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Quasi-linear system related to integrable Hamitonians

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1 The system:

$$U(t,q) = (u^1(t,q) \dots u^n(t,q))$$

$$U_t = A(U)\, U_q$$

$u^i(t,q)$ are assumed to be periodic in q and in t

$$\begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{pmatrix}_t = \underbrace{\begin{pmatrix} nu_0 & -1 & 0 & \dots & 0 \\ (n-1)u_1 & 0 & -1 & \dots & 0 \\ \vdots & & & & \vdots \\ & & & & -1 \\ u_{n-1} & 0 & \dots & & 0 \end{pmatrix}}_{A(U)} \begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ u_n \end{pmatrix}_q$$

2)

$$(u_k)_t = (n-k+1)\, u_{k-1} \cdot (u_1)_q - (u_{k+1})_q$$
$$k = 1, \dots, n$$

$$u_0 = \text{const}\,, \quad u_{n+1} \equiv 0$$

This system appears naturally in the following problem. Consider

$$H = \frac{1}{2} p^2 + u(q,t) \quad \begin{cases} \dot{q} = p \\ \dot{p} = -u_q(q,t) \end{cases}$$

Classical question: Find all those potentials $u(q,t)$ periodic in q, t such that the system is integrable, i.e. admits an additional integral

$$F = \frac{1}{n+1} p^{n+1} + u_0 p^n + u_1 p^{n-1} + \dots + u_n$$

Then one can easily show

3) $u_0 = \text{const}$, $u_1 = u$ is the potential and $u_1, \ldots, u_n$ satisfy the quasi-linear system

So we came to the following

Question Do there exist periodic solutions for $U_t = A(U)U_x$

Remark (misleading) Generally Hamiltonian systems do have many periodic solutions.

Known examples (1) $F \equiv p$ integral of momentum $\Longleftrightarrow u = \text{const}$

(2) F is quadratic $\Longleftrightarrow H = \frac{1}{2}p^2 + u(mag\ m)$

It is Natural to ask if (1) and (2) are the only examples of integrable cases

Or in other words is it true that the only periodic solutions for $U_t = A(U)U_x$ are simple waves

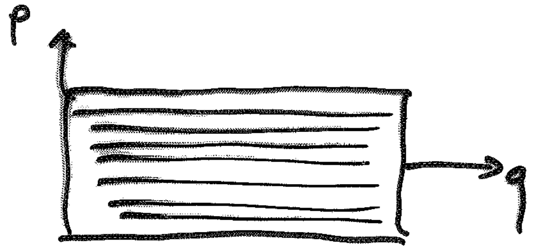
Related to the so called Birkhoff conjecture: The only integrable convex billiards in the plane are

circles $\quad$ and $\quad$ ellipses

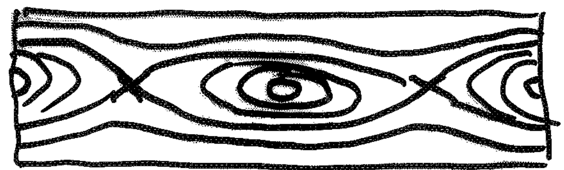
$\updownarrow$ $\qquad\qquad$ $\updownarrow$

(1) $\qquad\qquad$ (2)

Phase portraits



$F = p = \mathrm{const}$



F is quadratic

Known results

① $n=2$ that is $F = \frac{1}{3}p^3 + u_0 p^2 + u_1 p + u_2$

Assume $u_0 \equiv \mathrm{const}$ then u_1, u_2 satisfy

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} 2u_0 & -1 \\ u_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_q$$

Theorem 1 $\quad$ No periodic solutions only constants

Outline of the proof $\quad$ The system is equivalent to the second order eq.

$$u_{tt} - 2u_0 u_{tq} + u u_{qq} + u_q^2 = 0$$

1) Note that it follows from

5/ E. Hopf maximum principle that for non-constant periodic solution u $\max_{\mathbb{T}^2} u \leq u_0^2$ ($\max_{\mathbb{T}^2} u > u_0^2$ elliptic regime)

2) If one knows $u \leq u_0^2$ one has system of hyperbolic eq-s which is "almost" genuinly non-linear

$$d\lambda(r_\lambda) \neq 0.$$

$\overset{\text{P. Lax}}{\Longrightarrow}$ shock formation for any non-constant initial data.

(2) **Theorem 2** Let $u^1(q,t) \dots u^n(q,t)$ are smooth periodic solutions for $U_t = A(U)U_x$ such that the matrix $A(U)$ has no real eigenvalues

$$\Longrightarrow u^i = \text{const}$$

Proof Uses another idea by E. Hopf from Riemannian geometry

No real eigenvalues $\Longleftrightarrow F_p \neq 0$

One defines $\omega(p,q,t) = -\frac{F_q}{F_p}$

One has $\underbrace{\omega_t + p\omega_q - u_q \cdot \omega_p}_{\dot{\omega}} + \omega^2 + u_{qq} = 0$

Ricatti eq

6/ integrate for fixed p over q, t

$$\Rightarrow \quad + \frac{d}{dp}\left[\int u_q \cdot \omega\right] = \int \omega^2$$

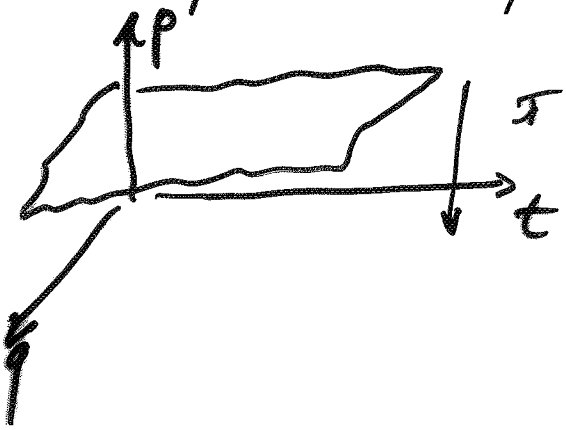
set $\varphi(p) = \int u_q \cdot \omega(p, q, t)\, dq\, dt$

$$\Rightarrow \quad \varphi' \geq K \varphi^2 \Longrightarrow \varphi = \text{const}$$

7)

Picture: Invariat tori $F = const$

$F_p \neq 0 \Rightarrow$ project diffeomophpically



ω = slope of the torus

Remark to 1 and 2

For $n > 2$ it is not necessarily true that the hyperbolic system is genuinly non-linear. That is one can not apply F. John results on shock formation.

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3 Most misterious fact about $U_t = A(U)U_x$: It is Hamiltonian and has ∞ many conservation laws

My hope is that if one knows many conserv. laws then one can restrict the class of periodic solutions.

But how? Even for Theorem 1 ($n=2$) I don't know how to use conservation laws

Here any conserv. law is a solution of Trikomi

8)
$$G_{u_1 u_1} + u_1 G_{u_2 u_2} = 0 \quad \text{(Trikomi)}$$

In the general case one can get conservation laws by the generating function $G(z, q, t)$:
$$F\left(G(z,q,t), q, t\right) = \frac{1}{n+1} z^{-(n+1)}$$

G – is holomorphic in the vecinity of $z=0$, $G = \frac{1}{z} + G_0 + G_1 z + \ldots$

all G_i are polynomials in $u_0, u_1, \ldots u_n$
$$G_0 = -u_0$$
$$k = 1, \ldots n; \quad G_k = -u_k - P_k(G_0, \ldots G_{k-1}, u_0, u_1, \ldots u_{k-1})$$
$$k = n+1, \ldots \quad G_k = P_k(G_0, \ldots G_{k-1}, u_0, u_1, \ldots u_n)$$

P_k polynomials

In the variables $G_1, \ldots G_n$ the system is
$$\begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & & & \\ G_1 & 0 & \ddots & & \\ G_2 & G_1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 1 \\ G_{n-1} & \cdots & G_2 & G_1 & 0 \end{pmatrix} \begin{pmatrix} G_1 \\ \vdots \\ G_n \end{pmatrix}_q = 0$$

$\boxed{G_t = M(\nabla \mathcal{H})_q}$ Hamiltonian system of hydrodynamics type $\boxed{\mathcal{H}(G_1, \ldots G_n)}$