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On differential equations of von Gehlen-Roan and Roan

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1 Introduction

Papers by von Gehlen-Roan and Roan.

Their motivation: study of zeroes of polynomials appearting in the study of Bethe Ansatz for the N-state superintegrable chiral Potts spin chain by using differential equations.

G. von Gehlen and S. S. Roan, The superintegrable chiral Potts quantum chain and generalized Chebyshev polynomials, in S. Pakuliak, G. von Gehlen (Eds.), Integrable Structure of Exactly Solvable Two-Dimensional Models of Quantum Field Theory, NATO Science Series II, vol. 35, Kluwer Academic Publisher, Dordrecht, 2001, pp. 155-172.

S. S. Roan, Structure of certain Chebyshev-type polynomials in Onsager's algebra representation, Journal of Computaional and Applied Mathematics, vol. 202 (2007), 88-1-4.

2 superintegrable chiral Potts hamiltonian

$N \geq 2$, $\omega = \exp(2\pi i/N)$,

$X, Z \in \mathrm{End}(\mathbf{C}^N), ZX = \omega XZ, X^N = Z^N = id.$

L: integer.

Consider on $(\mathbf{C}^N)^{\otimes L}$ the following operator

$$H(k') = -\sum_{l=1}^{L}\sum_{n=1}^{N-1} \frac{2}{1-\omega^{-n}} \left(X_l^n + k' Z_l^n Z_{l+1}^{N-n}\right)$$

where k' is a real parameter and X_l is the operator acting on the l-th component as X and for other components as identity.

If we write

$$H(k') = A_0 + k' A_1,$$

A_0 and A_1 satisfy the Dolan-Grady relation and give a representation of the Onsager algebra.

3 Polynomials

Define polynomials $F_j(s)$ by the relation.

$$\left(\frac{t^N - 1}{t - 1}\right)^L = \sum_{j=0}^{N-1} t^j F_{j+1}(s), \qquad s = t^N.$$

By the Bethe Ansatz eigenvalues are expressed in terms of zeroes of F_j.

In order to study these polynomials von-Gehlen-Roan derived a system of first order differential equations for

$$F(s) = {}^t\left(F_1, F_2, \ldots, F_N\right).$$

$$Ns(s-1)\frac{dF}{ds} = BF,$$
$$B = \begin{pmatrix} d_0 & -Ls & \cdots & -Ls \\ -L & d_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -Ls \\ -L & \cdots & -L & d_{N-1} \end{pmatrix},$$
$$d_j = L(N-1)s - j(s-1).$$

When $N = 2$ (the Ising case) each of polynomials satisfy Gauß hypergeometric differential equation. They also derive 3rd order differential equations for the case $N = 3$.

Eg.

$$27s^2(s-1)^2F_1^{'''} - 27s(s-1)((2L-4)s+2)F_1^{''}$$
$$+3(3L^2s(4s-1) - 3Ls(10s-7) + 2(s-1)(10s-1))F_1^{'}$$
$$-(L-1)(L(L(8s+1) - 4(s-1))F_1 = 0.$$

These systems have regular singular points only at $s = 0, 1, \infty$.

von Gehlen-Roan and Roan conjectured that each of F_j satisfies an N-th order ordinary differential equations of the form

Conjecture 1

$$N^N s^{N-1}(s-1)^{N-1}\frac{d^N F_j}{ds^N} + \sum_{k=1}^{N-1} N^k s^{k-1}(s-1)^{k-1} D_{jk}(s)\frac{d^k F_j}{ds^k} + D_{j0}F_j = 0$$

where D_{jk} are polynomials.

Among known Fuchsian differential equations of higher order there is a class called generalized hypergeometric differential equations.

After some calculation with small n, we find that defining G by $G(s) = (s-1)^{-L}F(s)$ the differential equations for G_j become a special kind of generalized hypergeometric differential equations.

4 A normal form of differential equations

The differential equations for G takes the following form

$$N\frac{dG}{ds} = \left(-\frac{L}{s-1}A_1 + \frac{1}{s}A_0\right)G,$$

$$A_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ L & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ L & \cdots & L & -N+1 \end{pmatrix}.$$

(In the original form the diagonal entries are $1-N$.)

Look for a N-th order matrix P and numbers a_j, b_j which satisfy the following relations.

$$\frac{1}{N}LPA_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ a_0 & a_1 - b_1 \cdots & a_{N-2} - b_{N-2} & a_{N-1} - b_{N-1} \end{pmatrix} P,$$

$$\frac{1}{N} P A_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 0 \\ 0 & & \cdots & 0 & 1 \\ 0 & -b_1 & \cdots & -b_{N-2} & -b_{N-1} \end{pmatrix} P$$

If we can find such nonsingular matrix P, $(PG)_1$ is annihilated by a generalized hypergeometric differential operator determined by a_j, b_j:

$$s \left(\sum_{j=0}^{N} a_j \vartheta^j \right) - \sum_{j=1}^{N} b_j \vartheta^j, \quad \vartheta = s \frac{d}{ds}, \quad a_N = 1, \, b_N = 1. \tag{1}$$

Factorize as

$$\begin{aligned} \sum_{j=0}^{N} a_j \vartheta^j &= \prod_{j=1}^{N} (\vartheta + \alpha_j) \\ \sum_{j=1}^{N} b_j \vartheta^j &= \vartheta \prod_{j=1}^{N-1} (\vartheta + \beta_j - 1) \, , \end{aligned} \tag{2}$$

the usual generalized hypergeometric differential operator.

5 Transformation Matrix (guess)

To find the matrices we made computation using the computer algebra system maxima. n-th order matrix Q_n with entries $q_{ij}^{(n)}$ that satisfy the following relations.

$$xQ_n \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ c_1 & \cdots & c_n \end{pmatrix} Q_n,$$

$$Q_n \begin{pmatrix} 0 & 0 & \cdots & 0 \\ x & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x & \cdots & x & n-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 0 \\ 0 & & \cdots & 0 & 1 \\ 0 & f_2 & \cdots & f_{n-1} & f_n \end{pmatrix} Q_n$$

where x is a parameter. By a computation for small n we may set

$$q_{i1}^{(n)} = (-1)^{n-1}(-x)^{i-1}, \quad q_{in}^{(n)} = (n-1)^{i-1}$$

without loss of generality.

The cases $n = 3,\ 4,\ 5$ look as follows:

$$Q_3 = \begin{pmatrix} 1 & -2 & 1 \\ -x & x-2 & 2 \\ x^2 & 3x-2 & 4 \end{pmatrix},$$
$$Q_4 = \begin{pmatrix} -1 & 3 & -3 & 1 \\ x & -2x+3 & x-6 & 3 \\ -x^2 & x^2-5x+3 & 5x-12 & 6 \\ x^3 & 6x^2-8x+3 & 19x-24 & 9 \end{pmatrix},$$
$$Q_5 = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 \\ -x & 3x-4 & -3x+12 & x-12 & 4 \\ x^2 & -2x^2+7x-9 & x^2-14x+24 & 7x-36 & 16 \\ -x^3 & x^3-9x^2+11x-4 & 9x^2-48x+48 & 37x-108 & 64 \\ x^4 & 10x^3-20x^2+15x-4 & 55x^2-140x+96 & 175x-324 & 256 \end{pmatrix}.$$

At the same time we can guess that

$$c_i = r(n, i-1, x) - s(n, i-1), \quad f_i = -s(n, i-1)$$

where $r(n, i, x)$ is defined by

$$\sum_{i=0}^{n} r(n, i, x) t^i = \prod_{j=1}^{n} (t + x - j + 1)$$

and $s(n,i)$ denotes the Stirling number of the first kind.
n-th order matrices $Q[k,n]$ with entries

$$q[k,n][i,j] = \sum_{l=0}^{k} \binom{k}{l} q_{ij}^{(n+k)}.$$

The original matrix Q_n is $Q[0,n]$.

From the results we immediately see that

$$q[k,n][i,l] = 0, \quad l = 1,\cdots,k,$$
$$q[k,n][i,k+1] = (-1)^{n-1}(k-x)^{i-1}$$

hold.

With this information we guessed the formula for $q[k,n,i,k+j]$ for $j=2$, 3.

$$q[k,n][i,k+2] = (-1)^{n+i+1}\sum_{s=0}^{i-1}(-1)^s\left((k+1)^s\binom{n-i+s}{1}\binom{i-1}{s}\right.$$
$$\left.+\left((s-k)(k+1)^s+k^{s+1}\right)\binom{i-1}{s+1}\right)x^{i-s-1},$$

$$
\begin{aligned}
q[k,n][i,k+3] &= (-1)^{n+i}\sum_{s=0}^{i-1}(-1)^s\left((k+2)^s\binom{n-i+s}{2}\binom{i-1}{s}\right.\\
&+\left((s-k+1)(k+2)^s+(k+1)^{s+1}\right)\binom{n-i+s-1}{1}\binom{i-1}{s+1}\\
&+\binom{i-1}{s+1}\sum_{t=0}^{s-2}(2+t2^{t+2})\binom{s+1}{t+3}k^{s-2-t}\\
&+\left.\binom{i-1}{s+2}\sum_{t=0}^{s-2}(t+3)(1+t2^{t+1})\binom{s+2}{t+4}k^{s-2-t}\right)x^{i-s-1}.
\end{aligned}
$$

Rewriting these expressions we arrived at the following conjectural form

$$
q[k,n][i,k+j]=\sum_{l=0}^{j-1}\binom{x+n-1}{j-1-l}\binom{-x}{l}(j+k-1-l-x)^{i-1}
$$

We checked this formula for the results we have.

6 Transformation matrix (check)

We set

$$q_{ij}^{(n)} = (-1)^{n+j} \sum_{s=0}^{j-1} \binom{x+n-1}{s} \binom{-x}{j-1-s} (s-x)^{i-1},$$

and consider the square matrix Q_n of order n with its (i,j)-entries $q_{ij}^{(n)}$.

Proposition 1 The matrix Q_n satisfies

$$xQ_n \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ c_1 & \cdots & c_n \end{pmatrix} Q_n, \tag{3}$$

$$Q_n \begin{pmatrix} 0 & 0 & \cdots & 0 \\ x & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x & \cdots & x & n-1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & & 0 \\ 0 & & \cdots & 0 & 1 \\ 0 & f_2 & \cdots & f_{n-1} & f_n \end{pmatrix} Q_n. \tag{4}$$

Proof.

The (i,j) component of the left hand side is

$$\sum_{j=1}^{n}\sum_{s=0}^{j-1}\binom{x+n-1}{s}\binom{-x}{j-1-s}(s-x)^{i-1}.$$

After some calculation we get

$$\begin{aligned}
&(-1)^{n-1}\binom{x+n-1}{n-1}\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}(l-x)^{i-1}\\
&=(-1)^{n+i}\binom{x+n-1}{n-1}\sum_{k=0}^{i-1}\binom{i-1}{k}(x-n+1)^{i-k-1}\\
&\quad\times\sum_{l=0}^{n-1}(-1)^l\binom{n-1}{l}(n-1-l)^k
\end{aligned}$$

The inner sum is $(n-1)!S(k,n-1)$ where $S(n,k)$ denotes the Stirling number of the second kind.
The sum is 0 if $i<n$ and when $i=n$ the sum is equal to

$$\prod_{j=1}^{n-1}(x+j).$$

The right hand side. If $i < n$ the result is obviously 0. When $i = n$ the term consists of two parts. After some calculation using the definition of $r(n, k, x)$, we get

$$\prod_{k=0}^{n-1}(x+n-1-k)\sum_{s=0}^{j-1}\binom{j-1}{s}\binom{x-1+s}{j-1}.$$

By the identity

$$(-1)^{k+l}\binom{j}{k} = \sum_{s}(-1)^{s}\binom{k-l}{s}\binom{j+s}{j-l}$$

above sum is equal to 1. Thus we proved the first equation. The second equation (4) is proved in a similar way. Here we use the relation

$$\sum_{j=1}^{n} q_{ij}^{(n)} = 0$$

which is already contained in the first equation.

7 Inverse matrix

By calculating the inverses of $q^{(n)}$ for severals ns, we can guess the answer.
Define $\bar{q}_{ij}^{(n)}(x)$ by the relation

$$\sum_{j=0}^{n-1} \bar{q}_{i,j+1}^{(n)}(x) t^j = \prod_{k=0}^{i-2} (t + x - k) \prod_{k=i}^{n-1} (t - k).$$

The matrix $\bar{Q}_n$ with entries $\bar{q}_{ij}^{(n)}$ satisfy the relation

$$\bar{Q}_n Q_n = \prod_{k=1}^{n} (x + k) I_n,$$

I_n: the identity matrix of order n.

8 Scalar differential operator

$N = n$, $x = -L$. Multiply the diagonal matrix of order N with the i-th component $(-N)^{-i+1}$ from the left. Assume $L > N$.

$$\begin{aligned} b_j &= (-1)^{N+j} N^{-N+j-1} s(N, j-1), \\ a_j &= (-1)^{N+j} N^{-N+j-1} r(N, j-1, -L). \end{aligned}$$

The corresponding N-th order differential operator (1) is

$$s \prod_{k=1}^{N} \left(\vartheta + \frac{L+k-1}{N} \right) - \prod_{k=1}^{N} \left(\vartheta + \frac{k-1}{N} \right).$$

Defining $H = PG$, we see that the function H_1 is annihilated by the above operator.

$$\begin{aligned} G_i &= \sum_{j=1}^{N} (-N)^{j-1} \bar{q}_{ij}^{(N)}(-L) H_j / \prod_{k=1}^{N} (k-L) \\ &= (-1)^{N-1} \prod_{k=0}^{i-2} (N\vartheta + L + k) \prod_{k=i}^{n-1} (N\vartheta + k) \, H_1. \end{aligned}$$

Defining

$$L_i = s\prod_{k=1}^{n}(N\vartheta + L + i + k - 2) - \prod_{k=1}^{n}(N\vartheta + i - k)$$

and using

$$\vartheta s = s(\vartheta + 1),$$

we have

Theorem 1

$$L_i G_i = 0.$$

Rewriting these differential equations those for F_j and assuming that L is a positive integer, we proved the conjecture of von Gehlen-Roan, Roan.

9 Power series solutions at $s = 0$

Here we assume that L is a positive integer. Generalized hypergeometric series

$$
\begin{aligned}
&F\begin{pmatrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n & \\ \beta_1, & \beta_2, & \cdots, & \beta_{n-1}, & 1 \end{pmatrix} | s \Bigg) \\
&= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_n)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_{n-1})_k k!} s^k, \\
&(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)
\end{aligned}
$$

where α_1,..., α_n, β_1, ..., β_{n-1} are parameters. The symbols $(\alpha)_k$ are sometimes called Pochhammer symbol.
As is known solutions of generalized hypergeometric differential equation (1) aroud $s = 0$ are given by

$$
F\left(\begin{matrix} \alpha_1, & \alpha_2, & \cdots, & \alpha_n & \\ \beta_1, & \beta_2, & \cdots, & \beta_{n-1}, & 1 \end{matrix} \Big| s\right),
$$

with α_1, ..., α_N, β_1, ..., β_{N-1} defined by the relations (2) and also

$$
s^{1-\beta_j} F\left(\begin{matrix} 1+\alpha_1-\beta_j, & \cdots, & 1+\alpha_{j-1}-\beta_j, & 1+\alpha_j-\beta_j, & 1+\alpha_{j+1}-\beta_j, & \cdots, & 1+ \\ 1+\beta_1-\beta_j, & \cdots, & 1+\beta_{j-1}-\beta_j, & 2-\beta_j, & 1+\beta_{j+1}-\beta_j, & \cdots, & 1+ \end{matrix}\right.
$$

for $j = 1, \ldots, N-1$. The power series solutions of $L_i f = 0$ are given by

$$F\left(\begin{matrix} \frac{L+i-1}{N}, & \frac{L+i}{N}, & \cdots, & \frac{L+N-1}{N}, & \cdots, & \frac{L+i+N-2}{N} \\ \frac{i}{N}, & \frac{i+1}{N}, & \cdots, & 1, & \cdots, & \frac{i+N-1}{N} \end{matrix} \middle| s\right).$$

The Pochhammer symbols in the coefficietns are simplified. We have the following series

$$\sum_{k=0}^{\infty} \frac{(L+i-1)_{kN}}{(i)_{kN}} s^k.$$

We see that these are essentially a sum of binominal series

$$\frac{1}{N} \sum_{j=0}^{N} f_i(\omega^j s^{1/N}), \quad \omega = \exp(2\pi i/N)$$

$$f_i(x) = \sum_{n=0}^{\infty} \frac{(L+i-1)_n}{(i)_n} x^n$$

$$= \frac{1}{\binom{-L}{i-1}} \left(x^{1-i}(1-x)^{-L} - x^{1-i} \sum_{k=0}^{i-2} \binom{-L}{k} (-x)^k \right).$$

Recalling the transformation we took, the analysis of power series solutions of generalized hypergeometric differential equations at $s = 0$ recovered our starting point.