Ideals of the rings of differential operators & integrable systems

Joint work with Yu. Berest

Motivation: Sergeev - Veselov 2004

$A = A_1 = \mathbb{C}[x; \frac{d}{dx}]$ Weyl algebra

$M \subset A$ right ideal ($M \cdot a \subset M \ \forall a$)

Problem: Classify M (up to iso)

Answer: VERY NICE moduli space (Berest - Wilson)

Relation to: KP hierarchy; classical int. systems; noncomm. geometry; bispectral problem; rings of diff. operators on singular curves, ...

One way to relate to (comm.) alg. geometry:

Cannings - Holland correspondence

$$M \xmapsto{\text{evaluation}} U = \langle a(f) : a \in M, f \in \mathbb{C}[x] \rangle$$

ideal $\qquad\qquad$ subspace of $\mathbb{C}[x]$

$$\{a \in A : a(\mathbb{C}[x]) \subset U\} \longleftarrow\!\shortmid \quad U \subset \mathbb{C}[x]$$

$$\left\{ \begin{array}{l} M \subset A \\ M \cap \mathbb{C}[x] \neq 0 \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{primary decomposable} \\ U \subset \mathbb{C}[x] \end{array} \right\}$$

Def: U primary decomposable if

$$U = \bigcap_{z \in \mathbb{C}} U_z \qquad \text{finite intersection}$$

$$(x-z)^n\, \mathbb{C}[x] \subset U_z \subset \mathbb{C}[x]$$

for some $n = n_z \geq 0$

Allow fractional ideals

$M \subset \mathbb{C}(x)[\frac{d}{dx}]$ $\quad f \cdot M \subset A$ for some $f \in \mathbb{C}[x]$

$M \xmapsto{\text{evaluation}} V \subset \mathbb{C}(x)$

Many interesting (families of) differential operators naturally arise as/in $\mathrm{End}_A M$

Example:

$$U = x^{-m} \cdot \mathbb{C}[x^2] \oplus x^{m+1}\mathbb{C}[x^2] \qquad m \in \mathbb{Z}_+$$

$$x^{-m} \cdot \mathbb{C}[x^2, x^{2m+1}]$$

$$\downarrow$$

$$M = \{ a \in \mathbb{C}[x][\tfrac{d}{dx}] : a(\mathbb{C}[x]) \subset U \}$$

- $x^m \in M$

- $L = \partial^2 - m(m+1)x^{-2}$

  $L(U) \subset U \Rightarrow L \in \mathrm{End}_A M$

  $\mathbb{C}[L] \cdot x^m \cdot A \subset M$ – a lot!

  $\Rightarrow$ can find $S = \partial^m + \dots \in M$

  $LS = SL_0 \qquad L_0 = \partial^2$

- $\mathrm{End}_A M$ contains 2 comm. subalgebras

  $Q = \{ q : qU \subset U \} = \mathbb{C}[x^2, x^{2m+1}]$

  $Q^+ = \mathbb{C}[L, B] / \{ B^2 = L^{2m+1} \}$

- $M = x^m \cdot A + S \cdot A$

Let's replace A_1 by

$$A = A_n = \mathbb{C}[x_1, \dots, x_n; \partial_1, \dots, \partial_n]$$

$$x = (x_1, \dots, x_n)$$

Q: What can one say about ideals?

(perhaps, under some restrictions on M: $M \cap \mathbb{C}[x] \neq 0$, M - projective, ...)

In particular, what replaces primary decomposable spaces?

Q′: Find $M \subset A$ s.t. $\mathrm{End}_A M$ contains a Schrödinger operator

$$L = \Delta - u(x)$$

$M \cap \mathbb{C}[x] \neq 0$

"Theorem": All such M can be completely characterized

C - Feigin - Veselov '99

Equivalent to 1 + 2 :

1 Singularities of $u(x)$ - <u>hyperplane arrangement</u>

$R = R_+ \sqcup -R_+$ normal vectors

and

$$u(x) = \sum_{\alpha \in R_+} \frac{m_\alpha (m_\alpha + 1)(\alpha,\alpha)}{(\alpha, x)^2}$$

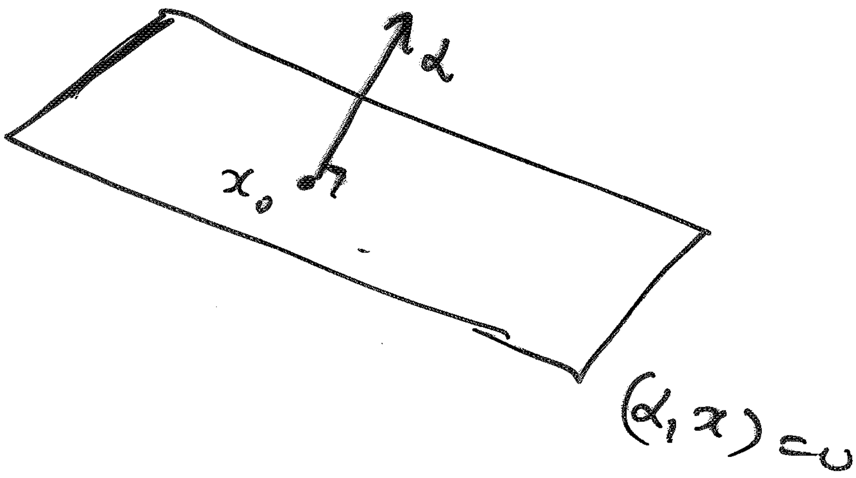
$m_\alpha \in \mathbb{Z}_+$

2 <u>Locus conditions</u> :

$\forall \alpha \in R \quad \forall x_0 : (\alpha, x_0) = 0$ generic

$[u(x_0 + \alpha t)] \in \mathbb{C}t^{-2} \oplus \mathbb{C}[[t^2; t^{2m_\alpha+1}]]$

formal series in t



Examples (CFV '99)

1. Coxeter case

R – root system of a Coxeter gr. W

$$m : R \to \mathbb{Z}_+ \qquad W\text{-inv.}$$
$$\alpha \mapsto m_\alpha$$

e.g. $W = S_n$ Calogero-Moser

$$u(x) = \sum_{i<j}^{n} \frac{2m(m+1)}{(x_i - x_j)^2} \qquad m \in \mathbb{Z}_+$$

2. Deformed root systems

$A_{n,1}(m)$ $A_{n,2}(m)$ $C_n(m, k, \ell)$

3. Berest-Loutsenko family : $\dim = 2$

4. $\dim = 1$ $u(x) = u(x;t)$ rat. sol. to KdV

Conjecturally, this is complete list

Suppose:

$$u(x) = \sum_{\alpha \in R_+} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \qquad m_\alpha \in \mathbb{Z}_+$$

locus configuration

Define: $\mathcal{U} \subset \mathbb{C}(x)$

$$\mathcal{U} := \left\{ f \in \mathbb{C}(x) \ : \ \forall \alpha \ \ \forall x_0 : (\alpha, x_0) = 0 \quad [f(x_0 + \alpha t)] \in t^{-m_\alpha} \cdot \mathbb{C}[[t^2, t^{2m_\alpha + 1}]] \right\}$$

$$M := \{ a \in \underset{\text{nth Weyl alg}}{\mathbb{C}(x)[\partial]} : a(\mathbb{C}[x]) \subset \mathcal{U} \}$$

M - right ideal of $A = A_n$

- $L = \Delta - u \in \mathrm{End}_A M$
- $\delta := \prod_{\alpha \in R_+} (\alpha, x)^{m_\alpha} \in M$

$\Rightarrow$ • $\exists\, S = \prod_{\alpha \in R_+} (\alpha, \partial)^{m_\alpha} + \ldots \in M$

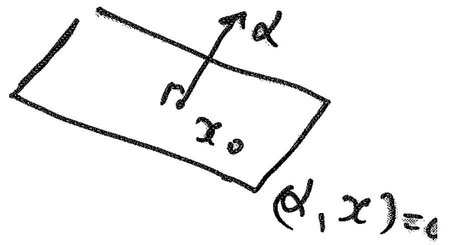
$$LS = S L_0, \qquad L_0 = \Delta$$

S : eigenfunctions of L_0 $\longrightarrow$ eigenfunctions of L

- Two large comm. subalgebras of $\mathrm{End}_A M$

$$Q := \{ q \in \mathbb{C}[x] : q\,\mathcal{U} \subset \mathcal{U} \}$$

$$[q(x_0 + \alpha t)] \in \mathbb{C}[[t^2, t^{2m_\alpha + 1}]]$$



Q^+ dual comm. subalgebra

$\forall q \in Q \quad \exists \quad L_q = q(\partial) + \dots \in \mathrm{End}_A M$

$[L_p, L_q] = 0 \quad \forall\, p, q \in Q$

- (Conjecture)

$$M = \delta \cdot A + \delta^* \cdot A$$

Sergeev – Veselov '2004 (CMP)

another type of deformed Calogero – Moser operators (related to Lie superalgebras)

E.g. Berest – Yakimov operator

$$V = \underset{x}{\mathbb{C}^n} \oplus \underset{y}{\mathbb{C}^m}$$

$$u = \sum_{i<j}^{n} \frac{2k(k+1)}{(x_i - x_j)^2} + \sum_{i<j}^{m} \frac{2k^{-1}(k^{-1}+1)}{(y_i - y_j)^2}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{2(k+1)}{(x_i - \sqrt{k}\, y_j)^2} \qquad k \in \mathbb{C}$$

$A(n, m)$

$BC(n, m)$ 3-parameter family

Exceptional cases: $G(1,2)$ $AB(1,3)$ $D(2,1,\lambda)$

<u>Thm (S–V '04):</u> $L = \Delta - u$ in cases $A(n,m)$, $BC(n,m)$ is completely integrable.

<u>Setting</u> : $L = \Delta - u$

$$u = \sum_{\alpha \in R_+} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

$R = \{\alpha\} \subset V = \mathbb{C}^n$ need not to be a root system

$$m : \begin{array}{ccc} R & \longrightarrow & \mathbb{C} \\ \alpha & \longmapsto & m_\alpha \end{array}$$

- $R_0 \subset R$ root system of $W_0 \subset GL\,V$ Coxeter group
- $\left.\begin{array}{l} R \\ m : R \to \mathbb{C} \end{array}\right\} W_0$-inv.
- $m_\alpha \in \mathbb{Z}_+$ for all $\alpha \in R \setminus R_0$
- locus conditions for all $\alpha \in R \setminus R_0$

<u>Thm</u> : (1) L is completely integrable

(2) $\exists\, S = \prod_{\alpha \in R \setminus R_0} (\alpha, \partial)^{m_\alpha} + \ldots$ s.t.

$$L S = S L_0$$

$L_0 = \Delta - u_0$ Calogero-Moser operator for R_0