Geometric algebra and quadrilateral lattices

Adam Doliwa
doliwa@matman.uwm.edu.pl

University of Warmia and Mazury in Olsztyn

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Outline

Geometric algebra

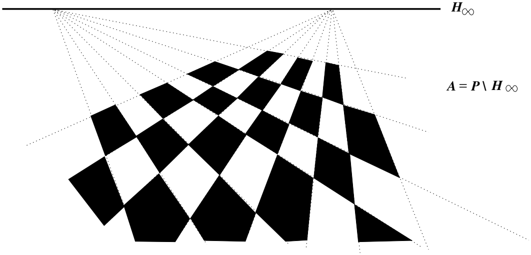
The quadrilateral lattice and geometric integrability scheme

The B-(Moutard) and C-(symmetric) quadrilateral lattices

The projective plane axioms

A projective plane is a set, whose elements are called points and a set of subsets, called lines, satisfying the following four axioms:

- P1 Two distinct points lie on one and exactly one line.
- P2 Two distinct lines meet in precisely one point.
- P3 There exist three noncollinear points.
- P4 Every line contains at least three points.



In analytic geometry one wants to get results, while in synthetic geometry one would like to get insight.

P1-P4 $\Rightarrow$ coordinatization in terms of a ternary ring

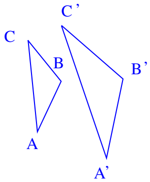
A ternary ring (Γ, T) is a set $\Gamma = \{0, 1, a, b, c, \dots\}$ together with a mapping $T : \Gamma \times \Gamma \times \Gamma \to \Gamma$ such that:

- **T1** For all $a, m, c \in \Gamma$, $T(0, m, c) = T(a, 0, c) = c$.
- **T2** For all $a \in \Gamma$, $T(a, 1, 0) = T(1, a, 0) = a$.
- **T3** If $m, m', b, b' \in \Gamma$ and $m \neq m'$, then the equation $T(x, m, b) = T(x, m', b')$ has a unique solution in Γ.
- **T4** If $a, a', b, b' \in \Gamma$ and $a \neq a'$, then the system of equations $T(a, x, y) = b$, $T(a', x, y) = b'$ has a unique solution in Γ.
- **T5** For all $a, m, c \in \Gamma$, the equation $T(a, m, x) = c$ has a unique solution in Γ.

addition: $a + b = T(a, 1, b)$
multiplication: $a \cdot b = T(a, b, 0)$
Example: A division ring $(\mathbb{D}, +, \cdot, 0, 1)$ is a ternary ring with $T(a, m, b) = a \cdot m + b$.

The Desargues axiom

P5 If two triangles are in perspective from a point then they are in perspective from a line.



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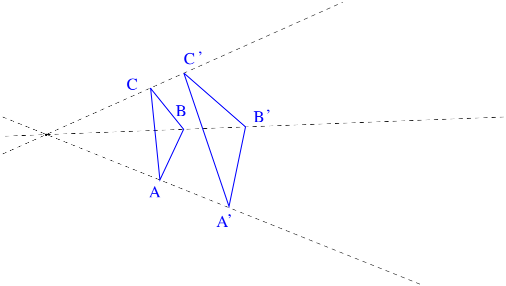

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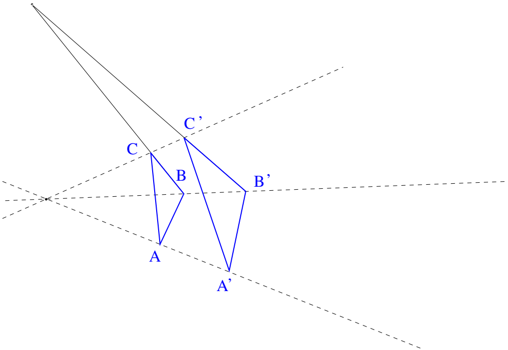

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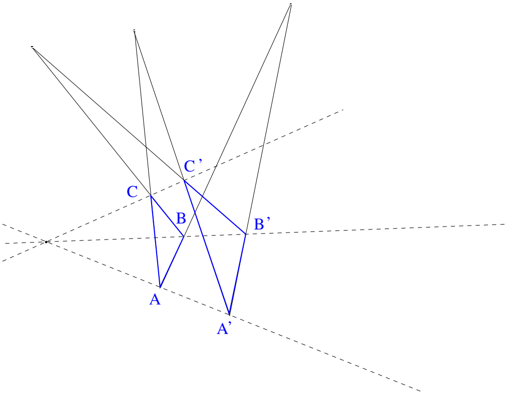

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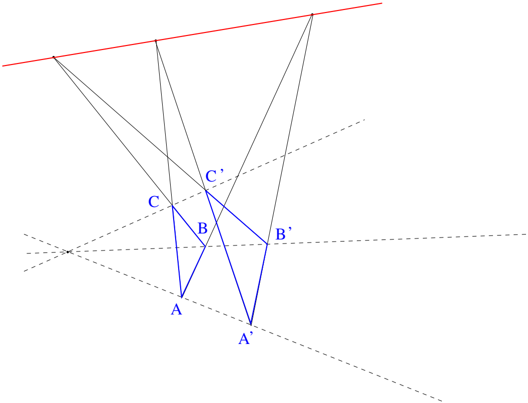

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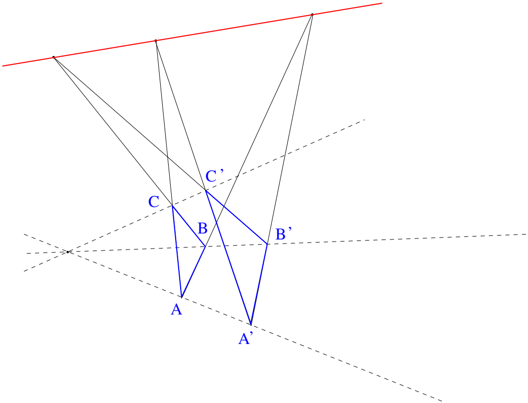

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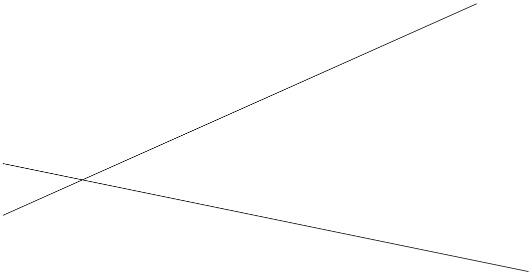

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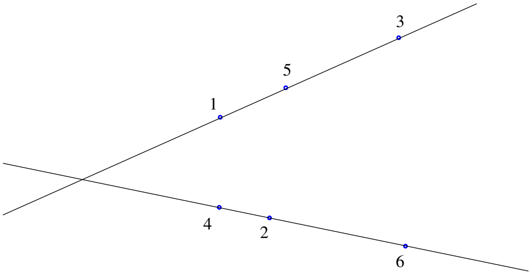

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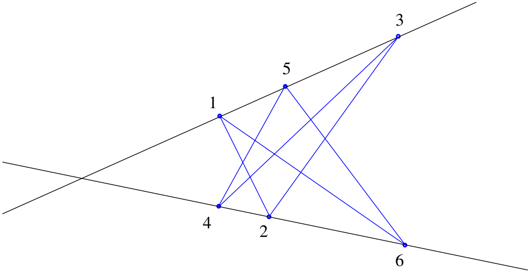

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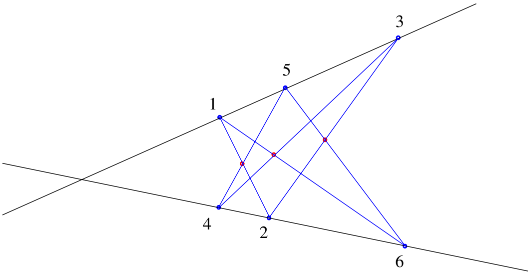

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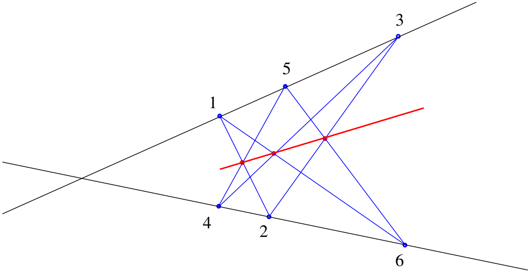

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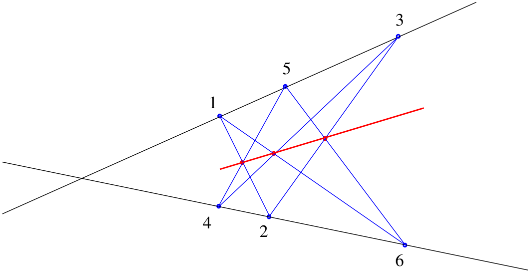

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The projective 3-space axioms

A projective 3-space is a set whose elements are called points, together with certain subsets called lines, and certain other subsets called planes, which satisfy the following axioms:

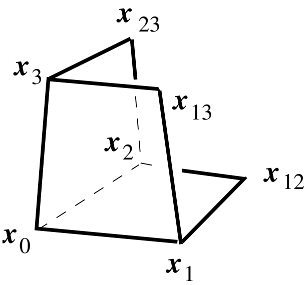
- S1 Two distinct points lie on one and only line.
- S2 Three noncollinear points lie on a unique plane.
- S3 A line meets a plane in at least one point.
- S4 Two planes have at least a line in common.
- S5 There exist four noncoplanar points, no three of which are collinear.
- S7 Every line has at least three points.

Theorem

Desargues' "axiom" holds in any projective 3*-space, where we do not necessarily assume that all the points lie in a plane.*

Geometric Integrability Scheme

Given generic points x_0, x_1, x_2 and x_3 in a projective 3-space, let x_{ij}, $1 \leq i < j \leq 3$, be generic points of the planes $\langle x_0, x_i, x_j \rangle$.



Then there exists exactly one point x_{123} which belongs simultaneously to the planes $\langle x_3, x_{13}, x_{23} \rangle$, $\langle x_2, x_{12}, x_{23} \rangle$ and $\langle x_1, x_{12}, x_{13} \rangle$.

Definition

A quadrilateral lattice is a map $x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D})$, $3 \leq N \leq M$, whose all elementary quadrilaterals are planar.

Geometric Integrability Scheme

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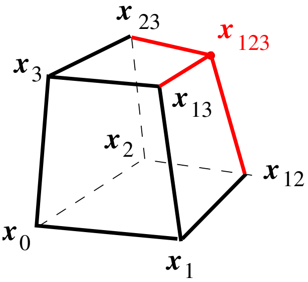

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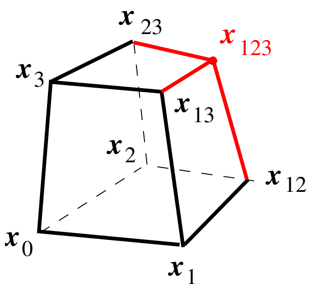

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The discrete Darboux equations (affine version)

In non-homogeneous coordinates $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{D}^M \sim \mathbb{P}(\mathbb{D}) \setminus H_\infty$,

$$\Delta_i \Delta_j \boldsymbol{x} = (\Delta_i \boldsymbol{x}) a^{ij} + (\Delta_j \boldsymbol{x}) a^{ji}, \quad 1 \leq i < j \leq N,$$

$$a^{ij} : \mathbb{Z}^N \to \mathbb{D}, \quad i \neq j.$$

Notation:

$\boldsymbol{x}_{(i)}(n_1, \ldots, n_i, \ldots, n_N) = \boldsymbol{x}(n_1, \ldots, n_i + 1, \ldots, n_N)$, $\Delta \boldsymbol{x} = \boldsymbol{x}_{(i)} - \boldsymbol{x}$.

The compatibility condition

$$\Delta_k a^{ij} + a^{ik} a^{ij}_{(k)} = a^{ij} a^{jk}_{(i)} + a^{ik} a^{kj}_{(i)}, \qquad i \neq j \neq k \neq i.$$

The $j \leftrightarrow k$ symmetry of the RHS implies the existence of functions $h^i : \mathbb{Z}^N \to \mathbb{D}$ such that $a^{ij} = (h^i)^{-1} \Delta_j h^i$, $i \neq j$.

In terms of

$$\boldsymbol{X}^i = (\Delta_i \boldsymbol{x})(h^i)^{-1}, \qquad \beta^{ij} = (\Delta_i h^j)(h^i_{(j)})^{-1}, \qquad i \neq j,$$

we have

$$\Delta_j \boldsymbol{X}^i = \boldsymbol{X}^j \beta^{ij}, \qquad \Delta_k \beta^{ij} = \beta^{kj} \beta^{ik}_{(j)}, \qquad i \neq j \neq k \neq i.$$

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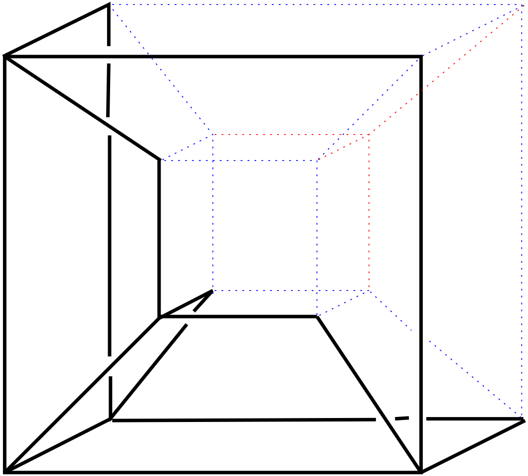
Multidimensional consistency of the quadrilateral lattice



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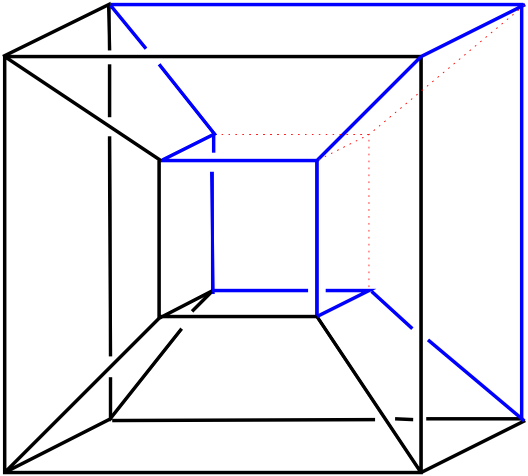

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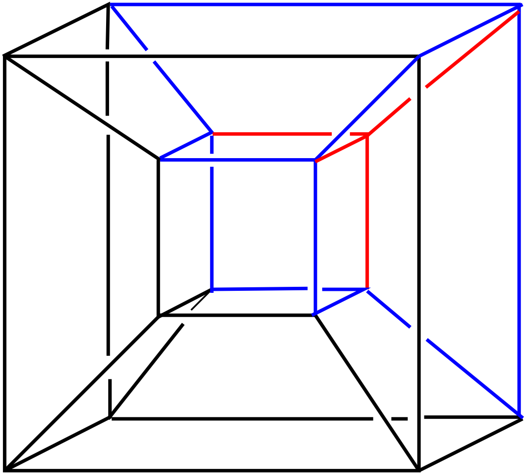


The vectorial fundamental transformation of Jonas

Given the column-vector solution $\boldsymbol{Y}^i : \mathbb{Z}^N \to \mathbb{D}^K$ of the linear problem

$$\Delta_j \boldsymbol{Y}^i = \boldsymbol{Y}^j \beta^{ij}, \qquad i \neq j,$$

and given the row-vector solution $\boldsymbol{Z}^i : \mathbb{Z}^N \to \mathbb{D}^K$ of its adjoint

$$\Delta_i \boldsymbol{Z}^j = \beta^{ij} \boldsymbol{Z}^i_{(j)}, \qquad i \neq j,$$

they allow to construct the $K \times K$ matrix-valued potential $\Omega[\boldsymbol{Y}, \boldsymbol{Z}]$ defined by

$$\Delta_i \Omega[\boldsymbol{Y}, \boldsymbol{Z}] = \boldsymbol{Y}^i \boldsymbol{Z}^i;$$

similarly one defines $\Omega[\boldsymbol{X}, \boldsymbol{Z}]$ and $\Omega[\boldsymbol{Y}, h]$. Then

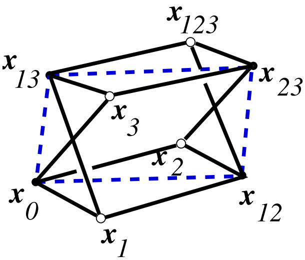
$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \Omega[\boldsymbol{X}, \boldsymbol{Z}] \Omega[\boldsymbol{Y}, \boldsymbol{Z}]^{-1} \Omega[\boldsymbol{Y}, h]$$

is a new quadrilateral lattice with the rotation coefficients

$$\tilde{\beta}^{ij} = \beta^{ij} - \boldsymbol{Z}^j \Omega[\boldsymbol{Y}, \boldsymbol{Z}]^{-1}_{(j)} \boldsymbol{Y}^i_{(j)}, \qquad i \neq j.$$

The B-quadrilateral lattice

Under hypotheses of the Geometric Integrability Scheme, assume that $\mathbb{D}$ is commutative and x_0, x_{12}, x_{13} and x_{23} are coplanar.



Then the points x_1, x_2, x_3 and x_{123} are coplanar as well.

Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{F})$, is called the B-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the points x, $x_{(ij)}$, $x_{(ik)}$ and $x_{(jk)}$ are coplanar.

A. D., 2007

The B-constraint implies existence of a function $\tau^B : \mathbb{Z}^N \to \mathbb{F}$ which satisfies Miwa's discrete BKP equation

$$\tau^B \tau^B_{(ijk)} = \tau^B_{(ij)}\tau^B_{(k)} - \tau^B_{(ik)}\tau^B_{(j)} + \tau^B_{(jk)}\tau^B_{(i)}, \quad 1 \leq i < j < k \leq N,$$

T. Miwa, 1982

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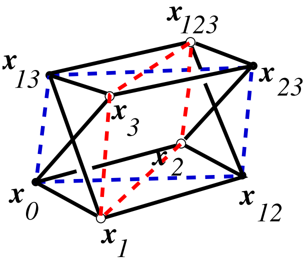

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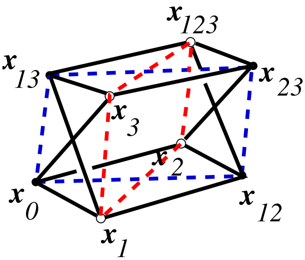

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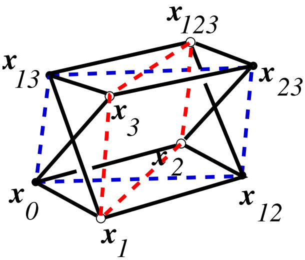

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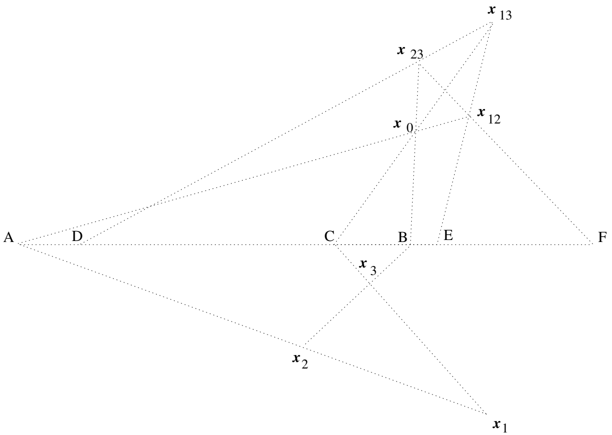
The Möbius theorem (1828)



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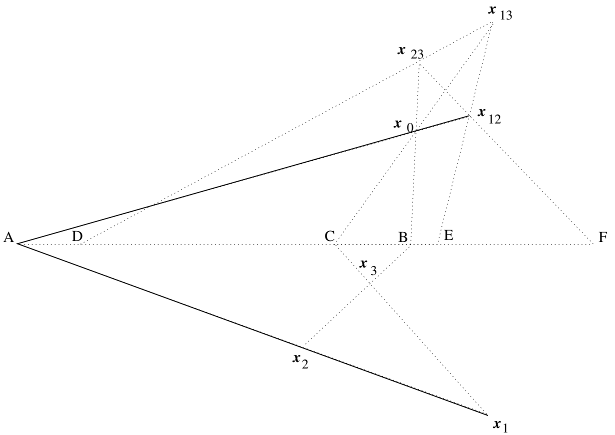

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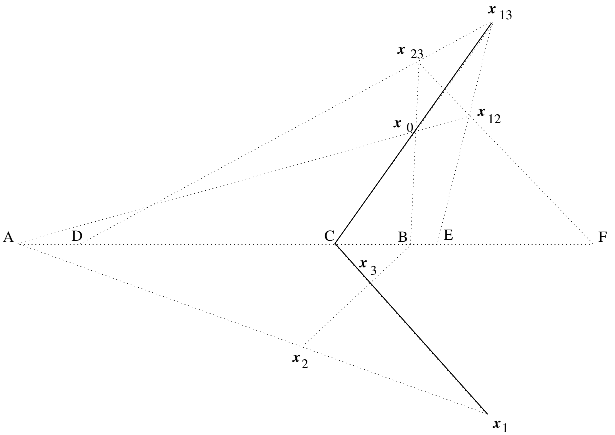

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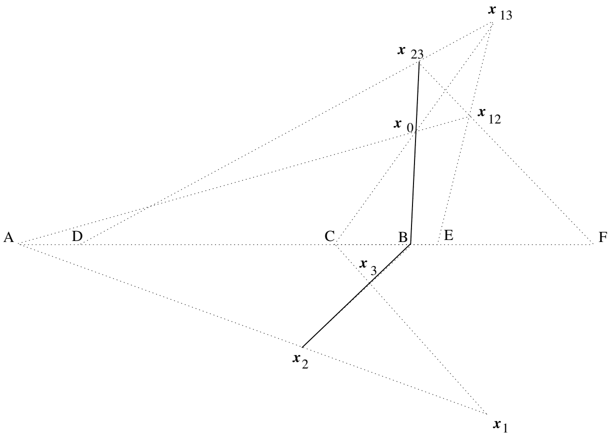

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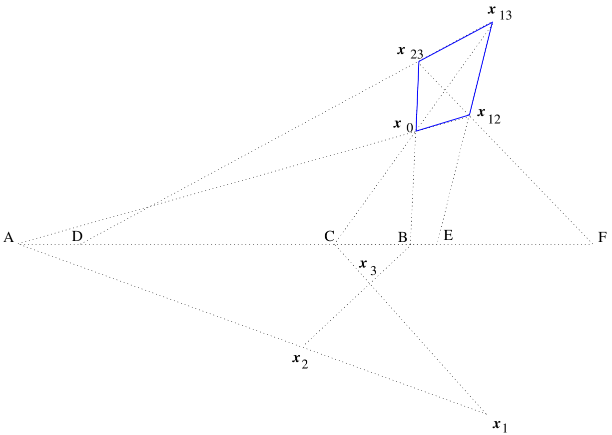

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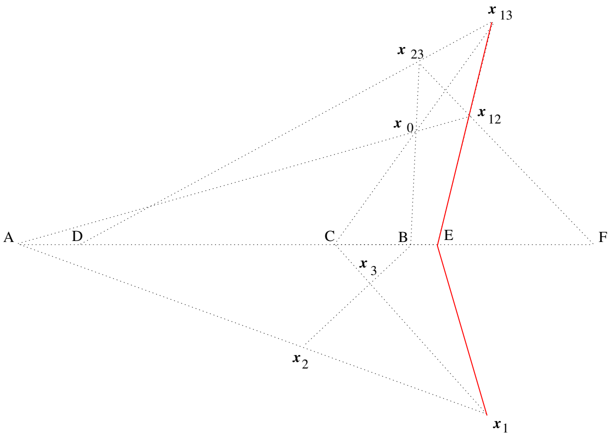

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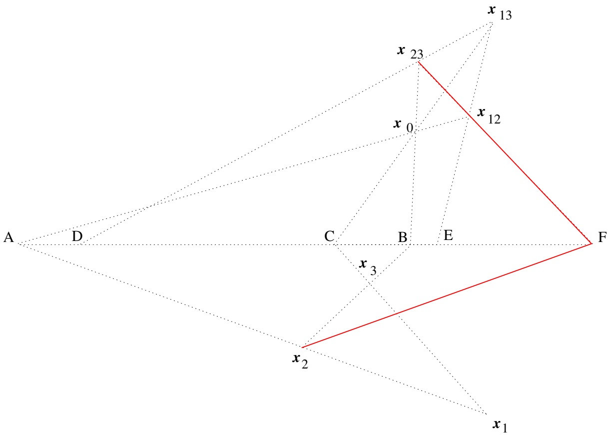

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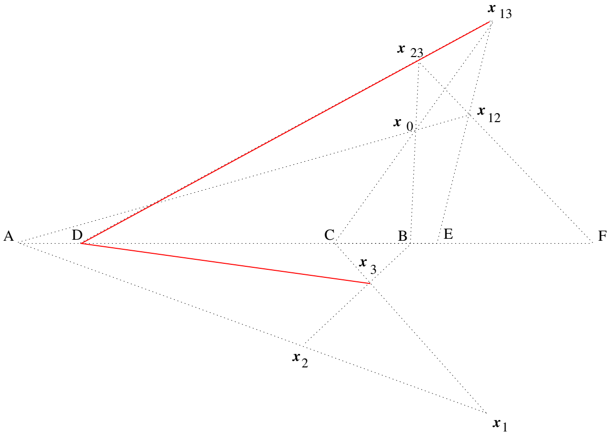

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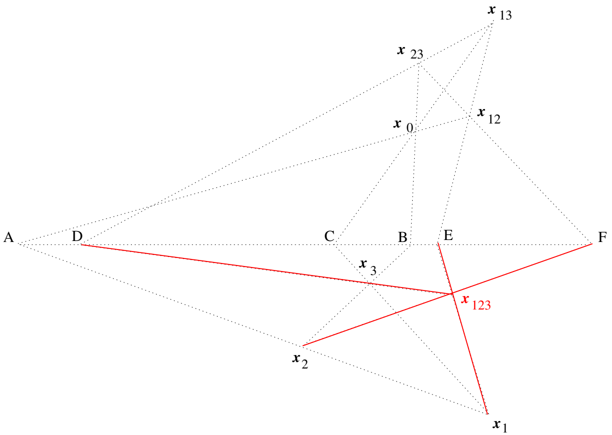

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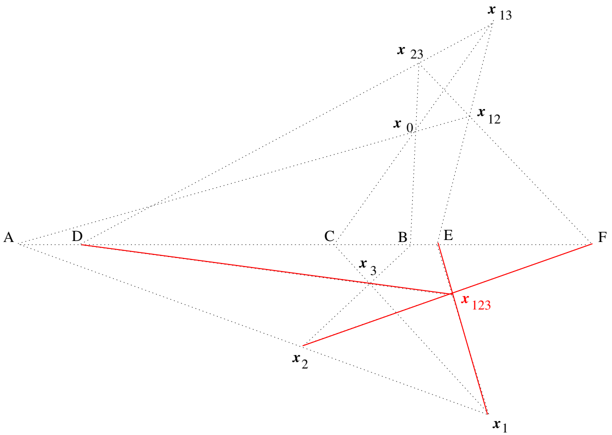

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The C-quadrilateral lattice

Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{A}^M(\mathbb{F}) = \mathbb{P}^M(\mathbb{F}) \setminus H_\infty$, is called the C-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the three intersection points of the common lines of the opposite planes of the corresponding hexahedron with the hyperplane at infinity are collinear.

3D constraint needs checking its 4D consistency

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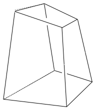

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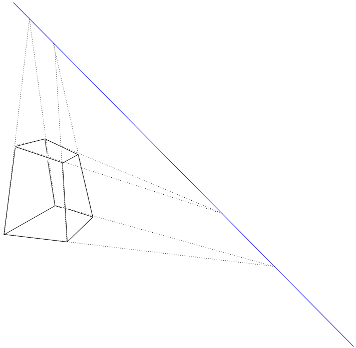

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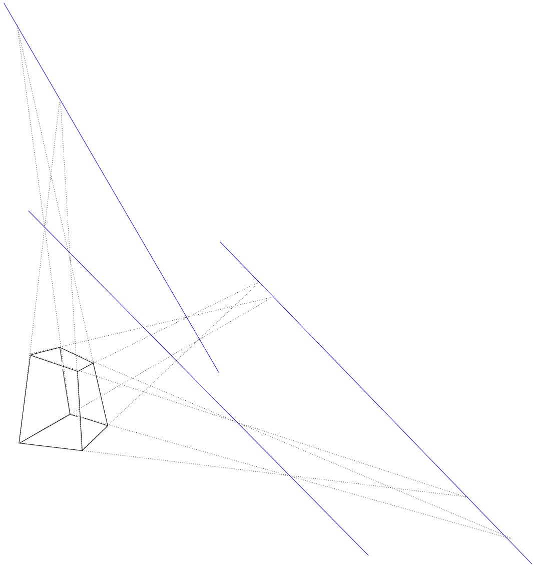

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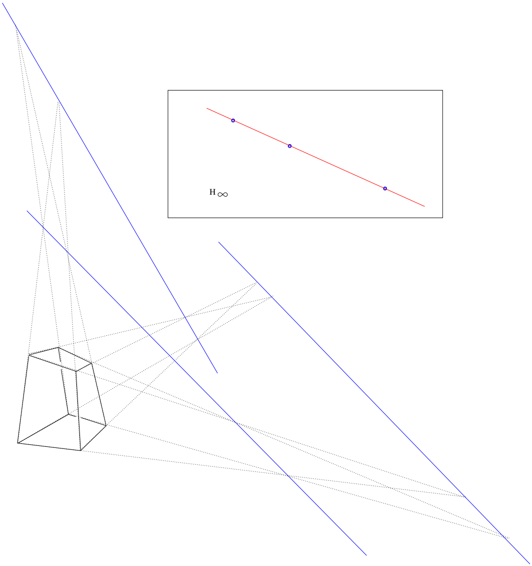

The discrete CKP equation

Algebraic characterization of the C-quadrilateral lattice

A quadrilateral lattice is subject to the C- reduction if and only if its rotation coefficients satisfy the constraint

$$\beta^{ij}\beta^{jk}\beta^{ki} = \beta^{kj}\beta^{ik}\beta^{ji}, \qquad i,j,k \quad \text{distinct.}$$

The symmetric lattice *W. K. Schief, A. D. & P. M. Santini, 2000*

The discrete CKP system *W. K. Schief, 2003*

$$\left(\tau\,\tau_{(ijk)} - \tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} - \tau_{(k)}\tau_{(ij)}\right)^2 = 4\left(\tau_{(i)}\tau_{(j)}\tau_{(ik)}\tau_{(jk)} + \tau_{(i)}\tau_{(k)}\tau_{(ij)}\tau_{(jk)} + \tau_{(j)}\tau_{(k)}\tau_{(ik)}\tau_{(ij)} - \tau_{(i)}\tau_{(j)}\tau_{(k)}\tau_{(ijk)} - \tau\,\tau_{(ij)}\tau_{(jk)}\tau_{(ik)}\right), \qquad i,j,k \quad \text{distinct.}$$

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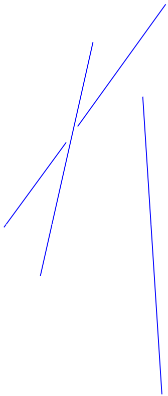
The Gallucci Theorem

If three skew lines all meet three other skew lines, any transversal to the first set of three meets any transversal to the second set.



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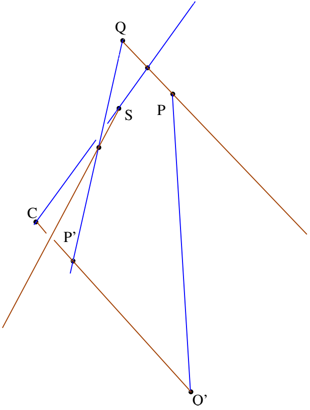

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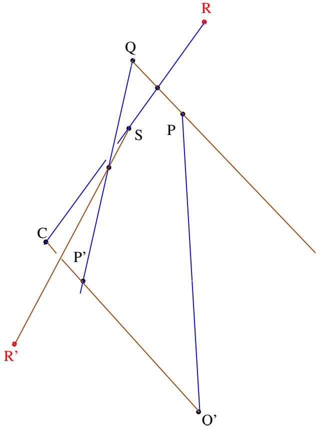

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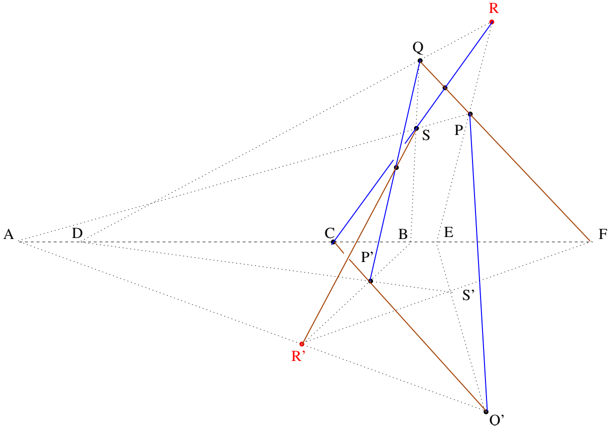

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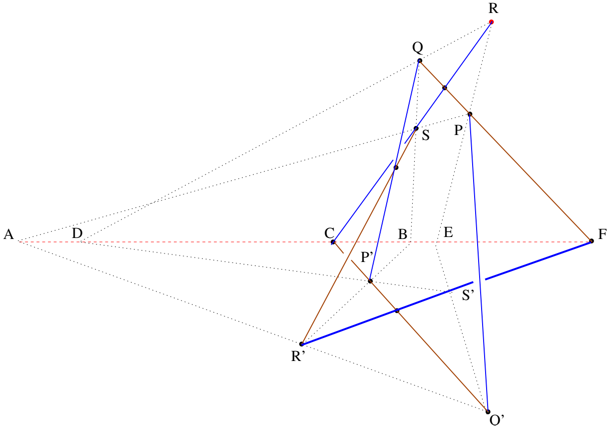

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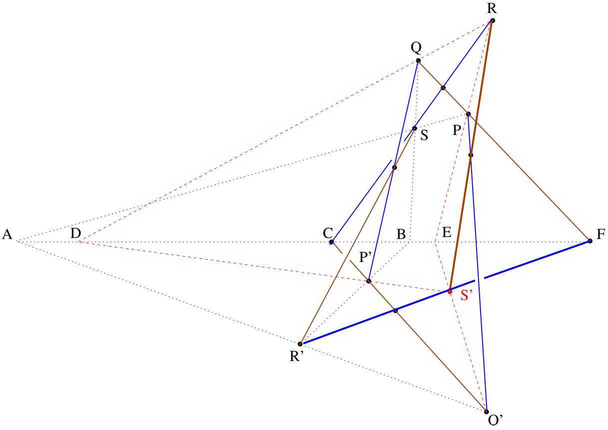

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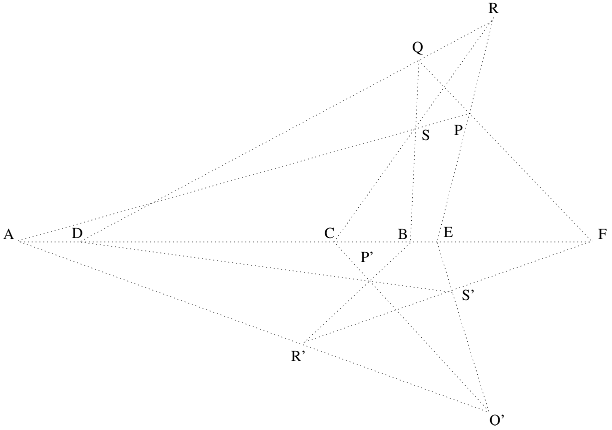

Theorem (16 point theorem)

Let $\mathbb{P}$ be a 3-dimensional projective space over the division ring $\mathbb{D}$. Let $\{g_1, g_2, g_3\}$ and $\{h_1, h_2, h_3\}$ be sets of skew lines with the property that each line g_i meets each line h_j. Then the following is true: $\mathbb{D}$ is commutative (hence a field) if and only if each transversal $g \notin \{g_1, g_2, g_3\}$ of $\{h_1, h_2, h_3\}$ intersects each transversal $h \notin \{h_1, h_2, h_3\}$ of $\{g_1, g_2, g_3\}$.