication follows from (4.3) by a similar argument.

If M' is $\mathrm{E}^\rho[m,n]$-trivial, then (4.1) and (4.3) become

$$\mathrm{E}''_{i,\lambda} \supseteq \mathrm{E}_{i,\lambda-\delta'_{i-1}} = \mathrm{E}_{i,\lambda+\delta'_i} \quad \text{and} \quad \mathrm{E}_{i,\lambda} \supseteq \mathrm{E}_{i,\lambda-\delta'_i}\,.$$

Replacing λ by $\lambda - \delta'_i$ in the first of these, we get

$$\mathrm{E}_{i,\lambda} = \mathrm{E}''_{i,\lambda-\delta'_i},$$

as required.

Theorem 4.7. *Suppose that M is $\mathrm{E}^\rho[m,n]$-linked $(0 < m < n)$. Then, for $m \le i < n$, M' is of type FP_i if, and only if, M'' is of type FP_{i+1}, and in this case*

$$\delta_i = \delta'_i - \delta''_{i+1}.$$

Furthermore, $\mathrm{E}'_{i,\delta'_i}$ contains a non-zero-divisor if, and only if, $\mathrm{E}''_{i+1,\delta''_i}$ does.

If M is in fact $\mathrm{E}^\rho[m,n]$-trivial, then

$$\mathrm{E}'_{i,\lambda} = \mathrm{E}''_{i+1,\lambda-\delta_i} \qquad (\lambda \in \mathbb{Z}).$$

Theorem 4.8. *Suppose that M'' is $\mathrm{E}^\rho[m,n]$-linked $(0 < m < n)$. Then, for $m \le i < n$, M is of type FP_i if, and only if, M' is, and in this case*

$$\delta_i = \delta'_i + \delta''_i.$$

Furthermore, E_{i,δ_i} contains a non-zero-divisor if, and only if, $\mathrm{E}'_{i,\delta'_i}$ does.

If M'' is in fact $\mathrm{E}^\rho[m,n]$-trivial, then

$$\mathrm{E}_{i,\lambda} = \mathrm{E}'_{i,\lambda-\delta''_i} \qquad (\lambda \in \mathbb{Z}).$$

The proofs of these two theorems are similar to that of Theorem 4.6, and are left to the reader.

As a consequence of the above three theorems, we obtain the following corollaries.

Theorem 4.9. *Suppose that* $0 < m < n$.

(i) *If* M' *and* M'' *are both* $\mathrm{E}^\rho[m,n]$*-linked or both* $\mathrm{E}^\rho[m,n]$*-trivial, then so is* M.

(ii) *If* M *is* $\mathrm{E}^\rho[m,n]$*-linked (respectively, -trivial), and* M' *is* $\mathrm{E}^\rho[m-1,n-1]$*-linked (respectively, -trivial), then* M'' *is* $\mathrm{E}^\rho[m,n]$*-linked (respectively, -trivial).*

(iii) *If* M *is* $\mathrm{E}^\rho[m,n]$*-linked (respectively, -trivial), and* M'' *is* $\mathrm{E}^\rho[m+1,n+1]$*-linked (respectively, -trivial), then* M' *is* $\mathrm{E}^\rho[m,n]$*-linked (respectively, -trivial).*

Proof. (i) Suppose that M', M'' are $\mathrm{E}^\rho[m,n]$-linked. Using Theorem 4.6, we deduce, for $m < i \leq n$, that M is of type FP_i, that E_{i,δ_i} contains a non-zero-divisor, and that $\delta_i = \delta_i' + \delta_i''$. We deduce the same for $m \leq i < n$ using Theorem 4.8. Thus, $\delta_i + \delta_{i+1} = 0$ for $m \leq i < n$, and so M is $\mathrm{E}^\rho[m,n]$-linked. If M', M'' are in fact $\mathrm{E}^\rho[m,n]$-trivial, then we deduce additionally from Theorem 4.6 that, for $m < i \leq n$, $\mathrm{E}_{i,\lambda} = \mathrm{E}''_{i,\lambda-\delta_i''}$ ($\lambda \in \mathbb{Z}$), and from Theorem 4.8 that, for $m \leq i < n$, $\mathrm{E}_{i,\lambda} = \mathrm{E}_{i,\lambda-\delta_i''}$ ($\lambda \in \mathbb{Z}$), and it follows that M is $\mathrm{E}^\rho[m,n]$-trivial.

(ii) can be proved in a similar manner using Theorems 4.6 and 4.7, and (iii) can be proved using Theorems 4.7 and 4.8.

Corollary 4.2. *If any two of* M', M, M'' *are eventually* E^ρ*-linked or any two are eventually* E^ρ*-trivial, then so is the third, and*

$$\delta^\rho(M) = \delta^\rho(M') + \delta^\rho(M'').$$

Corollary 4.3. *If*

$$0 \to M_l \to \cdots \to M_1 \to M_0 \to M \to 0$$

is an exact sequence with $M_0, M_1, \ldots, M_l$ *all eventually* E^ρ*-linked or all eventually* E^ρ*-trivial, then* M *is too, and*

$$\delta^\rho(M) = \sum_{i=0}^{l} (-1)^i \delta^\rho(M_i).$$

4.3 Projective modules

Let Q be a finitely-generated projective C-module. Then there is another finitely-generated projective C-module Q' such that $Q \oplus Q'$ is a free C-module Φ. We have the projection maps

$$\begin{aligned} \pi : \Phi \to \Phi; &\quad (q,q') \mapsto (q,0), \\ \pi' : \Phi \to \Phi; &\quad (q,q') \mapsto (0,q'). \end{aligned}$$

Let D, D' be the matrices of π, π' (with respect to some basis of Φ).

The *Hattori–Stallings rank* $\mathrm{h}_C(Q)$ of Q is defined to be the trace of D. Thus, if $D = [d_{ij}]_{i,j}$, then

$$\mathrm{h}_C(Q) = \sum_i d_{ii} \in C.$$

This is well defined in that it is independent of the choice of Q' and the choice of basis of Φ [60]. This rank function is *normalized* (that is, $\mathrm{h}_C(C) = 1$), and is *additive* (that is $\mathrm{h}_C(Q_1 \oplus Q_2) = \mathrm{h}_C(Q_1) + \mathrm{h}_C(Q_2)$ for two finitely-generated projective C-modules Q_1, Q_2).

Now suppose that C is *indecomposable* (that is, C cannot be written in the form $J \oplus J'$ for ideals $J, J' \neq C, 0$). For any prime ideal I of C, $C_I \otimes_C Q$ is a finitely generated free C_I-modules (here, C_I is the localisation of C at I), and the rank of this module is the same for all primes I. This common value is denoted by $\mathrm{rk}_C(Q)$. This rank function is also normalised and additive. We have

$$\mathrm{h}_C(Q) = \mathrm{rk}_C(Q) \cdot 1 \tag{4.5}$$

(see [14]), where 1 is the identity of C. (For commutative rings which are not indecomposable, there is a more complicated formula for $\mathrm{h}_C(Q)$ – see [14, p. 238].) If D, D' are the matrices above, we have [2]

$$J_{\mathrm{rk}_C(Q)}(D) = C = J_{\mathrm{rk}_C(Q')}(D'). \tag{4.6}$$

If we have a finite chain complex $\mathcal{Q} = (Q_i)_{i=0}^l$ of finitely-generated projective C-modules, then we define $\mathrm{h}_C(\mathcal{Q})$ to be $\sum_{i=0}^l (-1)^i \mathrm{h}_C(Q_i)$. Also, in the case when C is indecomposable, we define $\mathrm{rk}_C(\mathcal{Q})$ to be $\sum_{i=0}^l (-1)^i \mathrm{rk}_C(Q_i)$. Notice that, if $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by a finitary Tietze transformation, then $\mathrm{h}_C(\mathcal{Q}') = \mathrm{h}_C(\mathcal{Q})$, and also, for C indecomposable, $\mathrm{rk}_C(\mathcal{Q}') = \mathrm{rk}_C(\mathcal{Q})$.

Now suppose that P is a finitely-generated projective R-module. Then $C^k \otimes_R P$ is a finitely-generated projective C-module. We define $\mathrm{h}^\rho(P)$ to be $\mathrm{h}_C(C^k \otimes_R P)$, and, for C indecomposable, $\mathrm{rk}^\rho(P)$ to be $\mathrm{rk}_C(C^k \otimes_R P)$.

Recall that an R-module M is said to be of *type FP* if it has a finite resolution

$$(P_i)_{i=0}^l : \quad 0 \to P_l \to \cdots \to P_1 \to P_0 \to M \to 0$$

of finitely-generated projective R-modules. For such a module, we define $\mathrm{h}^\rho(M)$ to be $\mathrm{h}_C((C^k \otimes_R P_i)_{i=0}^l)$. This is well defined, since (using an argument like that at the start of this section) if $(P_i')_{i=0}^{l'}$ is another finite projective resolution, then there is a third finite projective resolution obtainable from both by a finite number of finitary Tietze transformations. Since applying $C^k \otimes_R -$ will preserve the Tietze transformations, we will have $\mathrm{h}_C((C^k \otimes_R P_i)_{i=0}^l) = \mathrm{h}_C((C^k \otimes_R P_i')_{i=0}^{l'})$.

Similarly, in the case when C is indecomposable, we can unambiguously define $\mathrm{rk}^\rho(M)$ to be $\mathrm{rk}_C((C^k \otimes_R P_i)_{i=0}^l)$. From (4.5), we have

$$\mathrm{h}^\rho(M) = \mathrm{rk}^\rho(M) \cdot 1.$$

Theorem 4.10. *If P is a finitely-generated projective R-module and C is indecomposable, then P is $\mathrm{E}^\rho[0, \infty]$-trivial and $\delta^\rho(P) = \mathrm{rk}^\rho(P)$.*

Proof. Let P' be a finitely-generated projective R-module such that $F = P \oplus P'$ is free, of rank s, say. Let ∂, ∂' be the projections onto the first and second factors respectively. Then we have the free resolution of P

$$\cdots \xrightarrow{\partial'} F \xrightarrow{\partial} F \xrightarrow{\partial'} F \xrightarrow{\partial} P \to 0.$$

Applying $C^k \otimes_R -$ and using (4.6), we find that for $n \geq 0$

$$\begin{aligned} \mathrm{E}^{\rho}_{n,ks-\mathrm{rk}_C(C^k \otimes_R P')}(P) &= C \qquad (n \text{ even}), \\ \mathrm{E}^{\rho}_{n,-\mathrm{rk}_C(C^k \otimes_R P)}(P) &= C \qquad (n \text{ odd}). \end{aligned}$$

Note that $ks - \mathrm{rk}_C(C^k \otimes_R P') = \mathrm{rk}_C(C^k \otimes_R P)$.

Now, for n odd and $\lambda < -\mathrm{rk}_C(C^k \otimes_R P)$, we have, from Theorem 3.2 and the above,

$$\begin{aligned} 0 &= \mathrm{E}^{\rho}_{n,\lambda}(P)\, \mathrm{E}^{\rho}_{n+1,\mathrm{rk}_C(C^k \otimes_R P)}(P) \\ &= \mathrm{E}^{\rho}_{n,\lambda}(P) C \\ &= \mathrm{E}^{\rho}_{n,\lambda}(P). \end{aligned}$$

Similar, for n even and $\lambda < \mathrm{rk}_C(C^k \otimes_R P)$, $\mathrm{E}^{\rho}_{n,\lambda}(P) = 0$, and the theorem follows.

Using Corollary 4.3, we extend this to modules of type FP (cf. Theorem 4.4).

Theorem 4.11. *If M is of type FP and C is indecomposable, then M is eventually E^{ρ}-trivial and $\delta^{\rho}(M) = \mathrm{rk}^{\rho}(M)$.*

4.4 Eventual E-linkage and eventual E-triviality

Let $\mathbf{FL}(R)$, $\mathbf{FP}(R)$, $\mathbf{ET}^{\rho}(R)$ and $\mathbf{EL}^{\rho}(R)$ denote, respectively, the classes of R-modules which are of type FL, type FP, eventually E^{ρ}-trivial and eventually E^{ρ}-linked. Then we have the following diagram of containments and associated rank functions:

$$\begin{array}{ccccc} & (\mathbf{EL}^{\rho}(R), \delta^{\rho}) & & & \\ & \cup\!\!| & & & \\ & (\mathbf{ET}^{\rho}(R), \delta^{\rho}) & & & \\ \swarrow\!\!\!\!\!\!\hookrightarrow & & & \searrow \text{(when } C \text{ is indecomposable)} & \\ (\mathbf{FL}(R), k\chi) & \subseteq & & (\mathbf{FP}(R), \mathrm{h}^{\rho}, \mathrm{rk}^{\rho} \ \ (C \text{ indecomposable})) . & \end{array}$$

All of these classes have the property that, if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence and any two of M', M, M'' are in the class, then so is the third, and then

$$\mathrm{r}(M) = \mathrm{r}(M') + \mathrm{r}(M'')$$

for the rank(s) r associated with the class.

By Theorem 4.4, δ^ρ extends $k\chi$. If C is indecomposable, then rk^ρ extends $k\chi$ (easily shown), and δ^ρ extends rk^ρ (by Theorem 4.11). In general, h^ρ 'nearly' extends $k\chi$ in that, for $M \in \mathbf{FL}(R)$, $\mathrm{h}^\rho(M) = k\chi(M) \cdot 1$, where 1 is the identity of C; thus, if 1 has infinite order in C, then $\mathrm{h}^\rho(M)$ and $k\chi(M)$ are essentially the same.

We will discuss the above inclusions further in the context of groups in Section 5.7.

5 Groups

5.1 Chains arising from abelianising functions

Let G be a group, K a commutative ring (with 1), and let KG be the corresponding group ring (with coefficient ring K). We can regard K as a KG-module with G acting trivially (so $g \cdot x = x$ for all $g \in G,\ x \in K$). This module will be denoted by ${}_GK$, or simply as K. The group G is said to be of *type FP_n over K* if ${}_GK$ is of type FP_n. For such a group and for any representation $\rho : KG \to \mathrm{Mat}_k(C)$, we then have the chain $\mathrm{E}_n^\rho({}_GK)$ (which we will denote by $\mathrm{E}_n^\rho(G)$) and the corresponding numbers $\nu_n^\rho({}_GK)$ and $\delta_n^\rho({}_GK)$ (which we will denote by $\nu_n^\rho(G)$ and $\delta_n^\rho(G)$ respectively). These chains and numbers are invariants of the group G, but are not, in general, invariants of the *isomorphism type* of G. To get such *group invariants*, we need to consider representations which are canonical in some sense.

To this end, we introduce the notion of an *abelianising function*, ^{T}, on groups. Such a function assigns to each group G an abelian group G^T together with an epimorphism $\tau_G^T : G \to G^T$, with the property that, if $\alpha : H \to G$ is an isomorphism, then there is an isomorphism $\alpha^T : H^T \to G^T$ such that $\tau_G^T\alpha = \alpha^T\tau_H^T$. We do not require ^{T} to be functorial, although in practice it will be. The main examples of such functions which are of interest to us here (and which are all functorial) are: *abelianisation*, ${}^{\mathrm{ab}}$, where G^{ab} is the quotient of G by its derived subgroup G'; *torsion-free abelianisation*, ${}^{\mathrm{tf}}$, where G^{tf} is obtained from G^{ab} by factoring out its torsion subgroup; and *trivialisation*, ${}^{\mathrm{triv}}$, where G^{triv} is the trivial group. Other examples include: *n-abelianisation*, ${}^{n\text{-}\mathrm{ab}}$, for a positive integer n, where $G^{n\text{-}\mathrm{ab}}$ is the quotient of G by G^nG' (thus, $G^{1\text{-}\mathrm{ab}}$ is just G^{ab}); *factoring out π-torsion*, ${}^{\pi\text{-}\mathrm{tf}}$, for a set of primes π, where $G^{\pi\text{-}\mathrm{tf}}$ is the quotient of G^{ab} obtained by factoring out all those torsion elements whose orders are π-numbers (thus, when π is the set of all primes, $G^{\pi\text{-}\mathrm{tf}}$ is just G^{tf}). Another example, which will feature in Section 5.8, is *weak torsion-free abelianisation*, ${}^{\mathrm{wtf}}$, where G^{wtf} is the quotient of G obtained by factoring out the subgroup generated by the elements of G of finite order, and then abelianising.

For any group homomorphism $\phi : G \to G_1$, we will denote the induced ring homomorphism $KG \to KG_1$ by the same symbol ϕ. Thus, in particular, in the above we obtain induced ring homomorphisms α, α^T, τ_G^T and τ_H^T such that $\tau_G^T\alpha = \alpha^T\tau_H^T$ still holds.

Let G be of type FP_n over K. If ^{T} is an abelianising function, then we have the one-dimensional representation $\tau_G^T : KG \to KG^T$, and so we have the chain $\mathrm{E}_n^{\tau_G^T}({}_GK)$. We denote this chain by $\mathrm{E}_n^T(G,K)$, and denote its λth term by $\mathrm{E}_{n,\lambda}^T(G,K)$.

Lemma 5.1. *If $\alpha : H \to G$ is an isomorphism, then α^T carries $\mathrm{E}_{n,\lambda}^T(H,K)$ isomorphically to $\mathrm{E}_{n,\lambda}^T(G,K)$ for all $\lambda \in \mathbb{Z}$.*

Proof. Let $\mathcal{F}$ be a free resolution of ${}_HK$ and, for $i \geq 0$, let $\boldsymbol{z}_i$ be a basis for F_i. Now, KH acts on KG on the right via α, and so we can consider the free chain complex $KG \otimes_{KH} \mathcal{F}$. This is a KG-free chain complex which, since α is an isomorphism, is a resolution of $KG \otimes_{KH} {}_HK \cong {}_GK$.

If $\mathcal{F}$ is of type FP_n, then so is $KG \otimes_{KH} \mathcal{F}$, and the matrix of the $(i+1)$th boundary map of $KG \otimes_{KH} \mathcal{F}$ with respect to the induced bases $1 \otimes \boldsymbol{z}_{i+1}$, $1 \otimes \boldsymbol{z}_i$ can be obtained from the matrix of ∂_{i+1} with respect to $\boldsymbol{z}_{i+1}$, $\boldsymbol{z}_i$ by applying α to each entry. Thus, since $\tau_G^T \alpha = \alpha^T \tau_H^T$, the isomorphism α^T will carry $\mathrm{E}_{n,\lambda}^T(H,K)$ to $\mathrm{E}_{n,\lambda}^T(G,K)$ for all $\lambda \in \mathbb{Z}$.

Remarks. (i) The above lemma, though easily proved, is of basic importance. It shows that if H and G are isomorphic groups (of type FP_n), then there is a ring isomorphism $KH^T \to KG^T$, which is induced by a group isomorphism $H^T \to G^T$ and which carries the chain $\mathrm{E}_n^T(H,K)$ isomorphically to the chain $\mathrm{E}_n^T(G,K)$. It is in this sense that $\mathrm{E}_n^T(-,K)$ is a group invariant. Moreover, we obtain a method for showing that two groups are not isomorphic: if there is no ring isomorphism $KH^T \to KG^T$, induced by a group isomorphism $H^T \to G^T$, which carries the chain $\mathrm{E}_n^T(H,K)$ to $\mathrm{E}_n^T(G,K)$, then $H \ncong G$. We will give examples which illustrate this below (see Examples 5.1 and 5.2 and Theorem 5.9).

(ii) With a suitable definition of an *abelianising representation*, we can extend the above lemma to higher-dimension representations $\rho : KG \to \mathrm{Mat}_k(C)$. Many of the following results also extend to the higher-dimensional situation. However, for simplicity, we concentrate on the one-dimensional case.

We will say that an abelianising function ^{T} is *compatible with monomorphisms* if for any *monomorphism* $\alpha : H \to G$ there is a homomorphism $\alpha^T : H^T \to G^T$ (not necessarily a monomorphism) such that $\tau_G^T \alpha = \alpha^T \tau_H^T$. All of the above examples are compatible with monomorphisms.

The following variation on Lemma 5.1 will be needed when we come to discuss E-ideals for certain group constructions.

Lemma 5.2. *If $\alpha : H \to G$ is a monomorphism and if ^{T} is compatible with monomorphisms, then for any left KH-module M of type FP_n,*

$$\alpha^T\, \mathrm{E}_n^{\tau_H^T}(M) = \mathrm{E}_n^{\tau_G^T}(KG \otimes_{KH} M).$$

Proof. In the proof of Lemma 5.1, replace ${}_HK$ by M and let $\mathcal{F}$ be a free resolution of type FP_n of M. Then $KG \otimes_{KH} \mathcal{F}$ is a free resolution of $KG \otimes_{KH} M$ (since KG is a free right KH-module).

Note that ${}^{\mathrm{ab}}$ is the strongest abelianising function. If ^{T} is any abelianising function, then $\tau_G^T = \beta_G \tau_G^{\mathrm{ab}}$ for some epimorphism $\beta_G : G^{\mathrm{ab}} \to G^T$. Thus,

$$\mathrm{E}_n^T(G, K) = \beta_G \, \mathrm{E}_n^{\mathrm{ab}}(G, K). \tag{5.1}$$

At the other end of the spectrum, ${}^{\mathrm{triv}}$ is the weakest abelianising function, and for any ^{T} we have

$$\mathrm{E}_n^{\mathrm{triv}}(G, K) = \mathrm{aug}\, \mathrm{E}_n^T(G, K), \tag{5.2}$$

where $\mathrm{aug} : KG^T \to K$ is the augmentation map. In general, if ^{T}, ${}^{T'}$ are abelianising functions and if, for a given group G, $\tau_G^{T'}$ factors through τ_G^T (that is, $\tau_G^{T'} = \gamma_G \tau_G^T$ for some γ_G), then

$$\mathrm{E}_n^{T'}(G, K) = \gamma_G E_n^T(G, K). \tag{5.3}$$

The most important coefficient ring is $\mathbb{Z}$, due to the following 'universal coefficient lemma'. Let $\iota_G : \mathbb{Z}G \to KG$ be the ring homomorphism which sends $1 \in \mathbb{Z}$ to $1 \in K$ and sends g to g ($g \in G$).

Lemma 5.3. *If G is of type FP_n over $\mathbb{Z}$, then, for any K, G is of type FP_n over K and*

$$\mathrm{E}_n^T(G, K) = \iota_{G^T}\, \mathrm{E}_n^T(G, \mathbb{Z}).$$

Proof. If $\mathcal{F}$ is a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ of type FP_n, then $K \otimes_{\mathbb{Z}} \mathcal{F}$ (with G acting diagonally) is a KG-free resolution of $K \otimes_{\mathbb{Z}} \mathbb{Z} \cong K$ of type FP_n (see, for example, [10, p. 4]). A choice of bases for F_{n+1}, F_n induces a choice of bases for $K \otimes_{\mathbb{Z}} F_{n+1}$, $K \otimes_{\mathbb{Z}} F_n$, and the matrix D_K for $1 \otimes \partial_{n+1}$ will then be obtained from the matrix D for ∂_{n+1} by applying ι_G to its entries. We then have

$$\begin{aligned}
\mathrm{E}_{n,\lambda}^T(G, K) &= J_{\overrightarrow{\chi}_n(K \otimes_{\mathbb{Z}} \mathcal{F}) - \lambda}\left(D_K^{\tau_G^T}\right) \\
&= J_{\overrightarrow{\chi}_n(\mathcal{F}) - \lambda}\left(D^{\tau_G^T \iota_G}\right) \\
&= J_{\overrightarrow{\chi}_n(\mathcal{F}) - \lambda}\left(D^{\iota_{G^T} \tau_G^T}\right) \quad (\text{since } \iota_{G^T} \tau_G^T = \tau_G^T \iota_G) \\
&= \left(\iota_{G^T}\, \mathrm{E}_{n,\lambda}^T(G, \mathbb{Z})\right),
\end{aligned}$$

as required.

For a group G of type FP_n over K, we define the associated invariants

$$\begin{aligned}
\nu_n^T(G, K) &= \nu_n^{\tau_G^T}({}_G K) = \min\left\{\lambda \in \mathbb{Z} : \mathrm{E}_{n,\lambda}^T(G, K) = KG^T\right\}, \\
\delta_n^T(G, K) &= \delta_n^{\tau_G^T}({}_G K) = \min\left\{\lambda \in \mathbb{Z} : \mathrm{E}_{n,\lambda}^T(G, K) \neq 0\right\}.
\end{aligned}$$

From (5.1) and (5.2), we deduce that

$$\delta_n^{\mathrm{ab}}(G, K) \leq \delta_n^T(G, K) \leq \delta_n^{\mathrm{triv}}(G, K) \leq \nu_n^{\mathrm{triv}}(G, K) \leq \nu_n^T(G, K) \leq \nu_n^{\mathrm{ab}}(G, K), \tag{5.4}$$

and, from Lemma 5.3, we deduce that

$$\delta_n^T(G,\mathbb{Z}) \leq \delta_n^T(G,K) \leq \nu_n^T(G,K) \leq \nu_n^T(G,\mathbb{Z}).$$

We also define $\mathrm{Q}_n^T(G,K)$ to be $\mathrm{Q}_n^{\tau_G^T}({}_GK)$ and $\mathrm{L}_n^T(G,K)$ to be $\mathrm{L}_n^{\tau_G^T}({}_GK)$, when defined.

Terminology and notation. When working with the basic coefficient ring $\mathbb{Z}$, we will usually omit reference to it in terminology and notation. For example, we will say that G is of type FP_n if it is of type FP_n over $\mathbb{Z}$, as is usual, and we will then denote the chain $\mathrm{E}_n^T(G,\mathbb{Z})$ by $\mathrm{E}_n^T(G)$ and denote its λth term by $\mathrm{E}_{n,\lambda}^T(G)$. We will also write $\nu_n^T(G)$ instead of $\nu_n^T(G,\mathbb{Z})$ and $\delta_n^T(G)$ instead of $\delta_n^T(G,\mathbb{Z})$.

Similarly, when ^{T} is ${}^{\mathrm{ab}}$, we will usually omit reference to it in notation. Thus, we will write $\mathrm{E}_n(G,K)$ instead of $\mathrm{E}_n^{\mathrm{ab}}(G,K)$, etc.

As discussed above, the most powerful chains of ideals are the chains $\mathrm{E}_n(G)$, and so we will tend to concentrate on these in the sequel. Note, however, that, because they carry the most information, they can also be the hardest to compute, and in practical situations we quite often pass to a weaker abelianising function or a different coefficient ring. Moreover, we may have a group which is not of type FP_n (so $\mathrm{E}_n(G)$ does not exist), but is of type FP_n over some other coefficient ring K.

When G is of type FP_n over K, where KG^T is a p.i.d., we have (by Theorem 4.3)

$$\mathrm{L}_i^T(G,K) \cong \mathrm{Tor}_i^{KG}(KG^T,K)$$

for $0 \leq i \leq n$. In particular, $\mathrm{L}_i^{\mathrm{triv}}(G) \cong \mathrm{H}_i(G)$, and we can deduce the following result, where $\mathrm{rk}_{\mathbb{Z}}(\cdot)$ denotes the $\mathbb{Z}$-rank, and $\mathrm{d}(\cdot)$ the minimal number of generators.

Lemma 5.4. *If G is of type FP_n, then*

$$\begin{aligned}
\delta_n^{\mathrm{triv}}(G) &= \mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_n(G) - \mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_{n-1}(G) + \cdots + (-1)^n\,\mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_0(G),\\
\nu_n^{\mathrm{triv}}(G) &= \mathrm{d}(\mathrm{H}_n(G)) - \mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_{n-1}(G) + \cdots + (-1)^n\,\mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_0(G).
\end{aligned}$$

Remark. We amplify a remark made at the start of Section 4.1 in the context of groups. For any group G, it is straightforward to show that $\mathbb{Z}G^{\mathrm{ab}} \otimes_{\mathbb{Z}G} {}_G\mathbb{Z}$ is isomorphic as a $\mathbb{Z}G^{\mathrm{ab}}$-module to ${}_{G^{\mathrm{ab}}}\mathbb{Z}$. Thus, the chain of n-dimensional Fitting ideals of $\mathbb{Z}G^{\mathrm{ab}} \otimes_{\mathbb{Z}G} {}_G\mathbb{Z}$ is the same as $\mathrm{E}_n(G^{\mathrm{ab}})$. In general, this chain is quite different from $\mathrm{E}_n(G)$. For example, if G^{ab} is an infinite cyclic group $\langle t\rangle$, then, for $n \geq 1$,

$$\mathrm{E}_{n,\lambda}(G^{\mathrm{ab}}) = \begin{cases} \mathbb{Z}\langle t\rangle, & \lambda \geq 0,\\ 0, & \lambda < 0,\end{cases}$$

whereas the chains $\mathrm{E}_n(G)$ can be quite complicated (see Example 5.2, Theorem 5.9).

5.2 Low-dimensional chains

We consider the chains $\mathrm{E}^T_n(-)$ for small values of n.

For any group G, let $\mathrm{I}G$ denote its augmentation ideal, that is, $\mathrm{I}G$ is the kernel of the augmentation map

$$\mathrm{aug} : \mathbb{Z}G \to \mathbb{Z}; \quad g \mapsto 1 \qquad (g \in G).$$

If $\boldsymbol{s}$ is a set of generators of G, then $\{1 - s : s \in \boldsymbol{s}\}$ is a set of generators of $\mathrm{I}G$ as a left $\mathbb{Z}G$-module [14], and we have the partial free resolution

$$\mathbb{Z}G^{|\boldsymbol{s}|} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0, \tag{5.5}$$

where ∂_1 is given by the $|\boldsymbol{s}| \times 1$ column matrix $\begin{bmatrix}1 - s\end{bmatrix}_{s \in \boldsymbol{s}}$. We easily deduce the following lemma.

Lemma 5.5. *For any group G,*

$$\mathrm{E}^T_{0,\lambda}(G) = \begin{cases} \mathbb{Z}G^T, & \lambda > 0, \\ \mathrm{I}G^T, & \lambda = 0, \\ 0, & \lambda < 0. \end{cases}$$

Now, let

$$\mathcal{P} = \langle \boldsymbol{x}; \boldsymbol{r} \rangle \tag{5.6}$$

be a group presentation. Here, $\boldsymbol{x}$ is a set (the generating symbols) and $\boldsymbol{r}$ is a set of non-empty, cyclically reduced words on $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ (the defining relators). The group $G = G(\mathcal{P})$ defined by $\mathcal{P}$ is the quotient of the free group $F(\boldsymbol{x})$ on $\boldsymbol{x}$ by the normal closure $N = N(\boldsymbol{r})$ of $\boldsymbol{r}$ in F. We have the ring homomorphism

$$\overline{} : \mathbb{Z}F(\boldsymbol{x}) \to \mathbb{Z}G; \quad W \mapsto WN = \overline{W} \qquad (W \in F).$$

For an abelianising function ^{T} and for $x \in \boldsymbol{x}$, we let $\partial^T/\partial x$ denote the composition

$$\frac{\partial^T}{\partial x} : \mathbb{Z}F(\boldsymbol{x}) \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F(\boldsymbol{x}) \xrightarrow{\overline{}} \mathbb{Z}G \xrightarrow{\tau^T_G} \mathbb{Z}G^T,$$

where $\partial/\partial x$ is (left) Fox derivation on the free group F [30] [24]. Assuming that $\boldsymbol{x}$ is finite, we let $\mathrm{A}^T_\lambda(\mathcal{P})$ $(\lambda \in \mathbb{Z})$ be the $((|\boldsymbol{x}| - 1) - \lambda)$th elementary ideal of the $|\boldsymbol{r}| \times |\boldsymbol{x}|$ matrix

$$\left[\frac{\partial^T R}{\partial x}\right]_{\substack{R \in \boldsymbol{r} \\ x \in \boldsymbol{x}}}. \tag{5.7}$$

As usual, when ^{T} is ${}^{\mathrm{ab}}$, we omit it, and simply write $\mathrm{A}_\lambda(\mathcal{P})$; this ideal is the $(\lambda + 1)$th *Alexander ideal* of $\mathcal{P}$ [30], [24].

Associated with the presentation $\mathcal{P}$, there is a standard partial free resolution (see, for example, [14]):

$$\mathbb{Z}G^{|\boldsymbol{r}|} \xrightarrow{\partial_2} \mathbb{Z}G^{|\boldsymbol{x}|} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0, \tag{5.8}$$

where ∂_1 is as in (5.5) (with $\boldsymbol{s} = \{\overline{x} : x \in \boldsymbol{x}\}$) and ∂_2 is given by the matrix

$$\left[\overline{\frac{\partial R}{\partial x}}\right]_{\substack{R\in\boldsymbol{r}\\ x\in\boldsymbol{x}}}.$$

The next theorem then follows immediately.

Theorem 5.1. *Let $\mathcal{P}$ be a group presentation as in* (5.6)*, and let G be the group defined by $\mathcal{P}$. If $\boldsymbol{x}$ is finite, then $\mathrm{E}^T_{1,\lambda}(G) = \mathrm{A}^T_\lambda(\mathcal{P})$ $(\lambda \in \mathbb{Z})$.*

A consequence of this theorem is the invariance of the Alexander ideals [30], [24].

Example 5.1. For $l = 1, 2, 3, \ldots$, let

$$\mathcal{P}_l = \langle a, b, t; a^3, b^3, (ab)^7, t^l a t^{-l} a^{-1}\rangle$$

and let $G_l = G(\mathcal{P}_l)$. Since $G_l \cong G_1 *_{t=x^l} \langle \boldsymbol{x}\rangle$, these groups all have the same integral homology by the Mayer–Vietoris sequence for amalgamated products [36], [10]. Now, for each l,

$$G_l^{\mathrm{ab}} = \langle \hat{a}\rangle \times \langle \hat{t}\rangle \cong \mathbb{Z}_3 \times \mathbb{Z},$$

where $\hat{a}$, $\hat{t}$ correspond to a, t respectively. The matrix (5.7) (taking ^{T} to be $^{\mathrm{ab}}$) is then

$$\begin{bmatrix} 1+\hat{a}+\hat{a}^2 & 0 & 0 \\ 0 & 1+\hat{a}^2+\hat{a} & 0 \\ 7 & 7\hat{a} & 0 \\ \hat{t}^l - 1 & 0 & \sum_{i=0}^{l-1} \hat{t}^i(1-\hat{a}) \end{bmatrix}.$$

From this we obtain (after some simplification)

$$\mathrm{E}_{1,\lambda}(G_l) = \mathrm{A}_\lambda(\mathcal{P}_l) = \begin{cases} \mathbb{Z}G_l^{\mathrm{ab}}, & \lambda \geq 2, \\ \left(1+\hat{a}+\hat{a}^2, 7, \sum_{i=0}^{l-1} \hat{t}^i\right), & \lambda = 1, \\ \left(1+\hat{a}+\hat{a}^2, 7(\hat{t}^l - 1), 7\sum_{i=0}^{l-1} \hat{t}^i(1-\hat{a})\right), & \lambda = 0, \\ 0, & \lambda < 0. \end{cases}$$

It proves more convenient (using (5.1) and Lemma 5.3) to consider

$$\mathrm{E}^{\mathrm{tf}}_{1,\lambda}(G_l, \mathbb{Z}_3) = \begin{cases} \mathbb{Z}_3\langle \hat{t}\rangle, & \lambda \geq 1, \\ \left(\hat{t}^l - 1\right), & \lambda = 0, \\ 0, & \lambda < 0. \end{cases}$$

Suppose that $l \neq l'$. If $G_l \cong G_{l'}$, then, by Lemma 5.1, there would be an automorphism of $\mathbb{Z}_3 G_l^{\mathrm{tf}} = \mathbb{Z}_3 G_{l'}^{\mathrm{tf}} = \mathbb{Z}_3\langle \hat{t}\rangle$, induced by an automorphism of $\langle \hat{t}\rangle$, which carries $\mathrm{E}^{\mathrm{tf}}_{1,\lambda}(G_l, \mathbb{Z}_3)$ to $\mathrm{E}^{\mathrm{tf}}_{1,\lambda}(G_{l'}, \mathbb{Z}_3)$ for each λ. However, neither of the possible automorphisms

$$\mathbb{Z}_3\langle \hat{t}\rangle \to \mathbb{Z}_3\langle \hat{t}\rangle; \quad \hat{t} \mapsto \hat{t}^{\pm 1}$$

carries $\mathrm{E}^{\mathrm{tf}}_{1,0}(G_l, \mathbb{Z}_3)$ to $\mathrm{E}^{\mathrm{tf}}_{1,0}(G_l, \mathbb{Z}_3)$. Thus, $G_l \ncong G_{l'}$.

Notational remark. Throughout the rest of the paper, we will adopt the convention used above: that is, for a group G given by a presentation $\langle \boldsymbol{x}; \boldsymbol{r} \rangle$, the element of G corresponding to a word W on $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$ will be denoted by $\overline{W}$, and the image of $\overline{W}$ under τ_G^T will be denoted by $\widehat{W}$.

Associated with $\mathcal{P}$ is a (left) $\mathbb{Z}G$-module $\pi_2(\mathcal{P})$, called the *second homotopy module*. Elements of $\pi_2(\mathcal{P})$ can be represented by geometric objects called *spherical pictures*. A collection of spherical pictures which represents a set of module generators of $\pi_2(\mathcal{P})$ is called a set of *generating pictures*. There is now a well established calculus for computing a set of generating pictures for a group presentation [6], [12], [51].

Let $\mathbb{D}$ be a spherical picture over $\mathcal{P}$. Each disc Δ has a basepoint 0_Δ and, when we read clockwise round Δ from 0_Δ, we obtain a word $R_\Delta^{\varepsilon(\Delta)}$, where $R_\Delta \in \boldsymbol{r}$ and $\varepsilon(\Delta) = \pm 1$. Choose a point 0 outside $\mathbb{D}$ and let γ_Δ be a transverse path from 0 to 0_Δ. Let W_Δ be the label on this path. If we choose another transverse path, we get a different word W'_Δ, but W_Δ and W'_Δ represent the same element of G. Thus, for each $R \in \boldsymbol{r}$, we have a well-defined element

$$\frac{\partial \mathbb{D}}{\partial R} = \sum_{\Delta : R_\Delta = R} \varepsilon_\Delta \overline{W_\Delta}$$

of $\mathbb{Z}G$. For an abelianising function ^{T}, we write $\partial^T \mathbb{D}/\partial R$ for the image of $\partial \mathbb{D}/\partial R$ under $\tau_G^T : \mathbb{Z}G \to \mathbb{Z}G^T$.

Let $\boldsymbol{d}$ be a set of generating pictures for $\pi_2(\mathcal{P})$. Assuming that $\mathcal{P}$ is finite, we let $\mathrm{B}_\lambda^T(\mathcal{P})$ $(\lambda \in \mathbb{Z})$ be the $((|\boldsymbol{r}| - |\boldsymbol{x}| + 1) - \lambda)$th elementary ideal of the $|\boldsymbol{d}| \times |\boldsymbol{r}|$ matrix

$$\left[\frac{\partial^T \mathbb{D}}{\partial R} \right]_{\substack{\mathbb{D} \in \boldsymbol{d} \\ R \in \boldsymbol{r}}}$$

(strictly speaking, we should denote this ideal by $\mathrm{B}_\lambda^T(\mathcal{P}, \boldsymbol{d})$, but we will omit $\boldsymbol{d}$). When ^{T} is ${}^{\mathrm{ab}}$, we simply write $\mathrm{B}_\lambda(\mathcal{P})$.

Now, the partial resolution (5.8) can be extended to

$$\mathbb{Z}G^{|\boldsymbol{d}|} \xrightarrow{\partial_3} \mathbb{Z}G^{|\boldsymbol{r}|} \xrightarrow{\partial_2} \mathbb{Z}G^{|\boldsymbol{x}|} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0, \tag{5.9}$$

where ∂_3 is given by the matrix

$$\left[\frac{\partial \mathbb{D}}{\partial R} \right]_{\substack{\mathbb{D} \in \boldsymbol{d} \\ R \in \boldsymbol{r}}} \tag{5.10}$$

(see [18]), and we deduce the following result.

Theorem 5.2. *If $\mathcal{P}$ is a finite presentation and if G is the group defined by $\mathcal{P}$, then $\mathrm{E}_{2,\lambda}^T(G) = \mathrm{B}_\lambda^T(\mathcal{P})$ $(\lambda \in \mathbb{Z})$.*

This then gives a method for computing the E_2-ideals for finitely-presented groups; note, however, that there are groups which are of type FP_2, but which have no finite presentation [9].

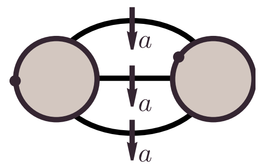

Figure 1: A dipole for the relator a^3.

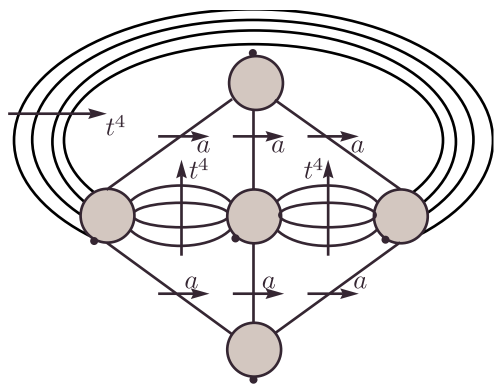

Figure 2: A spherical picture over $\mathcal{Q}_l$ for $l = 4$.

Example 5.2. For $l = 1, 2, \ldots$, let

$$\mathcal{Q}_l = \langle a, b, t; a^3, b^2, (ab)^7, t^l a t^{-l} a^{-1} \rangle$$

and let $H_l = G(\mathcal{Q}_l)$. Notice that these presentations differ from those in Example 5.1 only by the power of b. As in Example 5.1, all these groups have the same homology. For each l,

$$H_l^{\mathrm{ab}} = \langle \hat{t} \rangle \cong \mathbb{Z},$$

and the matrix (5.7) (with ^{T} as ${}^{\mathrm{ab}}$) is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 7 & 7 & 0 \\ \hat{t}^l - 1 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\mathrm{E}_{1,\lambda}(H_l) = \begin{cases} \mathbb{Z}\langle \hat{t} \rangle, & \lambda \geq 0, \\ 0, & \lambda < 0 \end{cases}$$

and, consequently, the E_1-ideals are unable to distinguish the groups H_l.

The calculus of pictures [12] gives four generating pictures for $\mathcal{Q}_l$, namely three 'dipoles' as in Figure 1 corresponding to the three relators which are proper powers, together with an extra picture, as illustrated in Figure 2. We then find that the matrix (5.10) is

$$\begin{bmatrix} 1-\overline{a} & 0 & 0 \\ 0 & 1-\overline{b} & 0 \\ 0 & 0 & 1-\overline{a}\overline{b} \\ 1-\overline{t}^{l} & 0 & 1+\overline{a}+\overline{a}^2 \end{bmatrix}.$$

When $\tau_{H_l}^{\mathrm{ab}}$ is applied to this, the first three rows become zero, and the last row becomes $\begin{bmatrix} 1-\hat{t}^l & 0 & 3 \end{bmatrix}$. Thus,

$$\mathrm{E}_{2,\lambda}(H_l) = \mathrm{B}_\lambda(\mathcal{Q}_l) = \begin{cases} \mathbb{Z}\langle \hat{t} \rangle, & \lambda \geq 2, \\ \left(1-\hat{t}^l, 3\right), & \lambda = 1, \\ 0, & \lambda < 1. \end{cases}$$

By an argument like that at the end of Example 5.1, if $l \neq l'$, then $H_l \ncong H_{l'}$.

The fact that the E_1-ideals cannot distinguish the groups in this example is an illustration of the following general situation.

A *relative presentation* is a triple

$$\langle H, \boldsymbol{t}; \boldsymbol{w} \rangle, \tag{5.11}$$

where H is a group (the 'coefficient group') and $\boldsymbol{w}$ is a set of non-empty, cyclically reduced words in $H * \langle \boldsymbol{t} \rangle \setminus H$. The group G defined by the relative presentation is the quotient of $H * \langle \boldsymbol{t} \rangle$ by the normal closure of $\boldsymbol{w}$. If $\mathcal{P}_H = \langle \boldsymbol{a}; \boldsymbol{v} \rangle$ is a presentation for H, then a presentation for G is given by

$$\mathcal{P} = \langle \boldsymbol{a}, \boldsymbol{t}; \boldsymbol{v}, \boldsymbol{w} \rangle \tag{5.12}$$

(where we assume that the elements of $\boldsymbol{w} \subseteq H * \langle \boldsymbol{t} \rangle$ are expressed as words on $(\boldsymbol{a} \cup \boldsymbol{t})^{\pm 1}$). Groups defined by relative presentations have been widely studied, particularly in the context of equations over groups.

For $W \in \boldsymbol{w}$, let W° be the word obtained from W by deleting all letters from $\boldsymbol{a}^{\pm 1}$ and cyclically reducing. Let

$$\mathcal{P}^\circ = \langle \boldsymbol{t}; W^\circ \ (W \in \boldsymbol{w}) \rangle$$

and let G° be the group defined by $\mathcal{P}^\circ$. There is then an epimorphism $\alpha : G \to G^\circ$.

Theorem 5.3. *If H is perfect, then $\alpha^{\mathrm{ab}} : G^{\mathrm{ab}} \to G^{\circ\mathrm{ab}}$ is an isomorphism and $\alpha^{\mathrm{ab}}\,\mathrm{E}_1(G) = \mathrm{E}_1(G^\circ)$.*

The proof of this is left to the reader; it is made easier by choosing a pre-abelian presentation [47] for H at the outset.

Now, if we have two presentations $\mathcal{P}$, $\widetilde{\mathcal{P}}$ as in (5.12) arising from perfect groups H, $\widetilde{H}$ respectively, then the E_1-ideals will only be able to determine non-isomorphism of $G(\mathcal{P})$ and $G(\widetilde{\mathcal{P}})$ if the E_1-ideals can determine non-isomorphism of $G(\mathcal{P}^\circ)$ and $G(\widetilde{\mathcal{P}}^\circ)$.

5.3 (Relatively) aspherical groups

In certain favourable circumstances, the partial resolution (5.9) can be usefully extended further.

For example, there is a notion of *asphericity* of relative presentations (see [5], [11], [28], [37]). If a relative presentation as in (5.11) is aspherical and 'orientable' (that is, no element of $\boldsymbol{w}$ is conjugate to its inverse), then the natural map $\iota : H \to G$ is injective and the partial resolution (5.8) corresponding to an ordinary presentation $\mathcal{P}$ for G as in (5.12) can be extended to a full resolution as follows.

Let

$$\cdots \xrightarrow{\partial_4^H} F_3^H \xrightarrow{\partial_3^H} F_2^H = \mathbb{Z}H^{|\boldsymbol{v}|} \xrightarrow{\partial_2^H} \mathbb{Z}H^{|\boldsymbol{a}|} \xrightarrow{\partial_1^H} \mathbb{Z}H \xrightarrow{\text{aug}} {}_H\mathbb{Z} \to 0$$

be a free resolution extending the partial resolution (5.8) arising from $\mathcal{P}_H$. For $W \in \boldsymbol{w}$, write $W = W_0^{p(W)}$, where W_0 is not a proper power and $p(W) \geq 1$ – we call W_0 the *root* of W and $p(W)$ the *period* of W. Let $\boldsymbol{w}' = \{W : W \in \boldsymbol{w},\ p(W) > 1\}$. For $W \in \boldsymbol{w}'$ and $\varepsilon = \pm 1$, let $\xi_\varepsilon(W) \in \mathbb{Z}G$ be defined to be $1 - \overline{W_0}$ if $\varepsilon = +1$ and $1 + \overline{W_0} + \cdots + \overline{W_0}^{p(W)-1}$ if $\varepsilon = -1$. Let F be the free $\mathbb{Z}G$-module with basis e_W $(W \in \boldsymbol{w}')$ and let ϕ_ε $(\varepsilon = \pm 1)$ be the map

$$\phi_\varepsilon : F \to F; \quad e_W \mapsto \xi_\varepsilon(W) e_W \qquad (W \in \boldsymbol{w}').$$

Then the partial free resolution (5.8) arising from $\mathcal{P}$ extends to a free resolution, the *generalised Lyndon resolution*, thus:

$$\cdots \to F_4 \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} \mathbb{Z}G^{|\boldsymbol{a}|+|\boldsymbol{t}|} \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\text{aug}} {}_G\mathbb{Z} \to 0, \tag{5.13}$$

where $F_2 = (\mathbb{Z}G \otimes_{\mathbb{Z}H} F_2^H) \oplus \mathbb{Z}G^{|\boldsymbol{w}|}$, $F_i = (\mathbb{Z}G \otimes_{\mathbb{Z}H} F_i^H) \oplus F$ $(i \geq 3)$ and $\partial_i = (1 \otimes \partial_i^H) \oplus \phi_{-\varepsilon(i)}$ $(i \geq 3)$, with $\varepsilon(i) = +1$ or -1 according to whether i is even or odd.

From this resolution we can compute the E_n^T-ideals for G in terms of those for H. Let $\xi_\varepsilon^T(W)$ be the image of $\xi_\varepsilon(W)$ under the map $\mathbb{Z}G \to \mathbb{Z}G^T$, and let $I_{\varepsilon,j}^T$ be the ideal generated by all products $\xi_\varepsilon^T(W_1)\xi_\varepsilon^T(W_2)\cdots\xi_\varepsilon^T(W_j)$ for j distinct elements $W_i \in \boldsymbol{w}'$. Assuming that $\boldsymbol{t}$ and $\boldsymbol{w}$ are finite, we let $\mathrm{E}_\varepsilon^T(\boldsymbol{w})$ $(\varepsilon = \pm 1)$ be the chain of ideals

$$\mathrm{E}_{\varepsilon,\lambda}^T(\boldsymbol{w}) = \begin{cases} \mathbb{Z}G^T, & \lambda \geq \chi_\varepsilon, \\ I_{\varepsilon,\chi_\varepsilon-\lambda}^T, & -\chi_{-\varepsilon} \leq \lambda < \chi_\varepsilon, \\ 0, & \lambda < -\chi_{-\varepsilon}, \end{cases}$$

where $\chi_1 = |\boldsymbol{w}| - |\boldsymbol{t}|$, $\chi_{-1} = |\boldsymbol{w}'| - |\boldsymbol{w}| + |\boldsymbol{t}|$. A straightforward calculation, which we omit, then gives the following theorem.

Theorem 5.4. *Let $n \geq 2$ and let H be a finitely generated group of type FP_n. Let $\langle H, \boldsymbol{t}; \boldsymbol{w}\rangle$ be an orientable, aspherical presentation over H, with $\boldsymbol{t}$ and $\boldsymbol{w}$*

finite. Let G be the group defined by this relative presentation, and let $\iota : H \to G$ be the natural embedding. Then G is of type FP_n and

$$\mathrm{E}_n^T = \iota^T\, \mathrm{E}_n^T(H) * \mathrm{E}_{\varepsilon(n)}^T(\boldsymbol{w}).$$

In the special case when $H = 1$, the relative presentation essentially reduces to an ordinary presentation, and asphericity is what is normally called *combinatorial asphericity* (CA) (for ordinary presentations, the term *asphericity* (A) is reserved for CA presentations in which each relator has period 1). The resolution (5.13) is then the standard *Lyndon resolution* [46], [38].

Since, when H is trivial, we have

$$\mathrm{E}_{n,\lambda}^T(H) = \begin{cases} \mathbb{Z}H^T, & \lambda \geq \varepsilon(n), \\ 0, & \lambda < \varepsilon(n) \end{cases} \quad \text{(for all } n\text{)},$$

we deduce the following.

Corollary 5.1. *If G is the group defined by a finite CA presentation $\langle \boldsymbol{t}; \boldsymbol{w} \rangle$, then for $n \geq 2$*

$$\mathrm{E}_{n,\lambda}^T(G) = \mathrm{E}_{\varepsilon(n),\lambda-\varepsilon(n)}^T(\boldsymbol{w}) \qquad (\lambda \in \mathbb{Z}).$$

Example 5.3. For instance, let p be an integer greater than 1 and let $\boldsymbol{t} = \{s, t\}$. Suppose that for each $W \in \boldsymbol{w}$, $p(W) = p$, the exponent sum of s in W_0 is 1, and the exponent sum of t in W_0 is 0. Using small cancellation theory, CA presentations of this form can easily be constructed for $\boldsymbol{w}$ arbitrarily large. The group G^{ab} is then the direct sum of a cyclic group of order p (generated by a, say) and an infinite cyclic group, and for $n \geq 2$ we have

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda \geq |\boldsymbol{w}| - 1, \\ \left((1-a)^{|\boldsymbol{w}|-1-\lambda}\right), & -1 \leq \lambda < |\boldsymbol{w}| - 1, \\ 0, & \lambda < -1 \end{cases} \quad (n \text{ even}),$$

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda \geq 1, \\ \left((1 + a + \cdots + a^{p-1})^{1-\lambda}\right), & 1 - |\boldsymbol{w}| \leq \lambda < 1, \\ 0, & \lambda < 1 - |\boldsymbol{w}| \end{cases} \quad (n \text{ odd}).$$

In particular, assuming that $(1-a)^{|\boldsymbol{w}|} \neq 0$ (as is certainly the case when $|\boldsymbol{w}| \leq p$), for $i \geq 2$, $\delta_i(G) + \delta_{i+1}(G) = -|\boldsymbol{w}|$. This demonstrates that the conditions in Theorem 3.4(i) cannot, in general, be relaxed. Notice that if we pass to G^{tf} by killing a, then we have

$$\mathrm{E}_{n,\lambda}^{\mathrm{tf}}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{tf}}, & \lambda \geq |\boldsymbol{w}| - 1, \\ 0, & \lambda < |\boldsymbol{w}| - 1 \end{cases} \quad (n \text{ even}),$$

$$\mathrm{E}_{n,\lambda}^{\mathrm{tf}}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{tf}}, & \lambda \geq 1, \\ \left(p^{1-\lambda}\right), & 1 - |\boldsymbol{w}| \leq \lambda < 1, \\ 0, & \lambda < 1 - |\boldsymbol{w}| \end{cases} \quad (n \text{ odd}).$$

The conditions of Theorem 3.4(i) now hold, and we see that, for $i \geq 2$, $\delta_i(G) + \delta_{i+1}(G) \geq 0$, as expected.

We discuss the phenomenon in Example 5.3 in more detail in Section 5.8.

Remark on knot groups. If X is a (tame) knot in $\mathbb{R}^3$, then the corresponding knot group G is the fundamental group of the complement $\mathbb{R}^3 \backslash X$. The Wirtinger presentation [24] of this group has the same (finite) number of defining relators as generators. Any one of these defining relators is a consequence of the others, and so we can remove one of them, to obtain a *deleted* Wirtinger presentation. This presentation is aspherical [50], and so (since G^{ab} in infinite cyclic, generated by t, say) we find that for $n \geq 2$

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}\langle t\rangle, & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases}$$

Although these chains are 'trivial', they are not useless, since they are still invariants of the group, and hence the knot.

Of course, $\mathrm{E}_1(G)$ is just the chain of Alexander ideals of the knot. In particular, $\mathrm{E}_{1,-1}(G) = 0$ and $\mathrm{E}_{1,0}(G)$ is a principal ideal generated by the knot polynomial $\mathrm{e}(t)$ (see [24] and also Section 5.10, where we discuss invariant polynomials in greater depth). Since $\mathrm{H}_0(G) = \mathrm{H}_1(G) = \mathbb{Z}$, we must have, by Lemma 5.4, that $\delta_0^{\mathrm{triv}}(G) = 1$ and $\delta_1^{\mathrm{triv}}(G) = \nu_1^{\mathrm{triv}}(G) = 0$, so $\mathrm{E}_{1,0}^{\mathrm{triv}}(G) = (\mathrm{e}(1)) = \mathbb{Z}$. That is, we recover the well-known fact that *the knot polynomial evaluated at* 1 *is* ± 1 [24].

5.4 The R. Thompson group

Let G be the group given by the presentation

$$\langle x_0, x_1, x_2, \ldots ; x_i x_j = x_{j+1} x_i \ (i < j)\rangle.$$

This group was originally defined by R. Thompson and has been much studied subsequently (see [19] for a survey). The abelianisation of G is free abelian of rank 2. In [17], Brown and Geoghegan showed that G is of type FP_∞ and gave a resolution $\mathcal{F} = (F_i, \partial_i)_{i \geq 0}$ of type FP_∞, with F_0 of rank 1 and F_i of rank 2 for $i > 0$ (see below). We will use this to give the following result.

Theorem 5.5. *If G is the R. Thompson group, then, for all n,*

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda > 0, \\ \mathrm{I}G^{\mathrm{ab}}, & \lambda = 0, \\ 0, & \lambda < 0. \end{cases}$$

Proof. Let ϕ be the endomorphism of G induced by $x_i \mapsto x_{i+1}$ $(i \geq 0)$. A map $\alpha : M \to M'$ of $\mathbb{Z}G$-modules will be said to be *ϕ-semi-linear* if $\alpha(g \cdot m) = \phi(g) \cdot \alpha(m)$, for $g \in G$, $m \in M$. We define ϕ^2-semi-linearity similarly.

Let F_0 be a free $\mathbb{Z}G$-module of rank 1 with basis element $z^{(0)}$ and, for $n > 0$, let F_n be free of rank 2 with basis $\{z_0^{(n)}, z_1^{(n)}\}$. We now define $\mathbb{Z}G$-linear maps $\partial_n : F_n \to F_{n-1}$ $(n > 0)$, $\partial_0 : F_0 \to \mathbb{Z}$ inductively, using ϕ-semi-linear maps $\psi_n : F_n \to F_n$ and ϕ^2-semi-linear maps $h_n : F_n \to F_{n+1}$:

$$h_0(z^{(0)}) = z_0^{(1)}, \qquad h_n(z_0^{(n)}) = 0, \qquad h_n(z_1^{(n)}) = z_0^{(n+1)} \qquad (n > 0),$$

$$\psi_0(z^{(0)}) = z^{(0)}, \qquad \psi_n(z_0^{(n)}) = z_1^{(n)} \quad (n > 0),$$

$$\partial_0(z^{(0)}) = 1, \qquad \partial_1(z_0^{(1)}) = (1 - \overline{x_0})z^{(0)},$$

$$\partial_{n+1}(z_1^{(n+1)}) = \psi_n \partial_{n+1}(z_0^{(n+1)}) \qquad (n \geq 0),$$

$$\psi_n(z_1^{(n)}) = h_{n-1}\partial_n(z_0^{(n)}) + \overline{x_0} z_1^{(n)} \qquad (n > 0),$$

$$\partial_{n+1}(z_0^{(n+1)}) = \psi_n^2(z_1^{(n)}) - \overline{x_0}\psi_n(z_1^{(n)}) - h_{n-1}\partial_n(z_1^{(n)}) \qquad (n > 0).$$

The complex $\mathcal{F} = (F_i, \partial_i)$ thus defined is a resolution of type FP_∞ for G [17].

Let D_n be the matrix associated with ∂_{n+1}, so

$$D_0 = \begin{bmatrix} 1 - \overline{x_0} \\ 1 - \overline{x_1} \end{bmatrix},$$

and suppose that for $n > 0$

$$D_n = \begin{bmatrix} \alpha_{11}^{(n)} & \alpha_{12}^{(n)} \\ \alpha_{21}^{(n)} & \alpha_{22}^{(n)} \end{bmatrix}.$$

Then

$$\begin{aligned}
\alpha_{11}^{(1)} &= (\overline{x_1} - \overline{x_0})(1 - \overline{x_2}) - (1 - \overline{x_3}),\\
\alpha_{12}^{(1)} &= (\overline{x_1} - \overline{x_0})\overline{x_0} + 1 - \overline{x_3},\\
\alpha_{21}^{(1)} &= ((\overline{x_2} - \overline{x_1})\overline{x_1} + 1 - \overline{x_4})(1 - \overline{x_2}),\\
\alpha_{22}^{(1)} &= (\overline{x_2} - \overline{x_1})(1 - \overline{x_3} + \overline{x_1 x_0}) + (1 - \overline{x_4})(\overline{x_0} - 1),
\end{aligned}$$

and the definitions of the boundary maps show that

$$\begin{aligned}
\alpha_{11}^{(n+1)} &= (\overline{x_1} - \overline{x_0})\phi^2(\beta^{(n)}) - \phi^2(\delta^{(n)}),\\
\alpha_{12}^{(n+1)} &= \phi^3(\beta^{(n)}) + (\overline{x_1} - \overline{x_0})\overline{x_0},\\
\alpha_{21}^{(n+1)} &= (\phi^4(\beta^{(n)}) + (\overline{x_2} - \overline{x_1})\overline{x_1})\phi^2(\beta^{(n)}),\\
\alpha_{22}^{(n+1)} &= (\phi^4(\beta^{(n)}) + (\overline{x_2} - \overline{x_1})\overline{x_1})\overline{x_0} + (\overline{x_2} - \overline{x_1})\phi^3(\beta^{(n)}) - \phi^3(\delta^{(n)}).
\end{aligned}$$

Now let us apply the abelianising map $\tau = \tau_G^{\mathrm{ab}} : \mathbb{Z}G \to \mathbb{Z}G^{\mathrm{ab}}$. Note that G^{ab} is free abelian of rank two, on generators $\hat{x}_0$, $\hat{x}_1$, and the homomorphism τ sends $\overline{x_i}$ to $\hat{x}_1$, for $i > 0$, and $\overline{x_0}$ to $\hat{x}_0$. Note also that, for $j \geq 1$, $\tau\phi^j = \tau\phi = \phi^{\mathrm{ab}}\tau$, where ϕ^{ab} is the map $\hat{x}_i \mapsto \hat{x}_1$ $(i = 0, 1)$. If we apply τ to each entry in D_{n+1}, we get the matrix

$$D^{\mathrm{ab}}_{n+1} = \begin{bmatrix} (\hat{x}_1 - \hat{x}_0)\phi^{\mathrm{ab}}\tau(\beta^{(n)}) - \phi^{\mathrm{ab}}\tau(\delta^{(n)}) & \phi^{\mathrm{ab}}\tau(\beta^{(n)}) + (\hat{x}_1 - \hat{x}_0)\hat{x}_0 \\ \phi^{\mathrm{ab}}\tau(\beta^{(n)})^2 & \phi^{\mathrm{ab}}\tau(\beta^{(n)})\hat{x}_0 - \phi^{\mathrm{ab}}\tau(\delta^{(n)}) \end{bmatrix}.$$

By induction, starting with

$$D^{\mathrm{ab}}_1 = \begin{bmatrix} (\hat{x}_1 - \hat{x}_0 - 1)(1 - \hat{x}_1) & (\hat{x}_1 - \hat{x}_0)\hat{x}_0 + (1 - \hat{x}_1) \\ (1 - \hat{x}_1)^2 & -(1 - \hat{x}_1)(1 - \hat{x}_0) \end{bmatrix},$$

we obtain

$$D^{\mathrm{ab}}_n = \begin{bmatrix} (1 - \hat{x}_0)(1 - \hat{x}_1) & (\hat{x}_1 - \hat{x}_0)\hat{x}_0 + (1 - \hat{x}_1) \\ (1 - \hat{x}_1)^2 & -(\hat{x}_1 - \hat{x}_0 - 1)(1 - \hat{x}_1) \end{bmatrix}$$

for even n and $D^{\mathrm{ab}}_n = D^{\mathrm{ab}}_1$ for odd n. Now, for $n > 0$, $\det(D^{\mathrm{ab}}_n) = 0$ and, for $n \geq 0$, the entries of D^{ab}_n generate the ideal $(1 - \hat{x}_0, 1 - \hat{x}_1)$, and so we obtain the result.

5.5 E-trivial groups

We will say that G is $\mathrm{E}^T[m,n]$*-trivial* $(0 \leq m \leq n \leq \infty)$ if ${}_G\mathbb{Z}$ is $\mathrm{E}^{\tau^T_G}[m,n]$-trivial; in other words, $\delta^T_i(G) = \nu^T_i(G)$ for each integer $m \leq i \leq n$ and $\delta^T_{i+1}(G) = -\delta^T_i(G)$ for $m \leq i < n$. As usual, when ^{T} is ${}^{\mathrm{ab}}$ we (usually) omit reference to it.

Lemma 5.6. (i) *If G is $\mathrm{E}[m,n]$-trivial, then it is $\mathrm{E}^T[m,n]$-trivial for any ^{T}.*

(ii) *If G is $\mathrm{E}^T[m,n]$-trivial for some ^{T}, then it is $\mathrm{E}^{\mathrm{triv}}[m,n]$-trivial; consequently, $\mathrm{H}_i(G) = 0$ for each integer $m < i \leq n$ and*

$$\delta^T_i(G) = \delta^{\mathrm{triv}}_i(G) = (-1)^i \sum_{j=0}^{m} (-1)^j \operatorname{rk}_{\mathbb{Z}} \mathrm{H}_j(G)$$

for $m \leq i \leq n$.

This follows from (5.1), (5.2), (5.4) and Lemma 5.4.

We can characterise $\mathrm{E}[0,n]$-trivial groups in terms of homology.

Theorem 5.6. *A group G is $\mathrm{E}[0,n]$-trivial $(n > 0)$ if, and only if, G is of type FP_n and $\mathrm{H}_i(G) = 0$ for each integer $0 < i \leq n$. For such a group, $\delta_i(G) = (-1)^i$ for $0 \leq i \leq n$.*

Proof. By Lemma 5.5, G is $\mathrm{E}[0,0]$-trivial if, and only if, $\mathrm{I}G^{\mathrm{ab}} = 0$, that is, $G^{\mathrm{ab}} = 1$, so $G^{\mathrm{ab}} = G^{\mathrm{triv}}$. Now use Proposition 3.2.

In particular, the $\mathrm{E}[0,1]$-trivial groups are the finitely generated perfect groups, the $\mathrm{E}[0,2]$-trivial groups are the super-perfect groups of type FP_2 (see, for example, [7] or [26]), and the $\mathrm{E}[0,\infty]$-trivial groups are the acyclic groups of type FP_∞ [8].

Later, in Example 5.6, we will give examples of groups which are $\mathrm{E}[m,n]$-trivial, but not $\mathrm{E}[m-1,n]$-trivial (when $m>0$) or $\mathrm{E}[m,n+1]$-trivial (when $m<\infty$).

One importance of E-triviality is that it enables us to get formulæ for the E-ideals for various group constructions in terms of the E-ideal of the groups used in the construction. We give an instance of this here, and more later (see Section 5.6).

Let G be an extension of G_0 by H, that is, there is a short exact sequence of groups

$$1 \to H \to G \xrightarrow{\alpha} G_0 \to 1.$$

Suppose that H is a finitely generated perfect group (that is, H is $\mathrm{E}[0,1]$-trivial). Since $H = H' \subseteq G'$, we have

$$G^{\mathrm{ab}} = G/G' \cong (G/H) \big/ (G'/H) \cong G_0/G_0' = G_0^{\mathrm{ab}}$$

and $\alpha^{\mathrm{ab}} : G^{\mathrm{ab}} \to G_0^{\mathrm{ab}}$ is an isomorphism. It follows from Theorem 5.3 that $\alpha^{\mathrm{ab}}\,\mathrm{E}_1(G) = \mathrm{E}_1(G_0)$. From Lemma 5.5, we also have $\alpha^{\mathrm{ab}}\,\mathrm{E}_0(G) = \mathrm{E}_0(G_0)$. This extends to higher dimensions as follows.

Theorem 5.7. *Let G be an extension of G_0 by H, as above. Suppose that H is $\mathrm{E}[0,n]$-trivial. Then G is of type FP_n if, and only if, G_0 is, and in that case $\alpha^{\mathrm{ab}}\,\mathrm{E}_i(G) = \mathrm{E}_i(G_0)$ for $0 \le i \le n$.*

Proof. It follows from [63] that if G_0 is of type FP_n, then so is G. Conversely, suppose that G has a resolution $\mathcal{F}$ of type FP_n. Consider the complex $\mathbb{Z}G_0 \otimes_{\mathbb{Z}G} \mathcal{F}$ (where we regard $\mathbb{Z}G_0$ as a right $\mathbb{Z}G$-module with G-action via α). Now, noting that $\mathbb{Z}G_0 \cong \mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G$ as right $\mathbb{Z}G$-modules, we have, for $1 \le i \le n$,

$$\begin{aligned} \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}G_0,\mathbb{Z}) &\cong \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}G,\mathbb{Z}) \\ &\cong \mathrm{Tor}_i^{\mathbb{Z}H}(\mathbb{Z},\mathbb{Z}) \\ &= \mathrm{H}_i(H) \\ &= 0. \end{aligned}$$

Thus, $\mathbb{Z}G_0 \otimes_{\mathbb{Z}G} \mathcal{F}$ is exact in dimensions $1,\dots,n$, and so gives rise to a partial resolution of length $n+1$ for $\mathbb{Z}G_0 \otimes_{\mathbb{Z}G} \mathbb{Z}$ ($\cong \mathbb{Z}$ as a left $\mathbb{Z}G_0$-module). If D_i ($0 \le i \le n$) is the matrix of the $(i+1)$th boundary map of $\mathcal{F}$, then that of $\mathbb{Z}G_0 \otimes_{\mathbb{Z}G} \mathbb{Z}$ is D_i^α, the matrix obtained from D_i by applying α to each entry. The result then follows, since $\tau_{G_0}^{\mathrm{ab}}\alpha = \alpha^{\mathrm{ab}}\tau_G^{\mathrm{ab}}$.

We can, of course, define $\mathrm{E}^T[m,n]$-trivial groups over any coefficient ring K by considering the chains $\mathrm{E}_i^T(G,K)$. By Lemma 5.3 (the universal coefficient lemma), if G is $\mathrm{E}^T[m,n]$-trivial (over $\mathbb{Z}$), then it is $\mathrm{E}^T[m,n]$-trivial over K for any K. However, as we would expect, a group can be $\mathrm{E}^T[m,n]$-trivial over some K without being $\mathrm{E}^T[m,n]$-trivial.

Example 5.4. Let G be given by the presentation

$$\langle a, t; a^p, t^{-1}at = a^q \rangle$$

with $1 < q < p$, $(p, q) = 1$. Then $G^{\mathrm{ab}} = \langle \hat{a} \rangle \times \langle \hat{t} \rangle \cong \mathbb{Z}_{(q-1,p)} \times \mathbb{Z}$ and, for $n > 1$,

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda \geq 1, \\ \left(1 - \hat{a}, p, 1 - q^l \hat{t}\right), & \lambda = 0, \\ 0, & \lambda < 0, \end{cases}$$

where $n = 2k$ or $2k-1$ (see Example 5.5, below). Thus, G is $\mathrm{E}[2,\infty]$-trivial over $\mathbb{Z}_q$, but it is not $\mathrm{E}[m,n]$-trivial for any $2 \leq m < n \leq \infty$.

5.6 Graphs of groups and other group constructions

If G is the fundamental group of a finite connected graph of groups [59] (with vertex groups G_v ($v \in \boldsymbol{v}$) and edge groups G_e ($e \in \boldsymbol{e}$)), then there is an associated short exact sequence of $\mathbb{Z}G$-modules [21]

$$0 \to \bigoplus_{e \in \boldsymbol{e}^+} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z} \xrightarrow{\iota} \bigoplus_{v \in \boldsymbol{v}} \mathbb{Z}G \otimes_{\mathbb{Z}G_v} \mathbb{Z} \to {}_G\mathbb{Z} \to 0 \tag{5.14}$$

(here $\boldsymbol{e}^+$ is an orientation [59] of the edge set $\boldsymbol{e}$ of the underlying graph). If we have resolutions $\mathcal{F}_e$ ($e \in \boldsymbol{e}^+$), $\mathcal{F}_v$ ($v \in \boldsymbol{v}$) for the edge and vertex groups, then

$$\mathcal{F}_{\boldsymbol{e}} = \bigoplus_{e \in \boldsymbol{e}^+} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathcal{F}_e, \qquad \mathcal{F}_{\boldsymbol{v}} = \bigoplus_{v \in \boldsymbol{v}} \mathbb{Z}G \otimes_{\mathbb{Z}G_v} \mathcal{F}_v$$

will be resolutions for

$$\bigoplus_{e \in \boldsymbol{e}^+} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}, \qquad \bigoplus_{v \in \boldsymbol{v}} \mathbb{Z}G \otimes_{\mathbb{Z}G_v} \mathbb{Z}$$

respectively, and ι will lift to a chain map $\iota : \mathcal{F}_{\boldsymbol{e}} \to \mathcal{F}_{\boldsymbol{v}}$. The mapping cylinder of this chain map will then give a $\mathbb{Z}G$-resolution of ${}_G\mathbb{Z}$ [58]. If each $\mathcal{F}_e$ is of type FP_{n-1} ($n > 0$), and each $\mathcal{F}_v$ is of type FP_n, then $\mathcal{F}$ will be of type FP_n. A matrix D_n for the $(n+1)$th boundary map of $\mathcal{F}$ will have the form

$$D_n = \begin{bmatrix} \Delta_n & 0 \\ X_n & -\Delta'_{n-1} \end{bmatrix},$$

where Δ_n is a matrix for the $(n+1)$th boundary map of $\mathcal{F}_{\boldsymbol{v}}$, Δ'_{n-1} is a matrix for the nth boundary map of $\mathcal{F}_{\boldsymbol{e}}$ and X_n is a matrix for the nth homomorphism of the chain map ι. From this, we can (in theory) compute the chain $\mathrm{E}_n(G)$.

We give an illustrative example.

Example 5.5. Let H be the group given by the finite, one-relator presentation $\langle \boldsymbol{x}; R_0^p \rangle$, where $p > 1$. Let H_0 be the subgroup of H generated by the image $\overline{R_0}$

of R_0 in H, so H_0 is cyclic of order p. For q co-prime to p, there is an embedding of H_0 in H induced by the map $R_0 \mapsto R_0^q$. Let G be the corresponding HNN-extension given by the presentation

$$\langle \boldsymbol{x}, t; R_0^p, t^{-1}R_0t = R_0^q \rangle.$$

The sequence (5.14) becomes

$$0 \to \mathbb{Z}G \otimes_{\mathbb{Z}H_0} \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z} \to \mathbb{Z} \to 0,$$

where $\iota(1 \otimes 1) = (1 - \bar{t}) \otimes 1$. Using the standard (Lyndon) resolutions for H and H_0, we obtain a resolution for G with

$$D_1 = \begin{bmatrix} \left[\xi \overline{\frac{\partial R_0}{\partial x}}\right]_{x \in \boldsymbol{x}} & 0 \\ \left[(1 - \bar{t}(1 + \overline{R_0} + \cdots + \overline{R_0}^{q-1}))\overline{\frac{\partial R_0}{\partial x}}\right]_{x \in \boldsymbol{x}} & -(1 - \overline{R_0}) \end{bmatrix},$$

$$D_n = \begin{cases} \begin{bmatrix} 1 - \overline{R_0} & 0 \\ 1 - q^k \bar{t} & -\xi \end{bmatrix}, & n = 2k, \\ \\ \begin{bmatrix} \xi & 0 \\ 1 - q^k \bar{t}(1 + \overline{R_0} + \cdots + \overline{R_0}^{q-1}) & -(1 - \overline{R_0}) \end{bmatrix}, & n = 2k + 1 \end{cases} \qquad (k \geq 1).$$

Here, $\xi = 1 + \overline{R_0} + \cdots + \overline{R_0}^{p-1}$. We then find that

$$\mathrm{E}_{1,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda \geq |\boldsymbol{x}|, \\ \left(1 - \widehat{R_0}, p\widehat{\frac{\partial R_0}{\partial x}}, (1 - q\hat{t})\widehat{\frac{\partial R_0}{\partial x}} \ (x \in \boldsymbol{x})\right), & \lambda = |\boldsymbol{x}| - 1, \\ 0, & \lambda < |\boldsymbol{x}| - 1, \end{cases}$$

and, for $n > 1$,

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda > (-1)^n(1 - |\boldsymbol{x}|), \\ \left(1 - \widehat{R_0}, p, 1 - q^k\hat{t}\right), & \lambda = (-1)^n(1 - |\boldsymbol{x}|), \\ 0, & \lambda < (-1)^n(1 - |\boldsymbol{x}|), \end{cases}$$

where $n = 2k$ or $2k - 1$.

For convenience, we set

$$\delta_n^T(\boldsymbol{e}) = \sum_{e \in \boldsymbol{e}^+} \delta_n^T(G_e), \qquad \delta_n^T(\boldsymbol{v}) = \sum_{v \in \boldsymbol{v}} \delta_n^T(G_v).$$

By Corollary 4.1, we have

$$\mathrm{E}_n^{\tau_G^T}\left(\bigoplus_{e \in \boldsymbol{e}^+} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}\right) = \mathop{*}_{e \in \boldsymbol{e}^+} \mathrm{E}_n^{\tau_G^T}\left(\mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}\right).$$

If ^{T} is compatible with monomorphisms, then by Lemma 5.2

$$\mathrm{E}_n^{\tau_G^T}(\mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}) = \iota_e^T \,\mathrm{E}_n^T(G_e)$$

(where $\iota_e : G_e \to G$ is the natural injection), and so, in this case, we get

$$\mathrm{E}_n^{\tau_G^T}\left(\bigoplus_{e\in \boldsymbol{e}^+} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}\right) = \mathop{*}_{e\in \boldsymbol{e}^+} \iota_e^T \,\mathrm{E}_n^T(G_e).$$

Similarly (again assuming that ^{T} is compatible with monomorphisms), we get

$$\mathrm{E}_n^{\tau_G^T}\left(\bigoplus_{v\in \boldsymbol{v}} \mathbb{Z}G \otimes_{\mathbb{Z}G_v} \mathbb{Z}\right) = \mathop{*}_{v\in \boldsymbol{v}} \iota_v^T \,\mathrm{E}_n^T(G_v).$$

Now, applying Theorems 4.6 and 4.7 to the short exact sequence (5.14), we obtain the following result.

Theorem 5.8. *Suppose that ^{T} is compatible with monomorphisms, and that each G_v $(v \in \boldsymbol{v})$ is of type FP_n and each G_e $(e \in \boldsymbol{e})$ is of type FP_{n-1}, so G is of type FP_n. Then*

(i) *if each G_e $(e \in \boldsymbol{e})$ is $\mathrm{E}^T[m-1, n-1]$-trivial $(m > 0)$,*

$$\mathrm{E}_i^T(G) = \mathop{*}_{v\in \boldsymbol{v}}{}^{[\delta_i^T(\boldsymbol{e})]} \left(\iota_v^T \,\mathrm{E}_i^T(G_v)\right),$$

for $m \le i < n$;

(ii) *if each G_v $(v \in \boldsymbol{v})$ is $\mathrm{E}^T[m, n]$-trivial $(m > 0)$,*

$$\mathrm{E}_i^T(G) = \mathop{*}_{e\in \boldsymbol{e}^+}{}^{[-\delta_i^T(\boldsymbol{v})]} \left(\iota_e^T \,\mathrm{E}_{i-1}^T(G_e)\right),$$

for $m < i \le n$.

In particular, since the trivial group is $\mathrm{E}[0, \infty]$-trivial, with $\delta_n(1) = (-1)^n$, we have the following corollary.

Corollary 5.2. *Suppose that ^{T} is compatible with monomorphisms. If G_1, G_2 are groups of type FP_n, then, for $n \ge 1$*

$$\mathrm{E}_n^T(G_1 * G_2) = \iota_i^T \,\mathrm{E}_n^T(G_1) *^{[(-1)^n]} \iota_2^T \,\mathrm{E}_n^T(G_2).$$

Remark. This was proved for the chains of Alexander ideals (the E_1-ideals) in [30].

Corollary 5.3. *Suppose that ^{T} is compatible with monomorphisms. If each G_v $(v \in \boldsymbol{v})$ is $\mathrm{E}^T[m, n]$-trivial and each G_e $(e \in \boldsymbol{e})$ is $\mathrm{E}^T[m-1, n-1]$-trivial $(m > 0)$, then G is $\mathrm{E}^T[m, n]$-trivial and, for $m \le i \le n$,*

$$\delta_i^T(G) = \delta_i^T(\boldsymbol{v}) - \delta_i^T(\boldsymbol{e}).$$

Example 5.6. For any m, n, with $0 \le m \le n \le \infty$, we find a group $G_{m,n}$ which is $\mathrm{E}[m,n]$-trivial, but neither $\mathrm{E}^T[m-1,n]$-trivial (when $m > 0$), nor $\mathrm{E}^T[m,n+1]$-trivial (when $n < \infty$) for any ^{T}.

For finite n, let H be a group of type FP_{n+1} which is $\mathrm{E}[0,n]$-trivial, but is not $\mathrm{E}[0,n+1]$-trivial. Such a group exists [48], [8], [39]. Notice that, by Theorem 5.6, $\mathrm{H}_{n+1}(H) \ne 0$.

Let $G_{m,n} = H \times \mathbb{Z}^m$. We show by induction on m that $G_{m,n}$ is $\mathrm{E}[m,n]$-trivial. By definition, $G_{0,n} = H$ is $\mathrm{E}[0,n]$-trivial. Now suppose that, for $m \ge 1$, $G_{m-1,n}$ is $\mathrm{E}[m-1,n]$-trivial. Then, from Corollary 5.3, since $G_{m,n}$ is an HNN-extension of $G_{m-1,n}$, it is $\mathrm{E}[m,n]$-trivial.

Now, since $\mathrm{H}_m(G_{m,n}) = \mathbb{Z} \ne 0$ and $\mathrm{H}_{n+1}(G_{m,n}) = \mathrm{H}_{n+1}(H) \ne 0$, by Lemma 5.6, $G_{m,n}$ can be neither $\mathrm{E}^{\mathrm{triv}}[m-1,n]$-trivial nor $\mathrm{E}^{\mathrm{triv}}[m,n+1]$-trivial, and so cannot be $\mathrm{E}^T[m-1,n]$- or $\mathrm{E}^T[m,n+1]$-trivial for any ^{T}.

To deal with the case when $n = \infty$, take H to be an acyclic group of type FP_∞, and proceed as above.

In Example 5.2, we exhibited groups which could not be distinguished by their integral homology or by their E_1-ideals, but which could be shown to be distinct by their E_2-ideals. We now prove a more general result of this kind.

Theorem 5.9. *For every n, there exists an infinite family of groups which have the same integral homology and, for $0 \le i \le n$, the same E_i-ideals, but whose E_{n+1}-ideals distinguish them.*

Proof. Again, let H be a group of type FP_n which is $\mathrm{E}[0,n]$-trivial, but not $\mathrm{E}[0,n+1]$-trivial. Let $G = H \times \langle t \rangle$, and, for $l \ge 1$, let G_l be the amalgamated product

$$G \underset{\langle t \rangle = \langle s^l \rangle}{*} \langle s \rangle.$$

Let $\alpha_l : G \to G_l$ be the natural injection. From the Mayer–Vietoris sequence for the homology of an amalgamated product [36], we deduce that $\mathrm{H}_*(G_l) = \mathrm{H}_*(G)$ for all l. Since the infinite cyclic group $\langle t \rangle$ is $\mathrm{E}[1,\infty]$-trivial, with $\delta_i(\langle t \rangle) = 0$ for $i \ge 1$, we obtain from Theorem 5.8(i) that, for $2 \le i \le n$,

$$\mathrm{E}_{i,\lambda}(G_l) = \left(\alpha_l^{\mathrm{ab}}\, \mathrm{E}_{i,\lambda}(G)\right).$$

By Theorem 5.7, for $i \le n$

$$\mathrm{E}_i(G) = \mathrm{E}_i(\langle t \rangle).$$

Thus, for $2 \le i \le n$,

$$\mathrm{E}_{i,\lambda}(G_l) = \begin{cases} \mathbb{Z}\langle \hat{s} \rangle, & \lambda \ge 0, \\ 0, & \lambda < 0. \end{cases}$$

A simple calculation, using a pre-abelian presentation for H [47], shows that this also holds for $i = 1$.

We compute $\mathrm{E}_{n+1}(G)$. Suppose that

$$\mathrm{H}_{n+1}(H) = \mathbb{Z}^q \oplus \left(\bigoplus_{j=1}^{p} \frac{\mathbb{Z}}{(c_j)} \right),$$

where each c_j is an integer greater than 1, with $c_j | c_{j+1}$. If $\mathcal{F}_0$ is a resolution of type FP_{n+1} for H, then, as in the proof of Theorem 3.5, bases may be chosen so that, if D_i is the matrix of ∂_{i+1}, then $D_i^{\mathrm{ab}} = D_i^{\mathrm{triv}}$ has the form (3.6). For $i = 0, \dots, n$, Δ_i in (3.6) is the $(\overrightarrow{\chi}_i(\mathcal{F}_0) - (-1)^i)$-square identity matrix, and Δ_{n+1} is the $(\overrightarrow{\chi}_{n+1}(\mathcal{F}_0) - q - (-1)^{n+1})$-square diagonal matrix with diagonal entries $1, \dots, 1, c_1, \dots, c_p$. The group $\langle t \rangle$ has the resolution

$$\mathcal{F}_1: \qquad 0 \to \mathbb{Z}\langle t \rangle \xrightarrow{t \mapsto 1-t} \mathbb{Z}\langle t \rangle \xrightarrow{\mathrm{aug}} \mathbb{Z} \to 0.$$

The $\mathbb{Z}G$-complex $\mathcal{F} = (\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathcal{F}_0) \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}\langle t \rangle} \mathcal{F}_1)$ is then a resolution of type FP_{n+1} for G, with $\overrightarrow{\chi}_i(\mathcal{F})$ equal to the rank r_i of the ith module of $\mathcal{F}_0$. The matrix of its $(n+2)$th boundary map is

$$\begin{bmatrix} D_{n+1} & 0 \\ (-1)^n(\bar{t} - 1)I_{r_n} & D_n \end{bmatrix}$$

with respect to the induced bases, whence

$$\mathrm{E}_{n+1,\lambda}(G) = \begin{cases} \mathbb{Z}\langle t \rangle, & \lambda \geq p+q, \\ \left(\left\{ c_1 \cdots c_k (\hat{t} - 1)^{p+q-\lambda-k} \right\}_{k=0}^{p+q-\lambda} \right), & q \leq \lambda < p+q, \\ \left(\left\{ c_1 \cdots c_k (\hat{t} - 1)^{p+q-\lambda-k} \right\}_{k=0}^{p} \right), & 0 \leq \lambda < q, \\ 0, & \lambda < 0. \end{cases}$$

Now,

$$\mathrm{E}_{n+1,\lambda}(G_l) = \left(\alpha_l^{\mathrm{ab}} \, \mathrm{E}_{n+1,\lambda}(G) \right),$$

and so, in particular, $\mathrm{E}_{n+1,1}(G_l) = (c_1, \hat{s}^l - 1)$. But, if $l \neq l'$, neither of the automorphisms

$$\mathbb{Z}\langle \hat{s} \rangle \to \mathbb{Z}\langle \hat{s} \rangle; \quad \hat{s} \mapsto \hat{s}^{\pm 1}$$

sends $(c_1, \hat{s}^l - 1)$ to $(c_1, \hat{s}^{l'} - 1)$. We conclude that, if $l \neq l'$, then $G_l \neq G_{l'}$.

Remark. There are certain other constructions where an analysis similar to that for graphs of groups can be used to obtain information about the E-ideals of the constructed group in terms of those of the groups used in the construction. However, an extra technique may be needed, namely that of *dimension shifting* (a well-known term in homological algebra) .

For example, in [52] an analysis was given of groups with presentations where the set of generator is partitioned into 'types' and each defining relator involves at most two types of generators. For such a group G, under certain circumstances, there is a short exact sequence similar to (5.14) except that the last term ${}_G\mathbb{Z}$ is replaced by the augmentation ideal $\mathrm{I}G$, and the summands in the other two terms have the form $\mathbb{Z}G \otimes_{\mathbb{Z}A} \mathrm{I}A$ for certain subgroups A of G (see [52] for precise details). In this case, we would need to use the fact (which can be deduced from (3.3)) that for any group H

$$\mathrm{E}_{n,\lambda}^{\tau_H^T}(\mathrm{I}H) = \mathrm{E}_{n+1,\lambda+(-1)^{n+1}}^T(H) \qquad (\lambda \in \mathbb{Z}).$$

A similar situation arises in connection with generalised graphs of groups [4], [12].

5.7 Eventually E-trivial groups and a question of Serre

We will say that a group G is *eventually* E^T*-trivial over* K if it is $\mathrm{E}^T[l,\infty]$-trivial over K for some l. We then have the invariant $\delta^T(G,K) = (-1)^l \delta_l^T(G,K)$.

We denote the class of groups of type FL over K by $\mathbf{FL}_K$, the class of groups of type FP over K by $\mathbf{FP}_K$, and the class of eventually E_K^T-trivial groups by $\mathbf{ET}_K^T$. As usual, when K is $\mathbb{Z}$, or ^{T} is $^{\mathrm{ab}}$, we omit it. For each of these classes $\mathbf{C}_K$, we have $\mathbf{C} \subseteq \mathbf{C}_K$. We also have the following inclusions:

$$\begin{array}{cl} & \mathbf{ET}_K^{\mathrm{triv}} \\ & \cup\!\! \mid \\ \mathbf{ET}_K \subseteq & \mathbf{ET}_K^T \\ \cup\!\! \mid & \cup\!\! \mid \text{(when } KG^T \text{ is indecomposable).} \\ \mathbf{FL}_K \subseteq & \mathbf{FP}_K \end{array}$$

The question of whether the containment $\mathbf{FL} \subseteq \mathbf{FP}$ is proper is a major open question raised by Serre [58]. At one stage, we hoped to show that this inclusion is proper using the following strategy: find a group G which is of type FP, but not obviously of type FL, construct an infinite free resolution of type FP_∞ for G, compute from this the E-ideals of G, and thereby, hopefully, show that $G \notin \mathbf{ET}$ (and thus $G \notin \mathbf{FL}$). However, this strategy cannot work, since the group ring with $\mathbb{Z}$-coefficients of any group is indecomposable (see [49]), so Theorem 4.10 applies, and we have

$$\mathbf{FL} \subseteq \mathbf{FP} \subseteq \mathbf{ET}.$$

So now, in addition to Serre's question of whether the first inclusion is proper, a further intriguing question arises.

Open Question 1. Is the inclusion $\mathbf{FP} \subseteq \mathbf{ET}$ proper?

In view of Theorem 5.6, a positive answer to this question could be obtained by answering the following question positively.

Open Question 2. Is there an acyclic group of type FP_∞ which is not of type FP?

A possible candidate for such an example is mentioned in [15].

Although the above strategy does not work for $\mathbb{Z}$-coefficients, it does work for slightly larger coefficient rings, as we now show – the strategy also works for monoids, as we will see in Section 6.

Theorem 5.10. *Let G be defined by a finite CA presentation $\langle \boldsymbol{t}; \boldsymbol{w}\rangle$ (as in Section 5.3). Let K be any ring containing $\mathbb{Z}[1/p(W)\ (W \in \boldsymbol{w})]$. Then*

(i) *$G \in \mathbf{FP}_K$; and*

(ii) *if the ideal I of KG^{ab} generated by the elements $\xi_{+1}^{\mathrm{ab}}(W)$ $(W \in \boldsymbol{w}')$ is non-zero, then $G \notin \mathbf{ET}_K$ (and so $G \notin \mathbf{FL}_K$).*

Proof. (i) From the Lyndon resolution (5.13), we get the short exact sequence

$$0 \to \operatorname{im} \partial_3 \to \bigoplus_{W \in \boldsymbol{w}} \mathbb{Z}G e_W \xrightarrow{\partial_2} \operatorname{im} \partial_2 \to 0,$$

where $\operatorname{im} \partial_3$ is the submodule generated by the elements $(1 - \overline{W_0}) e_W$ $(W \in \boldsymbol{w})$. Then, applying $K \otimes -$, we find that $K \otimes \operatorname{im} \partial_2$ is isomorphic to

$$\bigoplus_{W \in \boldsymbol{w}'} \frac{KG}{KG(1 - \overline{W_0})}.$$

But $KG/KG(1 - \overline{W_0})$ is a projective KG-module, for we have the map

$$KG \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \frac{KG}{KG(1 - \overline{W_0})}; \quad \alpha\beta = 1,$$

where α is the natural surjection, and

$$\beta(\xi + KG(1 - \overline{W_0})) = \frac{1}{p(W)} \xi \sum_{i=0}^{p(W)-1} \overline{W_0}^{\,i} \qquad (\xi \in KG).$$

Then $K \otimes \operatorname{im} \partial_2$ is projective, so applying $K \otimes -$ to the exact sequence (obtained from (5.8))

$$0 \to \operatorname{im} \partial_2 \to \mathbb{Z}G^{|\boldsymbol{t}|} \xrightarrow{\partial_1} \mathbb{Z}G \to {}_G\mathbb{Z} \to 0$$

gives a projective resolution of ${}_GK$.

(ii) Using Corollary 5.1 and Lemma 5.3, we find that, for even $n > 0$,

$$\mathrm{E}_{n, |\boldsymbol{w}| - |\boldsymbol{t}|}(G, K) = I.$$

Now, I lies in the augmentation ideal of KG^{ab}, so $I \neq KG^{\mathrm{ab}}$. Thus, if $I \neq 0$, then G cannot be $\mathrm{E}[l, \infty]$-trivial over K for any l.

Example 5.7. Let G be a one-relator group with torsion given by a presentation $\langle \boldsymbol{t}; R_0^p \rangle$ $(p > 1)$. If R_0 does not lie in the derived subgroup of the free group on $\boldsymbol{t}$ (that is, some $t \in \boldsymbol{t}$ occurs in R_0 with non-zero exponent sum), then G is of type FP over $\mathbb{Z}[1/p]$, but not of type FL over $\mathbb{Z}[1/p]$.

Example 5.8. Lee and Park [42] proved (by different methods) that a Fuchsian group G given by a presentation of the form

$$\langle x_1, \ldots, x_q; x_i^{n_i} \ (i = 1, \ldots, q), (x_i \ldots x_q)^{n_{q+1}} \rangle,$$

where $n_1, \ldots, n_{q+1} \geq 2$ and either $q > 2$ or $q = 2$ and $1/n_1 + 1/n_2 + 1/n_3 \leq 1$, is of type FP over $\mathbb{Q}$, but not of type FL over $\mathbb{Q}$. Provided G is not perfect, this result follows from our theorem.

In fact, let K be any commutative ring containing $\mathbb{Z}[1/n_i \ (1 \leq i \leq q+1)]$. Now, the above presentation is CA (shown, for example, by the weight test [12]). Also, I is the augmentation ideal of KG^{ab}, so, provided G is not perfect, $I \neq 0$, and then G will be of type FP over K, but not FL over K.

Remark. It is conceivable that the strategy described above for answering Serre's question for $\mathbb{Z}$-coefficients could be made to work by computing the chain $\mathrm{E}_n^\rho(G)$ for some representation $\rho : \mathbb{Z}G \to \mathrm{Mat}_k(C)$. Of course, by Theorem 4.10, this approach would only be feasible for C *not* indecomposable.

5.8 (Eventually) E-linked groups

We will say that a group G is $\mathrm{E}^T[m,n]$-*linked* if ${}_G\mathbb{Z}$ is and we will say that G is *eventually* E^T-*linked* if it is $\mathrm{E}^T[l,\infty]$-linked for some l. In the latter case, we define $\delta^T(G)$ to be $(-1)^l\delta_l^T(G)$.

When ^{T} is ${}^{\mathrm{triv}}$, these concepts have homological interpretations.

Proposition 5.1. *A group G is $\mathrm{E}^{\mathrm{triv}}[m,n]$-linked if, and only if, it is of type FP_n and $\mathrm{H}_i(G)$ is torsion for $m < i \leq n$. In this case,*

$$\delta_i^{\mathrm{triv}}(G) = (-1)^i \sum_{j=0}^{m} (-1)^j \operatorname{rk}_{\mathbb{Z}} \mathrm{H}_j(G)$$

for $m \leq i \leq n$.

This follows from Theorem 3.5.

In particular, a group is eventually $\mathrm{E}^{\mathrm{triv}}$-linked if, and only if, it is of type FP_∞ and $\mathrm{H}_*(G)$ is eventually torsion, and in this case

$$\begin{aligned}\delta^{\mathrm{triv}}(G) &= \sum_{i\geq 0}(-1)^i \operatorname{rk}_{\mathbb{Z}} \mathrm{H}_i(G)\\ &= \sum_{i\geq 0}(-1)^i \dim \mathrm{H}_i(G,\mathbb{Q}).\end{aligned}$$

Thus, the eventually $\mathrm{E}^{\mathrm{triv}}$-linked groups are the groups of type FP_∞ whose rational homology $\mathrm{H}_*(G,\mathbb{Q})$ is eventually 0, and $\delta^{\mathrm{triv}}(G)$ is then Brown's Euler characteristic $\tilde{\chi}(G)$ [13].

Proposition 5.2. *Suppose that G is $\mathrm{E}^{T'}[m,n]$-linked. Let ^{T} be such that $\tau_G^{T'}$ factors through τ_G^T and $\mathrm{E}^T_{i,\delta_i^T}(G)$ contains a non-zero-divisor for $m \leq i \leq n$. Then G is $\mathrm{E}^T[m,n]$-linked and*

$$\delta_i^T(G) = \delta_i^{T'}(G) \qquad (m \leq i \leq n).$$

Proof. If G is $\mathrm{E}^{T'}[m,n]$-linked, then, for $m \leq i < n$,

$$\delta_i^{T'}(G) + \delta_{i+1}^{T'}(G) = 0.$$

By Theorem 3.4(i), we also have

$$\delta_i^T(G) + \delta_{i+1}^T(G) \geq 0 \qquad (m \leq i < n).$$

By (5.3),

$$\delta_i^T(G) \le \delta_i^{T'}(G). \tag{5.15}$$

Thus,

$$0 \le \delta_i^T(G) + \delta_{i+1}^T(G) \le \delta_i^{T'}(G) + \delta_{i+1}^{T'}(G) = 0,$$

so G is $\mathrm{E}^T[m,n]$-linked.

To show the second part, we need only show that $\delta_m^T(G) = \delta_m^{T'}(G)$. From (5.15),

$$-\delta_m^T(G) = \delta_{m+1}^T(G) \le \delta_{m+1}^{T'}(G) = -\delta_m^{T'}(G),$$

so $\delta_m^{T'}(G) \le \delta_m^T(G)$, and thus $\delta_m^{T'}(G) = \delta_m^T(G)$, as required.

In particular, since $\mathbb{Z}G^{\mathrm{tf}}$ is an integral domain, and since G^{triv} factors through G^{tf}, we deduce the following.

Corollary 5.4. *If G is $\mathrm{E}^{\mathrm{triv}}[m,n]$-linked, then it is $\mathrm{E}^{\mathrm{tf}}[m,n]$-linked.*

The converse is false: the R. Thompson group is $\mathrm{E}^{\mathrm{tf}}[0,\infty]$-linked (with $\delta^{\mathrm{tf}}(G) = 0$), but it is not $\mathrm{E}^{\mathrm{triv}}[m,n]$-linked for any $0 \le m < n \le \infty$.

The E-linked property is actually quite subtle. We illustrate this in the context of CA groups.

Suppose that G is defined by a finite CA presentation $\langle \boldsymbol{t}; \boldsymbol{w}\rangle$, as in Section 5.3. Recall that for $W \in \boldsymbol{w}'$

$$\begin{aligned} \xi_{+1}(W) &= 1 - \overline{W}_0, \\ \xi_{-1}(W) &= 1 + \overline{W}_0 + \cdots + \overline{W}_0^{p(W)-1}. \end{aligned}$$

Now, since the E-ideals for G repeat with period 2 from dimension 2 onwards, it is enough to consider when G is $\mathrm{E}^T[2,3]$-linked (in which case it will be $\mathrm{E}^T[2,\infty]$-linked).

Note that $\xi_{-1}^{\mathrm{triv}}(W)$ is non-zero, so that $\xi_{-1}^T(W)$ is non-zero for any T. Thus, the ideal $\mathrm{E}_{3,-1+|\boldsymbol{t}|-|\boldsymbol{w}|}(G)$, which is generated by

$$\xi^T = \prod_{W \in \boldsymbol{w}'} \xi_{-1}^T(W)$$

is non-zero. Since $\mathrm{E}_{3,\lambda}^T(G) = 0$ for $\lambda < -1 + |\boldsymbol{t}| - |\boldsymbol{w}|$, we have

$$\delta_3^T(G) = -1 + |\boldsymbol{t}| - |\boldsymbol{w}|.$$

For $-(1 - |\boldsymbol{t}| + |\boldsymbol{w}| - |\boldsymbol{w}'|) \le \lambda < 1 - |\boldsymbol{t}| + |\boldsymbol{w}|$, $\xi^T \in \mathrm{Ann}(\mathrm{E}_{2,\lambda}^T(G))$, so all the elements of $\mathrm{E}_{2,\lambda}^T(G) \setminus \{0\}$ are zero-divisors. Thus, in order for G to be $\mathrm{E}^T[2,3]$-linked, the ideals $\mathrm{E}_{2,\lambda}^T(G)$ $(-(1 - |\boldsymbol{t}| + |\boldsymbol{w}| - |\boldsymbol{w}'|) \le \lambda < 1 - |\boldsymbol{t}| + |\boldsymbol{w}|)$ must be 0. Since the largest of these, $\mathrm{E}_{2,-|\boldsymbol{t}|+|\boldsymbol{w}|}^T(G)$, is generated by the elements $\xi_{+1}^T(W)$ $(W \in \boldsymbol{w}')$, the condition that all these ideals be 0 is just the condition that the image of $\overline{W}_0$ $(W \in \boldsymbol{w}')$ in G^T is 1. If this condition holds, then we have

$$\delta_2^T(G) = 1 - |\boldsymbol{t}| + |\boldsymbol{w}|.$$

Thus, we have proved the following result.

Theorem 5.11. *Let G be the group defined by the finite CA presentation $\langle \boldsymbol{t}; \boldsymbol{w}\rangle$. Then G is $\mathrm{E}^T[2,3]$-linked (and hence $\mathrm{E}^T[2,\infty]$-linked) if, and only if, the image of each element $\overline{W_0}$ $(W \in \boldsymbol{w}')$ in G^T is 1. In this case*

$$\delta_i^T(G) = (-1)^i(1 - |\boldsymbol{t}| + |\boldsymbol{w}|)$$

for all $i \geq 2$.

This result can be reformulated in terms of the abelianising function $^{\mathrm{wtf}}$.

By the torsion theorem for CA groups [38], the elements of finite order in G are powers of conjugates of the elements $\overline{W_0}$ $(W \in \boldsymbol{w}')$. Consequently, the quotient G_0 of G by the normal subgroup generated by the elements of finite order is given by the presentation $\langle \boldsymbol{t}; W_0\ (W \in \boldsymbol{w})\rangle$, and G^{wtf} is then G_0^{ab}. The condition that the image of each element $\overline{W_0}$ $(W \in \boldsymbol{w}')$ in G^T is 1 is just the condition that $\tau_G^T : G \to G^T$ factors through $\tau_G^{\mathrm{wtf}} : G \to G^{\mathrm{wtf}}$.

Theorem 5.12. *A group given by a finite CA presentation is $\mathrm{E}^T[2,\infty]$-linked if, and only if, $\tau_G^T : G \to G^T$ factors through $\tau_G^{\mathrm{wtf}} : G \to G^{\mathrm{wtf}}$.*

Example 5.9. Let G be given the CA presentation $\langle a, b; (aba^{-1}b^{q-1})^p\rangle$ $(p, q > 1)$. Then G^{ab}, G^{wtf} and G^{tf} are given by the presentations $\langle a, b; b^{pq}, aba^{-1}b^{-1}\rangle$, $\langle a, b; b^q, aba^{-1}b^{-1}\rangle$ and $\langle a;\rangle$ respectively. The group G is thus $\mathrm{E}^{\mathrm{wtf}}[2,\infty]$-linked and $\mathrm{E}^{\mathrm{tf}}[2,\infty]$-linked, but not $\mathrm{E}^{\mathrm{ab}}[2,\infty]$-linked.

Before we leave E-linked groups, we consider further the inclusions of classes in Sections 4.4 and 5.7. If we let $\mathbf{EL}_K^T$ denote the class of groups which are eventually E^T-linked over K, then we have the inclusions

$$\begin{array}{ccccc}
\mathbf{EL}_K \subseteq & \mathbf{EL}_K^T & \subseteq & \mathbf{EL}_K^{\mathrm{triv}} \\
\cup\!| & \cup\!| & & \cup\!| \\
\mathbf{ET}_K \subseteq & \mathbf{ET}_K^T & \subseteq & \mathbf{ET}_K^{\mathrm{triv}} \\
\cup\!| & \cup\!| & & \\
\mathbf{FL}_K \subseteq & \mathbf{FP}_K, & &
\end{array}$$

where, in general, the inclusion $\mathbf{FP}_K \subseteq \mathbf{ET}_K^T$ only holds when KG^T is indecomposable. Unlike the situation in Section 5.7, the inclusions $\mathbf{ET}_K^T \subseteq \mathbf{EL}_K^T$ are known to be proper in many cases – consider, for instance, CA groups and R. Thompson's group.

5.9 Finite conjugacy classes and a theorem like Gottlieb's

Let U be a finite conjugacy class of G and for any coefficient ring K let ζ_U be the element

$$|U| \cdot 1 - \sum_{u \in U} u$$

of KG. Then $\zeta_U \in Z(KG) \cap \mathrm{Ann}({}_G K)$. We thus deduce the following result from Theorem 4.2.

Theorem 5.13. *Let G be of type FP_n. Then, for any finite conjugacy class U of G,*

$$\lambda\zeta_U^T\,\mathrm{E}^T_{n,\lambda}(G,K)\subseteq \mathrm{E}^T_{n,\lambda-1}(G,K)\qquad(\lambda\in\mathbb{Z})$$

(here ζ_U^T is the image of ζ_U in KG^T).

Example 5.10. Consider the braid group $B_n (n\geq 3)$ with presentation

$$\begin{aligned}\langle a_1,\dots,a_{n-1};\ &a_ia_{i+1}a_i=a_{i+1}a_ia_{i+1}\ (1\leq i\leq n-2),\\ &a_ia_j=a_ja_i\ (1\leq i\leq n-2,\ i+1\leq j\leq n-1)\rangle.\end{aligned}$$

Then B_n^{ab} is infinite cyclic, generated by t, say. The centre of B_n is infinite cyclic, generated by $u=(\bar a_1\cdots\bar a_{n-1})^n$ [22]. Then $\zeta_u^{\mathrm{ab}}=1-t^{n(n-1)}$.

For $n=3,4$,

$$\mathrm{E}_{1,\lambda}(B_n)=\begin{cases}\mathbb{Z}\langle t\rangle, & \lambda\geq 1,\\ \left(1-t+t^2\right), & \lambda=0,\\ 0, & \lambda<0,\end{cases}$$

and, for $n\geq 5$,

$$\mathrm{E}_{1,\lambda}(B_n)=\begin{cases}\mathbb{Z}\langle t\rangle, & \lambda\geq 0,\\ 0, & \lambda<0.\end{cases}$$

For $\lambda>0$ (or $\lambda>1$ when $n=3,4$) or $\lambda<0$, we clearly have $\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{1,\lambda}(B_n)\subseteq \mathrm{E}_{1,\lambda-1}(B_n)$, and for $\lambda=0$ we clearly have $0\cdot\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{1,0}(B_n)=\mathrm{E}_{1,-1}(B_n)$. Since $1-t+t^2$ is a factor of $1-t^6$ (and hence of $1-t^{12}$), we also have $1\cdot\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{1,1}(B_n)\subseteq \mathrm{E}_{1,0}(B_n)$ for $n=3,4$.

Example 5.11. Consider the group G_l in the proof of Theorem 5.9. Taking $p=q=2$, for instance, we have

$$\mathrm{E}_{n+1,\lambda}(G_l)=\begin{cases}\mathbb{Z}\langle\hat s\rangle, & \lambda\geq 4,\\ \left(\hat s^l-1,c_1\right), & \lambda=3,\\ \left((\hat s^l-1)^2,c_1(\hat s^l-1),c_1c_2\right), & \lambda=2,\\ \left((\hat s^l-1)^3,c_1(\hat s^l-1)^2,c_1c_2(\hat s^l-1)\right), & \lambda=1,\\ \left((\hat s^l-1)^4,c_1(\hat s^l-1)^3,c_1c_2(\hat s^l-1)^2\right), & \lambda=0,\\ 0, & \lambda<0.\end{cases}$$

Now, $u=\bar s^l$ is central and $\zeta_u^{\mathrm{ab}}=1-\hat s^l$. We see that $\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{n+1,\lambda}(G_l)\subseteq \mathrm{E}_{n+1,\lambda-1}(G_l)$ $(2<\lambda)$, $\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{n+1,\lambda}(G_l)=\mathrm{E}_{n+1,\lambda-1}(G_l)$ $(0<\lambda\leq 2)$ and $0\cdot\zeta_u^{\mathrm{ab}}\,\mathrm{E}_{n+1,0}(G_l)=\mathrm{E}_{n+1,-1}(G_l)$.

Example 5.12. Let G be the group defined by the presentation

$$\langle x,y,a;a^2,xyx^{-1}y^{-1},xay^{-1}a^{-1}\rangle.$$

Then G^{ab} is a direct product of a cyclic group $\langle\hat a\rangle$ of order 2 (corresponding to a) and an infinite cyclic group generated by $\hat x$ corresponding to x (and y). The two elements $\overline{x}$ and $\overline{y}$ of G represented by x and y form a single conjugacy class U, and $\zeta_U^{\mathrm{ab}} = 2(1-\hat{x})$.

A free resolution of ${}_G\mathbb{Z}$ is

$$\cdots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\operatorname{rk}_{\mathbb{Z}G} F_0 = 1$, $\operatorname{rk}_{\mathbb{Z}G} F_1 = 3$, $\operatorname{rk}_{\mathbb{Z}G} F_2 = 3$, $\operatorname{rk}_{\mathbb{Z}G} F_n = 2$ $(n \geq 3)$ and $\partial_1, \partial_2, \partial_3, \partial_4, \partial_n$ $(n \geq 5)$ are given by the matrices

$$\begin{bmatrix} 1-\overline{x} \\ 1-\overline{y} \\ 1-\overline{a} \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1+\overline{a} \\ 1-\overline{y} & \overline{x}-1 & 0 \\ 1 & -\overline{a} & \overline{x}-1 \end{bmatrix},$$
$$\begin{bmatrix} 1-\overline{a} & 0 & 0 \\ -(\overline{x}-1)(\overline{y}-1) & 1+\overline{a} & (\overline{a}+1)(\overline{y}-1) \end{bmatrix},$$
$$\begin{bmatrix} (\overline{x}-1)(\overline{y}-1) & 1-\overline{a} \\ 1+\overline{a} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1-(-1)^n\overline{a} & (-1)^n(\overline{x}-1)(\overline{y}-1) \\ 0 & 1+(-1)^n\overline{a} \end{bmatrix}$$

respectively. We deduce that

$$\mathrm{E}_{1,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda > 0 \\ \left(1+\hat{a}, (1-\hat{a})(\hat{x}-1), (\hat{x}-1)^2\right), & \lambda = 0, \\ 0, & \lambda < 0, \end{cases}$$

and for $n > 1$

$$\mathrm{E}_{n,\lambda}(G) = \begin{cases} \mathbb{Z}G^{\mathrm{ab}}, & \lambda > 0, \\ \left(1-\hat{a}, 1+\hat{a}, (\hat{x}-1)^2\right) = \left(1-\hat{a}, 2, \hat{x}^2+1\right), & \lambda = 0, \\ 0, & \lambda < 0. \end{cases}$$

Note that

$$\zeta_U^{\mathrm{ab}} = (1-\hat{x}) \cdot (1+\hat{a}) - 1 \cdot (1-\hat{a})(\hat{x}-1),$$

so $\zeta_U^{\mathrm{ab}}\,\mathrm{E}_{1,1}(G) \subseteq \mathrm{E}_{1,0}(G)$, as expected. Clearly, also, for $n > 1$, $\zeta_U^{\mathrm{ab}}\,\mathrm{E}_{n,1}(G) \subseteq \mathrm{E}_{n,0}(G)$, as expected.

In the last example, we have $\delta_n(G) = 0$ for all n. This is illustrative of the following general result.

Theorem 5.14. *If G has a finite conjugacy class such that the image $\hat{u}$ in G^{ab} of some $u \in U$ has infinite order, then $\delta_n(G) = 0$, when defined.*

Proof. From Theorem 5.13, we have

$$\delta_n(G)|U|(1-\hat{u})e = 0$$

for any $e \in \mathrm{E}_{n,\delta_n(G)}(G)$. Since $\hat{u}$ has infinite order, $1-\hat{u}$ is not a zero-divisor in $\mathbb{Z}G^{\mathrm{ab}}$. Since $\mathrm{E}_{n,\delta_n(G)}(G)$ is non-trivial, we may choose e to be non-zero. Thus, we must have $\delta_n(G) = 0$.

Notice that in the above proof, if we could choose e to be a non-zero-divisor, then we would only need to require that $\hat{u}$ be non-trivial to arrive at a similar conclusion. Thus, we obtain the next result.

Theorem 5.15. *Suppose that G is of type FP_n and that $\mathrm{E}_{n,\delta_n(G)}(G)$ contains a non-zero-divisor. If G has a finite conjugacy class U such that the image of some $u \in U$ in G^{ab} is non-trivial, then $\delta_n(G) = 0$.*

Corollary 5.5. *If G is eventually* E*-linked and if $\delta(G) \neq 0$, then every finite conjugacy class of G is contained in the derived subgroup G' of G.*

In particular, if G is of type FL and if $\chi(G) \neq 0$, then every finite conjugacy class of G is contained in G'. However, in this case there is a stronger result, which follows from Stallings' proof [60] of Gottlieb's theorem, namely:

If G is of type FL and $\chi(G) \neq 0$, then G has no non-trivial finite conjugacy classes. (See [40, p. 145].)

Open Question 3. Is there an eventually E-linked group with $\delta \neq 0$ having a non-trivial finite conjugacy class?

5.10 Invariant polynomials

For a finitely generated group G, the group G^{tf} will be free abelian on a finite set $\boldsymbol{x}$, and an argument like that in [24, Section VIII.2] shows that, if K is a unique factorisation domain, then the Laurent polynomial ring $KG^{\mathrm{tf}} = K[x, x^{-1}\ (x \in \boldsymbol{x})]$ is a greatest common divisor domain in which the only units are the 'obvious' ones of the form cg (for c a unit of K, and $g \in G^{\mathrm{tf}}$).

If G is of type FP_n, then $\mathrm{E}^{\mathrm{tf}}_{n,\lambda}(G, K)$ $(\lambda \in \mathbb{Z})$ will be contained in a smallest principal ideal $\overline{\mathrm{E}}^{\mathrm{tf}}_{n,\lambda}(G, K)$. We let $\mathrm{e}_{n,\lambda}(G, K)$ denote a generator of this ideal. This polynomial is then unique up to multiplication by a unit of the Laurent polynomial ring KG^{tf}. We deduce from Lemma 5.1 that these polynomials are group invariants in the following sense.

Lemma 5.7. *If $\alpha : H \to G$ is an isomorphism, then α^{tf} carries $\mathrm{e}_{n,\lambda}(H, K)$ to $\mathrm{e}_{n,\lambda}(G, K)$ $(\lambda \in \mathbb{Z})$, up to multiplication by a unit.*

When $K = \mathbb{Z}$ and $n = 1$, these polynomials are, of course, just the standard Alexander polynomials [24].

Since $\mathrm{E}^{\mathrm{tf}}_{n,\lambda}(G, K) \subseteq \mathrm{E}^{\mathrm{tf}}_{n,\lambda+1}(G, K)$, we have $\overline{\mathrm{E}}^{\mathrm{tf}}_{n,\lambda}(G, K) \subseteq \overline{\mathrm{E}}^{\mathrm{tf}}_{n,\lambda+1}(G, K)$, and so

$$\mathrm{e}_{n,\lambda+1}(G, K) \,\big|\, \mathrm{e}_{n,\lambda}(G, K) \quad (\lambda \in \mathbb{Z}).$$

Example 5.13. Let G be given by the presentation

$$\langle a, b, t; [a, b], t^{-m}at^m = a^p, t^{-m}bt^m = b^q \rangle$$

where $m, p, q > 0$. The calculus of pictures [4] shows that this presentation has one generating picture, and the matrix (5.10) is then

$$\Big[(1 + \overline{a} + \cdots + \overline{a}^{p-1})(1 + \overline{b} + \cdots + \overline{b}^{q-1})t^m - 1 \quad 1 - b \quad a - 1\Big]$$

So

$$\mathrm{E}^{\mathrm{tf}}_{2,0}(G,\mathbb{Z}) = \begin{cases} \left(pq\hat{t}^m - 1\right), & p,q > 1, \\ \left(q\hat{t}^m - 1, \hat{a} - 1\right), & p = 1, q > 1, \\ \left(p\hat{t}^m - 1, \hat{b} - 1\right), & p > 1, q = 1, \\ \left(\hat{t}^m - 1, \hat{a} - 1, \hat{b} - 1\right), & p = q = 1, \end{cases}$$

and

$$\mathrm{e}_{2,0}(G,\mathbb{Z}) = \begin{cases} pq\hat{t}^m - 1, & p,q > 1, \\ 1, & p = 1 \text{ or } q = 1. \end{cases}$$

Example 5.14. Let G be the group in Example 5.12. Then $\mathrm{E}^{\mathrm{tf}}_{n,0}(G,\mathbb{Z}) = (2, \hat{x}^2 + 1)$ for all n, so $\mathrm{e}_{n,0}(G,\mathbb{Z})$ is just the constant polynomial 1. However, if we pass to $\mathbb{Z}_2$-coefficients we get the more interesting polynomial $\mathrm{e}_{n,0}(G,\mathbb{Z}_2) = \hat{x}^2 + 1$.

Example 5.15. For the group G_l in the proof of Theorem 5.9, let r be a prime divisor of c_1. Then

$$\mathrm{e}_{n+1,\lambda}(G_l,\mathbb{Z}_r) = \begin{cases} 1, & \lambda \geq p + q, \\ (\hat{s}^l - 1)^{p+q-\lambda}, & 0 \leq \lambda < p + q, \\ 0, & \lambda < 0. \end{cases}$$

5.11 Minimality of resolutions

If G is a group and $\mathcal{F}$ is a free resolution of ${}_G\mathbb{Z}$ of type FP_n, then, by the definition of the E-ideals,

$$\frac{1}{\dim(\rho)}\nu^{\rho}_n(G) \leq \overrightarrow{\chi}_n(\mathcal{F})$$

for any representation ρ of $\mathbb{Z}G$. Thus, for a group G of type FP_n,

$$\overrightarrow{\chi}_n(G) = \min\{\overrightarrow{\chi}_n(\mathcal{F}) : \mathcal{F} \text{ a free resolution of } \mathbb{Z} \text{ of type } \mathrm{FP}_n\}$$

exists and

$$\frac{1}{\dim(\rho)}\nu^{\rho}_n(G) \leq \overrightarrow{\chi}_n(G)$$

for any representation ρ.

Note in particular that taking the one-dimensional representation $\tau^{\mathrm{triv}}_G : \mathbb{Z}G \to \mathbb{Z}$ gives the second equality of Lemma 5.4,

$$\nu^{\mathrm{triv}}_n(G) = \mathrm{d}(\mathrm{H}_n(G)) - \mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_{n-1}(G) + \cdots + (-1)^n\,\mathrm{rk}_{\mathbb{Z}}\,\mathrm{H}_0(G),$$

which is the well-known lower bound for $\overrightarrow{\chi}_n(G)$ obtained in [61].

If G has an Eilenberg–Mac Lane complex $\mathcal{K}$ with finite n-skeleton (so G is of type F_n), then the chain complex of the universal cover of $\mathcal{K}$ gives a free resolution $\mathcal{F}(\mathcal{K})$ of ${}_G\mathbb{Z}$ of type FP_n with $\overrightarrow{\chi}_n(\mathcal{F}(\mathcal{K}))$ equal to the directed Euler characteristic $\overrightarrow{\chi}_n(\mathcal{K})$ of the n-skeleton of $\mathcal{K}$. Thus, if G is of type F_n, then

$$q_n(G) = \min\{\overrightarrow{\chi}_n(\mathcal{K}) : \mathcal{K} \text{ is a } K(G,1)\text{-space with finite } n\text{-skeleton}\}$$

exists and

$$\overrightarrow{\chi}_n(G) \leq q_n(G).$$

A resolution $\mathcal{F}$ (respectively, $K(G,1)$-space $\mathcal{K}$) is said to be *n-minimal* if it realises $\overrightarrow{\chi}_n(G)$ (respectively, $q_n(G)$), and we will say that it is *n-optimal* if $\overrightarrow{\chi}_n(\mathcal{F})$ (respectively, $\overrightarrow{\chi}_n(\mathcal{K})$) is equal to $(1/\dim(\rho))\nu_n^\rho(G)$ for some representation ρ. Obviously, n-optimality implies n-minimality.

The invariant $\overrightarrow{\chi}_n(G)$ was defined and studied by Swan [61] (where it is denoted by $\mu_n(G)$). The invariant q_2 (which is concerned with the minimality and 'efficiency' of group presentations) has been much studied (see [4], [34], the references cited there, and [43]), and the higher invariants q_n have been considered by Eckmann [27].

In [44] (see also [45]) Lustig gave a test for n-minimality as follows. Let $\mathcal{F} = (F_i, \partial_i)_{i\geq 0}$ be a free resolution of ${}_G\mathbb{Z}$ of type FP_n. The nth *Fox ideal*, $\Phi_n(\mathcal{F})$, of $\mathcal{F}$ is the (two-sided) ideal of $\mathbb{Z}G$ generated by the entries of the matrix of ∂_{n+1} with respect to a choice of bases for F_n, F_{n+1} (the Fox ideal is independent of the choice of bases).

Lustig's Test. *If there is a representation ρ of $\mathbb{Z}G$ with $\rho(\Phi_n(\mathcal{F})) = 0$, then $\mathcal{F}$ is n-minimal.*

Example 5.16. Let G be the R. Thompson group and consider the resolution $\mathcal{F}$ due to Brown and Geoghegan [17] described in Section 5.4. For each $n \geq 0$, the image of Φ_n under the abelianising map $\mathbb{Z}G \to \mathbb{Z}G^{ab}$ is $\mathrm{I}G^{ab}$, so the image of Φ_n under the trivialising map $\mathbb{Z}G \to \mathbb{Z}$ is 0. Thus, $\mathcal{F}$ is n-minimal for all n, and so $\overrightarrow{\chi}_n(G) = 1$ $(n \geq 0)$. It follows from [17] that $\mathcal{F}$ arises from a $K(G,1)$-space, so $q_n(G) = \overrightarrow{\chi}_n(G) = 1$ for all n.

Example 5.17. Let G be the group of Example 5.5, and consider the resolution described there. Then

$$\begin{aligned}
\Phi_0 &= \left(1 - \overline{x}\ (x \in \boldsymbol{x}), 1 - \overline{t}\right), \\
\Phi_1 &= \left(1 - \overline{R_0}, p\overline{\frac{\partial R_0}{\partial x}}, (1 - q\overline{t})\overline{\frac{\partial R_0}{\partial x}}\ (x \in \boldsymbol{x})\right) \\
\Phi_n &= \left(1 - \overline{R_0}, p, 1 - q^{[(n+1)/2]}\overline{t}\right) \qquad (n > 1).
\end{aligned}$$

Since q is coprime to p, its image in $\mathbb{Z}_p$ is a unit. The map

$$\rho_n : \mathbb{Z}G \to \mathbb{Z}_p; \quad x \mapsto 1, \quad t \mapsto q^{-[(n+1)/2]}$$

sends Φ_n to 0 $(n \geq 0)$. Thus, the resolution is n-minimal for all n, so

$$\overrightarrow{\chi}_0(G) = 1, \qquad \overrightarrow{\chi}_n(G) = (-1)^n(1 - |\boldsymbol{x}|) \quad (n > 0).$$

(Again, it can be shown that the resolution in question is the chain complex of a $K(G,1)$-space, so $q_n(G) = \overrightarrow{\chi}_n(G)$ for all n.)

We show that Lustig's test is, in fact, a test for n-optimality.

Theorem 5.16. *A resolution $\mathcal{F}$ will satisfy Lustig's test if, and only if, it is n-optimal*

Proof. Suppose that $\mathcal{F}$ is n-optimal, so $\nu_n^\rho(G) = k\overrightarrow{\chi}_n(\mathcal{F})$ for some representation $\rho : \mathbb{Z}G \to \mathrm{Mat}_k(C)$. If $D = [a_{ij}]$ is a matrix for the $(n+1)$th boundary map ∂_{n+1} of $\mathcal{F}$, then $J = \mathrm{E}^\rho_{n,\nu_n^\rho - 1}(G)$ is the ideal of C generated by the entries of all the matrices $\rho(a_{ij})$ as i, j vary, and $J \neq C$. Define $\overline{\rho}$ to be the composition

$$\mathbb{Z}G \xrightarrow{\rho} \mathrm{Mat}_k(C) \to \mathrm{Mat}_k(C/J).$$

Then $\overline{\rho}(a_{ij}) = 0$ for all i, j, so $\mathcal{F}$ satisfies Lustig's test.

Now suppose that $\mathcal{F}$ satisfies Lustig's test for some $\rho : \mathbb{Z}G \to \mathrm{Mat}_k(C)$. Then $\mathrm{E}_{n,k\overrightarrow{\chi}_n(\mathcal{F})-1}(G)$ is the ideal of C generated by the entries of the matrix $D^\rho = [\rho(a_{ij})]$, and this is the zero ideal, since $\rho(a_{ij}) = 0$ for each i, j. Thus, $\nu_n^\rho(G) = k\overrightarrow{\chi}_n(\mathcal{F})$, and so $\mathcal{F}$ is n-optimal.

If G is given by a finite CA presentation $\langle \boldsymbol{t}; \boldsymbol{w}\rangle$ (see Section 5.3), then for the Lyndon resolution $\mathcal{L}$ we find that $\Phi_n(\mathcal{L})$ $(n \geq 2)$ is generated by the elements $\xi_{\varepsilon(n)}(W)$ $(W \in \boldsymbol{w}')$. When n is even, $\xi_{\varepsilon(n)}(W) \in \mathrm{I}G$, so the one-dimensional representation $\tau_G^{\mathrm{triv}} : \mathbb{Z}G \to \mathbb{Z}$ sends $\Phi_n(\mathcal{L})$ to 0. The Lyndon resolution is thus n-minimal for $n \geq 2$, even.

Open Question 4. Is the Lyndon resolution n-minimal for $n \geq 3$, odd?

6 Monoids

We can easily extend the definition of the E-ideals from groups to monoids. However, we must distinguish a left- and a right-hand case. If S is a monoid, then by considering free left resolutions of type FP_n of the *left* $\mathbb{Z}S$-module ${}_S\mathbb{Z}$ we can obtain a chain of ideals $\mathrm{E}_n^{(l)}(S)$ in $\mathbb{Z}S^{\mathrm{ab}}$ (or, more generally, a chain of ideals $\mathrm{E}_n^{(l)T}(S)$ in $\mathbb{Z}S^T$ for ^{T} an abelianising function on monoids). If we instead consider resolutions of type FP_n of the trivial *right* $\mathbb{Z}S$-module $\mathbb{Z}_S$, we obtain a chain of ideals $\mathrm{E}_n^{(r)}(S)$.

When S is a group, the automorphism

$$\mathbb{Z}S^{\mathrm{ab}} \to \mathbb{Z}S^{\mathrm{ab}}; \quad sS' \mapsto s^{-1}S' \qquad (s \in S)$$

of $\mathbb{Z}S^{\mathrm{ab}}$ sends $\mathrm{E}^{(l)}_{n,\lambda}(S)$ to $\mathrm{E}^{(r)}_{n,\lambda}(S)$ for each $\lambda \in \mathbb{Z}$, which is why, for groups, we have only discussed the chains of ideals $\mathrm{E}_n(S) = \mathrm{E}_n^{(l)}(S)$. However, in general, the $\mathrm{E}_n^{(l)}$- and $\mathrm{E}_n^{(r)}$-ideals may be different. In fact, an example due to Cohen [23] shows that, for a given dimension $n > 0$, the chain of $\mathrm{E}_n^{(l)}$-ideals can exist while the chain of $\mathrm{E}_n^{(r)}$-ideals does not (and vice versa). Even when they do both exist, they can be different, as shown in the example below.

Given a monoid presentation $\mathcal{P} = [\boldsymbol{x}; \boldsymbol{r}]$ for $S = S(\mathcal{P})$, there is a concept of spherical pictures over $\mathcal{P}$ representing elements of a certain $(\mathbb{Z}S, \mathbb{Z}S)$-bimodule

[53], [54]. If we let $\boldsymbol{d}$ be a set of pictures representing generators for this bimodule, then there are partial left and right resolutions of $\mathbb{Z}$ analogous to (5.9) [54]. These partial resolutions may then be used to calculate the E_0-, E_1- and E_2-ideals of S. We mention that a calculus of pictures over monoid presentations is emerging [53], [55], [56], [64].

When S has a finite complete rewriting system, it is known that S is both left and right FP_∞. Moreover, the rewriting system can be used to explicitly compute a left or right free resolution of $\mathbb{Z}$ [3], [16], [32], [41]. This makes the computation of the $\mathrm{E}_n^{(l)}$- and $\mathrm{E}_n^{(r)}$-ideals for S tractable.

Example 6.1. Let S be the monoid defined by the complete rewriting presentation

$$\left[x, \theta; x\theta = \theta, \theta^2 = \theta\right].$$

Using the method of [16], we obtain a free right resolution

$$\cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z}_S \to 0,$$

where $F_0 = \mathbb{Z}S$, $F_n = z_x\mathbb{Z}S \oplus z_\theta\mathbb{Z}S$ $(n > 0)$, $\partial_1(z_x) = 1 - \overline{x}$, $\partial_1(z_\theta) = 1 - \overline{\theta}$ and, for $n > 1$,

$$\partial_n(z_x) = \begin{cases} z_x\overline{\theta}, & n \text{ even}, \\ z_x(1-\overline{\theta}), & n \text{ odd}, \end{cases} \qquad \partial_n(z_\theta) = \begin{cases} z_\theta\overline{\theta}, & n \text{ even}, \\ z_\theta(1-\overline{\theta}), & n \text{ odd}. \end{cases}$$

This then gives

$$\mathrm{E}_{0,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{\mathrm{ab}}, & \lambda \geq 1, \\ (1-\hat{\theta}, 1-\hat{x}) = (1-\hat{\theta}), & \lambda = 0, \\ 0, & \lambda < 0, \end{cases}$$

$$\mathrm{E}_{n,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{\mathrm{ab}}, & \lambda \geq 1, \\ (1-\hat{\theta}), & \lambda = -1, 0, \\ 0, & \lambda < -1 \end{cases} \quad (n > 0, \text{ even}),$$

$$\mathrm{E}_{n,\lambda}^{(r)}(S) = \begin{cases} \mathbb{Z}S^{\mathrm{ab}}, & \lambda \geq 1, \\ (\hat{\theta}), & \lambda = -1, 0, \\ 0, & \lambda < -1 \end{cases} \quad (n \text{ odd}).$$

In a similar way, we can obtain a free left resolution of ${}_S\mathbb{Z}$. This resolution may then have a *collapsing scheme* [16] applied to it (that is, a number of inverse Tietze transformations), giving the resolution

$$\cdots \xrightarrow{1 \mapsto \overline{\theta}} \mathbb{Z}S \xrightarrow{1 \mapsto 1-\overline{\theta}} \mathbb{Z}S \xrightarrow{1 \mapsto \overline{\theta}} \mathbb{Z}S \xrightarrow{1 \mapsto 1-\overline{\theta}} \mathbb{Z}S \xrightarrow{\mathrm{aug}} {}_S\mathbb{Z} \to 0.$$

This gives

$$\mathrm{E}_{n,\lambda}^{(l)}(S) = \begin{cases} \mathbb{Z}S^{\mathrm{ab}}, & \lambda \geq 1, \\ (1-\hat{\theta}), & \lambda = 0, \\ 0, & \lambda < 0 \end{cases} \quad (n \text{ even}), \qquad \begin{cases} \mathbb{Z}S^{\mathrm{ab}}, & \lambda \geq 0, \\ (\hat{\theta}), & \lambda = -1, \\ 0, & \lambda < -1 \end{cases} \quad (n \text{ odd}).$$

Notice that S is neither eventually $\mathrm{E}^{(l)}$-trivial, nor eventually $\mathrm{E}^{(r)}$-trivial, so S can be neither left nor right FL. However [33], $\mathrm{cd}^{(l)}(S) = 0$ and $\mathrm{cd}^{(r)}(S) = 1$. Thus, $\mathbb{Z}$ as a left $\mathbb{Z}S$-module and $\mathrm{I}S$ as a right $\mathbb{Z}S$-module are both projective, so S is left and right FP. Thus, we see that the monoid version of Serre's question (Open Question 1) can be answered (negatively), using the strategy described in Section 5.7.

Remark. The ideals found in the last example are a special case of a general result concerning monoids with a zero (here, θ is a right zero). This result, and others, will appear in [25].

References

[1] J. W. Alexander, *Topological invariants of knots and links*, Trans. Amer. Math. Soc. **30** (1928), 275–306.

[2] G. Almkvist, *Endomorphisms of finitely generated projective modules over a commutative ring*, Ark. Mat. **11** (1973), 263–301.

[3] D. J. Anick, *On the homology of associative algebras*, Trans. Amer. Math. Soc. **296** (1986), 641–659.

[4] Y.-G. Baik and S. J. Pride, *On the efficiency of Coxeter groups*, Bull. London Math. Soc. **29** (1997), 32–36.

[5] Y.-G. Baik, W. A. Bogley and S. J. Pride, *On the asphericity of length four relative group presentations*, Int. J. Algebra. Comput. **7** (1997) 227–312.

[6] Y.-G. Baik, J. Harlander and S. J. Pride, *The geometry of group extensions*, J. Group Theory **1** (1998), 395–416.

[7] G. Baumslag, *A remark on groups with trivial multiplicator*, Amer. J. Math. **97** (1975), 863–864.

[8] G. Baumslag, E. Dyer and A. Heller, *The topology of discrete groups*, J. Pure Appl. Algebra **16** (1980), 1–47.

[9] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.

[10] R. Bieri, *Homological Dimension of Discrete Groups*, Queen Mary College, 1976.

[11] W. A. Bogley and S. J. Pride, *Aspherical relative presentations*, Proc. Edinburgh Math. Soc. **35** (1992), 1–32.

[12] W. A. Bogley and S. J. Pride, *Calculating generators of* π_2 in Two-dimensional Homotopy and Combinatorial Group Theory, eds C. Hog-Angeloni, W. Metzler and A. J. Sieradski, Cambridge University Press, 1993, pp. 157–188.

[13] K. S. Brown, *Euler characteristics of discrete groups and G-spaces*, Invent. Math. **27** (1974), 229–264.

[14] K. S. Brown, *Cohomology of Groups*, Springer, 1982.

[15] K. S. Brown, *The geometry of finitely presented infinite simple groups* in Algorithms and Classification in Combinatorial Group Theory, eds G. Baumslag and C. F. Miller III, Springer, 1992, pp. 121–136.

[16] K. S. Brown, *The geometry of rewriting systems: A proof of the Anick–Groves–Squier theorem* in Algorithms and Classification in Combinatorial Group Theory, eds G. Baumslag and C. F. Miller III, Springer, 1992, pp. 137–163.

[17] K. S. Brown and R. Geoghegan, *An infinte-dimensional torsion-free* FP_∞ *group*, Invent. Math. **77** (1984), 367–381.

[18] R. Brown and J. Huebschmann, *Identities among relations* in Low-dimensional Topology, eds R. Brown and T. L. Thickstun, Cambridge University Press, 1982, pp.153–202.

[19] J. W. Cannon, W. J. Floyd and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. II. Sr. **42** (1996), 215–256.

[20] H. Cartan and S. Eilenberg, *Homological Algebra*, Oxford University Press, 1956.

[21] I. M. Chiswell, *Exact sequences associated with a graph of groups*, J. Pure. Appl. Algebra **8** (1976), 63–74.

[22] W. L. Chow, *On the algebraic braid group*, Ann. Math. **49** (1948), 654–658.

[23] D. Cohen, *A monoid which is right FP_∞ but not left FP_1*, Bull. London Math. Soc. **24** (1992), 340–342.

[24] R. H. Crowell and R. H. Fox, *Introduction to Knot Theory*, Ginn and Company, 1963.

[25] D. A. Cruickshank, New invariants for monoids, in preparation.

[26] M. N. Dyer, *Groups with no imperfect subgroups and aspherical 2-complexes*, Coment. Math. Helv. **68** (1993), 333-339.

[27] B. Eckmann, *Amenable groups and Euler characteristics*, Comment. Math. Helv. **67** (1992), 383–393.

[28] M. Edjvet, *On the asphericiaty of one-relator relative presentations*, Proc. R. Soc. Edinburgh A **124** (1994), 713–728.

[29] D. Eisenbud and M. L. Green, *Ideals of minors in free resolutions*, Duke Math. J. **75**, 339–352.

[30] R. H. Fox, *Free differential calculus. I: the isomorphism problem of groups*, Ann. Math. (2) **59** (1954), 196–210.

[31] N. D. Gilbert and T. Porter, *Knots and Surfaces*, The Clarendon Press, 1994.

[32] J. R. J. Groves, *Rewriting systems and homology of groups* in Groups-Canberra 1989, ed. L. G. Kovács, Springer, 1990, pp.114–141.

[33] V. S. Guba and S. J. Pride, *On the left and right cohomological dimension of monoids*, Bull. London Math. Soc. **30** (1998), 391–396.

[34] J. Harlander, *Some aspects of efficiency*, Preprint, University of Frankfurt, 1998.

[35] A. Hattori, *Rank element of a projective module*, Nagoya Math. J. **25** (1965), 113–120.

[36] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, 1971.

[37] J. Howie and V. Metaftsis, *On the asphericity of length five relative presentations*, to appear in Proc. London Math. Soc.

[38] J. Huebschmann, *Cohomology theory of aspherical groups and of small cancellation groups*, J. Pure Appl. Algebra **14** (1979), 137–143.

[39] D. M. Kan and W. P. Thurston, *Every connected space has the homology of a $K(G,1)$*, Topology **15** (1976), 253–258.

[40] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, 1970.

[41] Y. Kobayashi, *Complete rewriting systems and homology of monoid algebras*, J. Pure Appl. Algebra **65** (1990), 263–275.

[42] J. B. Lee and C.-Y. Park, *Finiteness properties of some Poincaré duality groups*, J. Korean Math. Soc. **32** (1995), 33–40.

[43] W. Lück, *L^2-Betti numbers of mapping tori and groups*, Topology **33** (1994), 203–214.

[44] M. Lustig, *On the rank, the deficiency and the homological dimension of groups: The computation of a lower bound via Fox ideals* in Topology and Combinatorial Group Theory, ed. P. Latiolais, Springer, 1990, pp.164–174.

[45] M. Lustig, *Fox ideals, $\mathcal{N}$-torsion and applications to groups and 3-manifolds*, in Two-dimensional Homotopy and Combinatorial Group Theory, eds C. Hog-Angeloni, W. Metzler and A. J. Sieradski, Cambridge University Press, 1993, pp.219–250.

[46] R. C. Lyndon, *Cohomology theory of groups with a single defining relation*, Ann. Math. II. **52** (1950), 650–665.

[47] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Dover Publications, Inc., New York, 1976.

[48] C. R. F. Maunder, *A short proof of a theorem of Kan and Thurston*, Bull. London Math. Soc. **13** (1981), 325–327.

[49] M. S. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540.

[50] C. D. Papakyriakopolous, *On Dehn's lemma and the asphericity of knots*, Ann. Math. (2) **66** (1957), 1–26.

[51] S. J. Pride, *Identities among relators of group presentations* in Group Theory from a Geometrical Viewpoint, eds E. Ghys, A. Haefliger and A. Verjovsky, World Scientific, 1991, pp.687–717.

[52] S. J. Pride, *The (co)homology of groups given by presentations in which each defining relator involves at most two types of generators*, J. Austral. Math. Soc. A **52** (1992), 205–218.

[53] S. J. Pride, *Geometric methods in combinatorial semigroup theory* in Semigroups, Formal Languages and Groups, ed. J. Fountain, Kluwer, 1995, pp.215–232.

[54] S. J. Pride, *Low dimensional homotopy theory for monoids*, Int. J. Algebra Comput. **5** (1995), 631–649.

[55] S. J. Pride and J. Wang, *Rewritting systems, finiteness conditions, and associated functions* in Algorithmic Problems in Groups and Semigroups, eds J.-C. Birget, S. Margolis, J. Meakin and M. Sapir, Birkhaüser, 2000, pp. 195–216.

[56] S. J. Pride and X. Wang, *Second order Dehn functions of groups and monoids*, Int. J. Algebra Comput. **10** (2000), 425–456.

[57] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, London, 1979.

[58] J.-P. Serre, *Cohomologie des groupes discrets*, Ann. Math. Stud. **70** (1971), 77–169.

[59] J.-P. Serre, *Trees*, Springer, 1980.

[60] J. Stallings, *Centerless groups—an algebraic formulation of Gottlieb's theorem*, Topology **4** (1965), 129–134.

[61] R. G. Swan, *Minimal resolutions for finite groups*, Topology **4** (1965), 193–208.

[62] L. Traldi, *Higher Fitting invariants*, Houston J. Math., **17** (1991), 55–61.

[63] C. T. C. Wall, *Resolutions for extensions of groups*, Proc. Cambridge Phil. Soc. **57** (1961), 251–255.

[64] J. Wang, *Rewriting Systems, Finiteness Conditions and Second Order Dehn Functions of Monoids*, Ph.D. Thesis, University of Glasgow, 1998.