Talalaev's quantum spectral curve and the Langlands correspondence

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The material below worked out for the Gaudin model, but potentially can be generalized to "ARBITRARY" integrable system. The aim is to approach QUANTUM integrable systems with the unifying concept QUANTUM SPECTRAL CURVE.

1 Gaudin model

The simplest example Let K be an aribitrary constant matrix,

$$L(z) = K + \sum_{i=1,...,k} \frac{1}{z - z_i} \begin{pmatrix} \hat{q}_{1,i} \\ ... \\ \hat{q}_{n,i} \end{pmatrix} \begin{pmatrix} \hat{p}_{1,i} & ... & \hat{p}_{n,i} \end{pmatrix} = K + \hat{Q}\ diag(\frac{1}{(z-z_1)}, ..., \frac{1}{(z-z_k)})\ \hat{P}^t \quad (1)$$

where $\hat{p}_{i,j}, \hat{q}_{i,j}$, $i = 1, ..., n; j = 1, ..., k$ are the standard generators of the standard Heisenberg algebra $[\hat{p}_{i,j}, \hat{q}_{k,l}] = \delta_{i,k}\delta_{j,l}$, $[\hat{p}_{i,j}, \hat{p}_{k,l}] = [\hat{q}_{i,j}, \hat{q}_{k,l}] = 0$; z_i are arbitrary constants ($z_i \neq z_j$).Matrices $\hat{Q}, \hat{P}$ are $n \times k$-rectangular matrices with the elements: $\hat{Q}_{i,j} = \hat{q}_{i,k}$, $\hat{P}_{i,j} = \hat{p}_{i,j}$.

The standard example Consider $gl_n \oplus ... \oplus gl_n$, denote by e^i_{kl} the standard basic element from the i-th copy of the direct sum $gl_n \oplus ... \oplus gl_n$. The standard Lax operator for the Gaudin model is the following:

$$L_{Gaudin\ standard}(z) = \sum_{i=1,...,k} \frac{1}{z - z_i} \begin{pmatrix} e^i_{1,1} & ... & e^i_{1,n} \\ ... & ... & ... \\ e^i_{n,1} & ... & e^i_{n,n} \end{pmatrix} \quad (2)$$

$L_{Gaudin\ standard}(z) \in Mat_n \otimes U(gl_n \oplus ... \oplus gl_n) \otimes \mathbb{C}(z)$, recall that $U(gl_n \oplus ... \oplus gl_n)$ is canonically isomorphic to $U(gl_n) \otimes ... \otimes U(gl_n)$.

2 Quantum integrals of motion (Talalaev 2004)

Theorem [Talalaev04, ChervovTalalaev04, ChervovTalalaev06-1] For the Lax matrix of the quantum Gaudin type model the following holds true. Consider the following expression (Talalaev's quantum spectral curve), which is now a differential operator in variable z:

$$det^{column}(\partial_z - L(z)) = \sum_i QH_i(z)\partial_z^i \quad (3)$$

then

$$\forall i, j, u \in \mathbb{C}, v \in \mathbb{C} \quad [QH_i(z)|_{z=u}, QH_j(z)|_{z=v}] = 0 \quad (4)$$

So taking $QH_i(z)$ for different i, z (or their residues at poles at points $(z - z_i)$) one obtains the set of quantum mutually commuting conserved integrals of motion.

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3 Quantum linear algebra and Manin's matrices

Main claims **Observation** Consider $Id\, \partial_z \pm L(z) \in Mat_n \otimes R \otimes \mathbb{C}((z))[\partial_z]$, where Id is the identity matrix, then

$$Id\, \partial_z - L(z) \text{ is a column Manin's matrix} \tag{5}$$
$$Id\, \partial_z + L(z) \text{ is a row Manin's matrix.} \tag{6}$$

Claim (partly proved) All the linear algebra (Cayley-Hamilton, Newton, Plucker, etc.) facts holds true for Manin's matrices.

Preliminaries

Definition 1 *Let us call a matrix M with elements in a associative ring R* column-Manin's *if the properties below are satisfied:*

- elements which belong to the same column of M commute among themselves
- $\forall p, q, k, l \;\; [M_{pq}, M_{kl}] = [M_{kq}, M_{pl}]$, i.e. for each 2×2 submatrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{of } M \text{ it is true that} \qquad [a, d] = [c, b] \tag{7}$$

Remark 1 See [?] chapter 6.1, especially formula 1.

Definition 2 *Matrix M is called* row-Manin's *iff transpose matrix M^t is column-Manin's .*

The proposition below goes back to [?] page 193 Proposition 4, [?]:

Proposition 1 *Coaction. Let A be a column-Manin's rectangular $n \times m$-matrix. Consider variables x_i, $i = 1, ..., m$ which commute among themselves and with elements $A_{p,q}$ of the matrix A and variables $\psi_1, ..., \psi_n$ which anticommute among themselves: $\psi_i^2 = 0, \psi_i\psi_j = -\psi_j\psi_i$ and commute with $A_{p,q}$. Then the following is true. Consider new variables $\tilde{x}_i$, $\tilde{\psi}_i$:*

$$\begin{pmatrix} \tilde{x}_1 \\ ... \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} A_{1,1} & ... & A_{1,m} \\ ... & & \\ A_{n,1} & ... & A_{n,m} \end{pmatrix} \begin{pmatrix} x_1 \\ ... \\ x_m \end{pmatrix} \qquad (\tilde{\psi}_1, ..., \tilde{\psi}_m) = (\psi_1, ..., \psi_n) \begin{pmatrix} A_{1,1} & ... & A_{1,m} \\ ... & & \\ A_{n,1} & ... & A_{n,m} \end{pmatrix} \tag{8}$$

If a matrix A is a column-Manin's matrix, then the new variables $\tilde{x}_i$ and $\tilde{\psi}_i$ form commutative polynomial and Grassman algebras respectively. Moreover converse is true: if either $\tilde{x}_i$ or $\tilde{\psi}_i$ form commutative polynomial or Grassman algebras respectively then A is column-Manin's matrix. For a row-Manin's matrix the same is true up to a change $A_{i,j} \to A_{j,i}$.

Proof. Straightforward calculation.

4 Spectrum of the model

Theorem (Varchenko et.al. 2005) Consider $v : QH_i(z)v = \lambda_i(z)$, then $\forall q(z)$: $\sum_i \lambda_i(z)\partial_z^i q(z) = 0 => q(z)$ rational function. Moreover converse is true.

Corollary Bethe ansatz equations for Bethe roots.

5 Eigenfunctions of the model and separation of variables

Conjecture There exists variables α_i, β_j, such that $\forall i : \det(\partial_z - L(z))|_{\partial_z -> \beta_i, z -> \alpha_i} = 0$ identically.

Corollary: Eigenfunctions Ψ in variables α_i can be factorized in terms of one particle wave functions.

$$\Psi(\alpha_i)\text{''} = \text{''} \prod_i \Psi_{1-particle}(\alpha_i), \tag{9}$$

$$det(\partial_z - L(z))\Psi_{1-particle}(z) = 0. \tag{10}$$

5.1 General Capelli identities

5.1.1 The Gaudin model

Let us consider the Lax operator defined by the formula 1. We denote by $L^{classical}(z)$ the expression given by the same formula 1 in which we substitute p, q for $\hat{p}, \hat{q}$. Variables p, q are considered as the standard coordinates on $\mathbb{C}^{2nk}$, so they are commuting variables.

The Capelli identities can be generalized and reformulated as follows:

Conjecture 1 *Let F be any GL-invariant polynomial made of matrix elements (for example $det(M), TrM^k, Tr_{in\ arbitrary\ representation}M$), then*

$$F(Wick(\lambda - L^{classical}(z))) = Wick(F(\lambda - L^{classical}(z)), \tag{11}$$

$$i.e. \quad F(\partial_z - L(z))) = Wick(F(\lambda - L^{classical}(z)), \tag{12}$$

where we use conventions that $Wick(\lambda) = \partial_z, Wick(p) = \hat{p}, Wick(q) = \hat{q}$ and one always puts $\hat{q}$ on the left from $\hat{p}$, and ∂_z on the left from z.

Remark 2 In particular: the case $F = det$ and $z_i = z_j$ is related to the classical Capelli identity; the case $F = det$ and $z_i \neq z_j$ is related to the Capelli identity of [?]; the case $z_i = z_j$, $F = Tr_{in\ representation}$ is related to the higher Capelli identities of [?]. For generic F one should clarify the ordering, this will be done in subsequent papers. Theorem 3 page 16 in [?] is probably related to some elliptic generalization of this identity for $F = det$ and $z_i \neq z_j$.

A companion conjecture is the following:

Conjecture 2 *In canonical variables p_i, q_i for arbitrary classical conservation law $H(q_i, p_i)$ of the Gaudin model it is true that Wick ordered (i.e. put $\hat{q}_i$ on the left from $\hat{p}_i$) expression $\hat{H}(\hat{q}_i, \hat{p}_i)$ defines a quantum conservation law, i.e. $\forall H_1, H_2 :\ [\hat{H}_1^{Wick}, \hat{H}_2^{Wick}] = 0$. In particular the following expressions form a linear basis among all[2] quantum commuting conservation laws:*

$$\sum_{i_1,...,i_m} \sum_{\alpha_1,...,\alpha_m} \sum_{\sigma \in S_m} \chi(\sigma) \hat{q}_{i_1,\alpha_1} ... \hat{q}_{i_m,\alpha_m} \frac{1}{z - z_{\alpha_1}} ... \frac{1}{z - z_{\alpha_m}} \hat{p}_{\sigma(i_1),\alpha_1} ... \hat{p}_{\sigma(i_m),\alpha_m} \tag{13}$$

[2] $GL(n)$ invariant (technical detail)

where χ is some character of S_m, m is arbitrary integer.

In classical limit these elements degenerates to $Tr_{V_\chi}\pi_{V_\chi}(L^{classical}(z))$, where (π, V_χ) is a representation of GL_n corresponding to the same Young diagram as the character χ of S_m.

References

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