On pattern structures of the N-soliton solution of the discrete KP equation over a finite field.

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J. Phys. A: Math. Theor. **40** (2007), 949-959.

M.B. in this work was supported by

Grant-in-Aid of the JSPS Postdoctoral Fellowship for Foreign Researchers (No.P05050)

and by

Polish Ministry of Science and Information Society Technologies (Grant 1 P03B 01728).

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N-soliton solutions over $\mathbb{F}$

$\mathbb{F} = \mathbb{F}_q$ - a fixed finite field and $\mathbb{L} \supset \mathbb{F}$ - a finite extension of $\mathbb{F}$, $G(\mathbb{L}/\mathbb{F})$ the Galois group.

Parameters (all parameters in the construction are distinct)

- $A_0, A_i \in \mathbb{F}$, $i = 1, 2, 3$,
- $C_\alpha \in \mathbb{L}$, where $\alpha = 1, \ldots, N$, the $\mathbb{F}$-rationality conditions

$$\forall \sigma \in G(\mathbb{L}/\mathbb{K}), \quad \sigma(C_\alpha) = C_{\alpha'},$$

- N pairs $D_\beta, E_\beta \in \mathbb{L}$, for $\beta = 1, \ldots, N$, satisfy the $\mathbb{K}$-rationality conditions

$$\forall \sigma \in G(\mathbb{L}/\mathbb{K}) : \quad \sigma(\{D_\beta, E_\beta\}) = \{D_{\beta'}, E_{\beta'}\},$$

Auxiliary functions ϕ_α, $\alpha = 1, 2, ...N$,

$$\phi_\alpha(t) = \frac{1}{t - C_\alpha} \prod_{k=1}^{3} \left(\frac{t - A_k}{C_\alpha - A_k} \right)^{n_k},$$

$N \times N$ matrix with element in row β and column α given by

$$[\phi_{\boldsymbol{A}}(\boldsymbol{D}, \boldsymbol{E})]_{\alpha\beta} = \phi_\alpha(D_\beta) - \phi_\alpha(E_\beta).$$

Theorem 1. *The function $\tau(n_1, n_2, n_3) : \mathbb{Z}^3 \to \mathbb{F}$ given by*

$$\boxed{\tau = \det \phi_{\boldsymbol{A}}(\boldsymbol{D}, \boldsymbol{E})}$$

is the $\mathbb{F}$-valued N-soliton solution of the discrete KP equation

$$Z_1(T_1\bar{\tau})(T_{23}\bar{\tau}) - Z_2(T_2\bar{\tau})(T_{13}\bar{\tau}) + Z_3(T_3\bar{\tau})(T_{12}\bar{\tau}) = 0$$

for $Z_1 = A_2 - A_3, \quad Z_2 = A_1 - A_3, \quad Z_3 = A_1 - A_2$.

T_i denotes a shift operator in a variable n_i, for example $T_2\tau(n_1, n_2, n_3) = \tau(n_1, n_2 + 1, n_3)$.

A gauge invariance: (for any constant $\alpha, \beta, \gamma, \delta$)

$$\tau \simeq \tau' = \alpha^{n_1}\beta^{n_2}\gamma^{n_3}\delta \cdot \tau.$$

Theorem 2. *Let q denote any fixed generator of $\mathbb{F}^*$ i.e. a multiplicative subgroup of the finite field $\mathbb{F}$. The N-soliton solution of the dKP equation over a finite field $\mathbb{F}$ admits the following form*

$$\tau' = \sum_{J\subset\{1,\ldots,N\}} (-1)^{\#J} \left(\prod_{i,i'\in J;\ i<i'} a_{ii'} \right) q^{\left(\sum_{j\in J} \hat{\eta}_j\right)},$$

where the sum is taken over all subsets of $\{1,\ldots,N\}$ and $\#J$ denotes the cardinality of J. The exponents are $\hat{\eta}_j = \eta_j + \eta_j^0$ where

$$\boxed{\eta_j := \sum_{k=1}^{3} p_j^k n_k.}$$

Moreover

$$a_{ij} := \frac{(D_i - D_j)(E_i - E_j)}{(D_i - E_j)(D_j - E_i)},$$

and the parameters p_j^k and phase constants η_j^0 are defined by

$$q^{p_i^k} := \frac{E_i - A_k}{D_i - A_k} \quad \text{and} \quad q^{\eta_i^0} := \prod_{p=1}^{N} \frac{(C_p - D_i)}{(C_p - E_i)} \prod_{p=1;\,p\neq i}^{N} \frac{(D_p - E_i)}{(D_p - D_i)}.$$

In example: a **two**-soliton solution

$$1 - q^{\hat{\eta}_1} - q^{\hat{\eta}_2} + a_{12} q^{\hat{\eta}_1+\hat{\eta}_2},$$

and a **three**-soliton solution

$$1 - q^{\hat{\eta}_1} - q^{\hat{\eta}_2} - q^{\hat{\eta}_3} + a_{12} q^{\hat{\eta}_1+\hat{\eta}_2} + a_{23} q^{\hat{\eta}_2+\hat{\eta}_3} + a_{13} q^{\hat{\eta}_1+\hat{\eta}_3} - a_{12}a_{23}a_{13} q^{\hat{\eta}_1+\hat{\eta}_3+\hat{\eta}_3}.$$

- No "$<$" relation, no analogue of wave amplitude; one may trace only the propagation of patterns;
- Since $q^{|\mathbb{F}|-1} = 1$, this implies periodicity of $\tau(n_1, n_2, n_3)$ with respect to each variable n_i.
- No asymptotic behaviour, all information about the solution in a finite *base cube* with the length of any edge at most $|\mathbb{F}| - 1$.
- The ith one-soliton component of the N-soliton solution is unchanged by a shift in the lattice by $\vec{n}^i = (n_1^i, n_2^i, n_3^i)$ for any n_1^i, n_2^i, n_3^i satisfying $q^{\eta_i} = 1$, or equivalently,

$$\eta_i = \sum_{k=1}^{3} p_i^k n_k^i \equiv 0 \mod (|\mathbb{F}| - 1).$$

  Since the formula for N-soliton solutions contains q^{η_i} for $i \in \{1, 2, \ldots, N\}$, a period vector $\vec{n} = (n_1, n_2, n_3)$ for this solution is a common solution for all i. In general, it is impossible to find a nonzero solution for $N \geq 3$ and it means there is no additional structure within the base cube in this case.

Examples

- The finite field $\mathbb{F} = \mathbb{F}_{17}$ (integers modulo 17). As a generator of $\mathbb{F}^*$ we choose $q = 3$.
- $A_1 = 7$, $A_2 = 4$, $A_3 = 3$; $C_1 = 11$, $D_1 = 6$, $E_1 = 9$; $C_2 = 10$, $D_2 = 12$, $E_2 = 14$; $C_3 = 8$, $D_3 = 13$, $E_3 = 15$.

$$\begin{aligned} \vec{p}_1 &= (p_1^1, p_1^2, p_1^3) = (6, 7, 14), \\ \vec{p}_2 &= (p_2^1, p_2^2, p_2^3) = (6, 9, 5), \\ \vec{p}_3 &= (p_3^1, p_3^2, p_3^3) = (11, 5, 10). \end{aligned}$$

Periods in variables n_1, n_2, n_3 are: $8, 16, 8$ for the soliton $_1$, $8, 16, 16$ for the soliton $_2$ and $16, 16, 8$ for $_3$.

Examples of period vectors: $\vec{n}_1 = (1, 0, 3)$, $\vec{n}_2 = (1, 0, 2)$ and $\vec{n}_3 = (1, 1, 0)$.

Period vectors $\vec{n}$ in the plane n_2, n_3 are $\vec{n}_{1a} = \vec{n}_{3a} = (0, 2, -1)$, $\vec{n}_{1b} = \vec{n}_{3b} = (0, 0, 8)$, $\vec{n}_{2a} = (0, 3, 1)$ and $\vec{n}_{2b} = (0, -1, 5)$.

Elements of $\mathbb{F}_{17}$ are represented on the scale below: from 0 - dark to 16 - light gray.

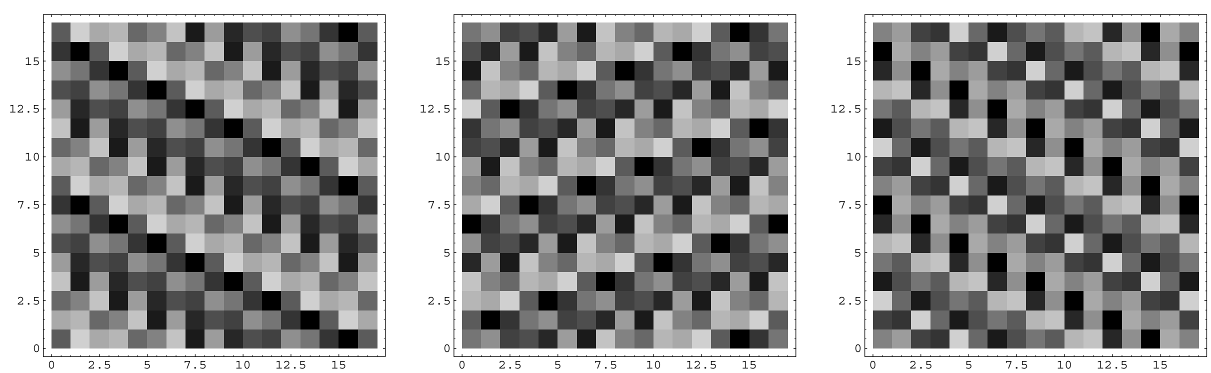

Figure 1: A plot of $\tau(n_1, n_2, n_3)$ function of a three one-soliton solutions. We fix $n_1 = 0$, and $n_2, n_3 \in \{0, \ldots, 16\}$. The n_2 axis is directed to the right and the n_3 axis is directed upwards.

Elements of $\mathbb{F}_{17}$ are represented on the scale below: from 0 - dark to 16 - light gray.

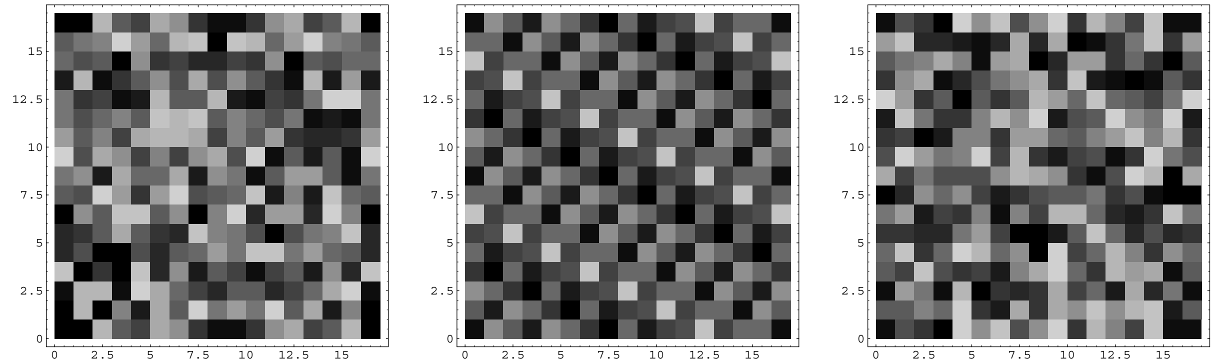

Figure 2: A plot of $\tau(n_1, n_2, n_3)$ for the three two-soliton interactions (AB, AC, BC) of one-soliton solutions presented in Figure 1. We fix $n_1 = 0$, and $n_2, n_3 \in \{0, \ldots, 16\}$. The n_2 axis is directed to the right and the n_3 axis is directed upwards.

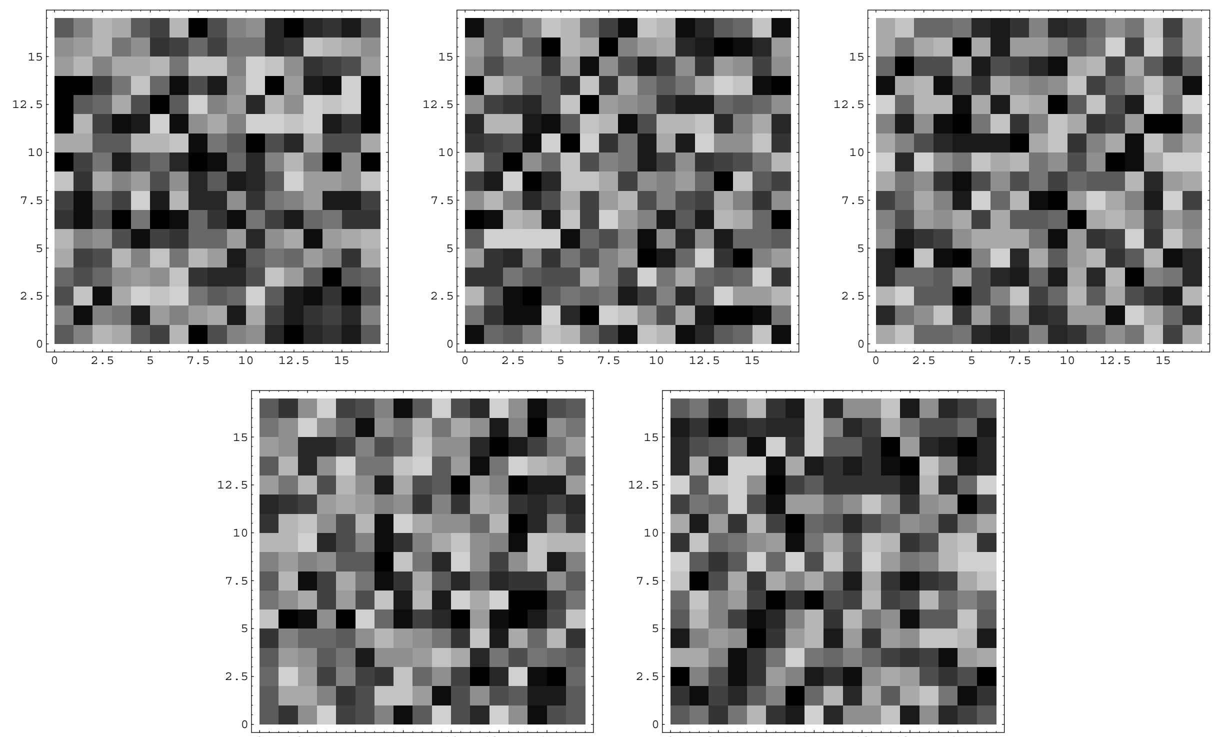

Figure 3: A three-soliton solution $\tau(n_1, n_2, n_3)$ being the solitonic sum of those from Figure 1 for $n_1 = 0, 1, 2, 4$ and 8. Axes: n_2 directed to the right, n_3 directed upward.