Geometric algebra and quadrilateral lattices

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Outline

Geometric algebra

The quadrilateral lattice and geometric integrability scheme

The B-(Moutard) and C-(symmetric) quadrilateral lattices

The projective plane axioms

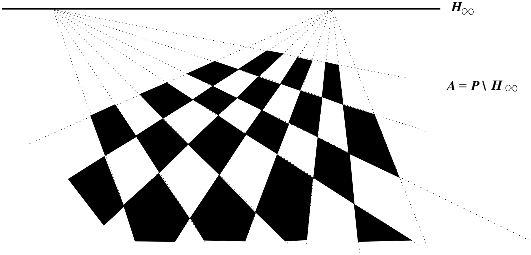
A projective plane is a set, whose elements are called points and a set of subsets, called lines, satisfying the following four axioms:

P1 Two distinct points lie on one and exactly one line.

P2 Two distinct lines meet in precisely one point.

P3 There exist three noncollinear points.

P4 Every line contains at least three points.



In analytic geometry one wants to get results, while in synthetic geometry one would like to get insight.

P1-P4 $\Rightarrow$ coordinatization in terms of a ternary ring

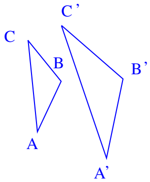
A ternary ring (Γ, T) is a set $\Gamma = \{0, 1, a, b, c, \dots\}$ together with a mapping $T : \Gamma \times \Gamma \times \Gamma \to \Gamma$ such that:

- **T1** For all $a, m, c \in \Gamma$, $T(0, m, c) = T(a, 0, c) = c$.
- **T2** For all $a \in \Gamma$, $T(a, 1, 0) = T(1, a, 0) = a$.
- **T3** If $m, m', b, b' \in \Gamma$ and $m \neq m'$, then the equation $T(x, m, b) = T(x, m', b')$ has a unique solution in Γ.
- **T4** If $a, a', b, b' \in \Gamma$ and $a \neq a'$, then the system of equations $T(a, x, y) = b$, $T(a', x, y) = b'$ has a unique solution in Γ.
- **T5** For all $a, m, c \in \Gamma$, the equation $T(a, m, x) = c$ has a unique solution in Γ.

addition: $a + b = T(a, 1, b)$
multiplication: $a \cdot b = T(a, b, 0)$
Example: A division ring $(\mathbb{D}, +, \cdot, 0, 1)$ is a ternary ring with $T(a, m, b) = a \cdot m + b$.

The Desargues axiom

P5 If two triangles are in perspective from a point then they are in perspective from a line.



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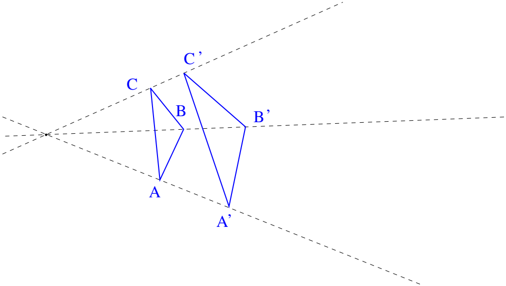

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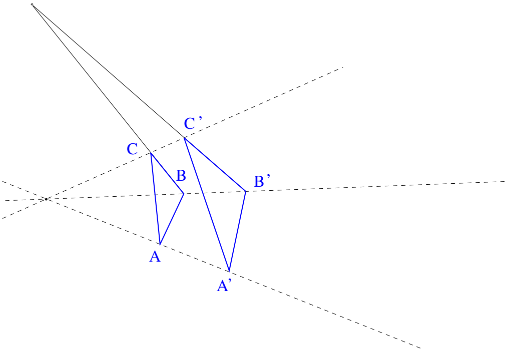

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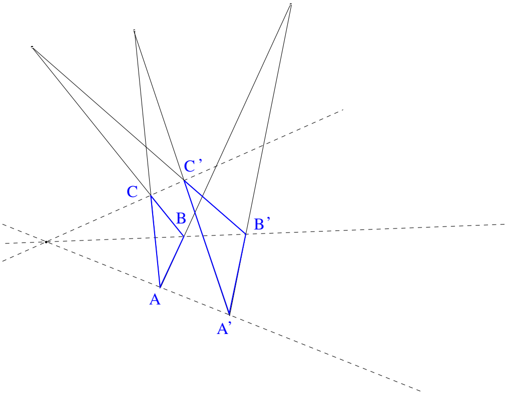

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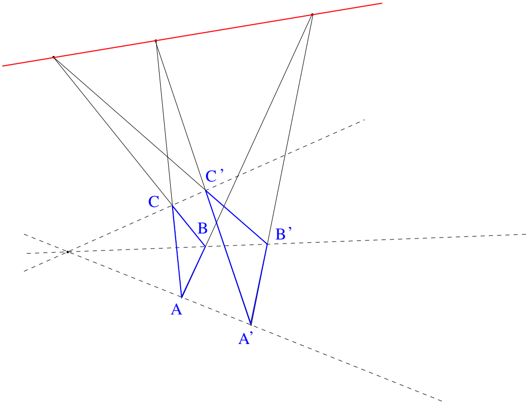

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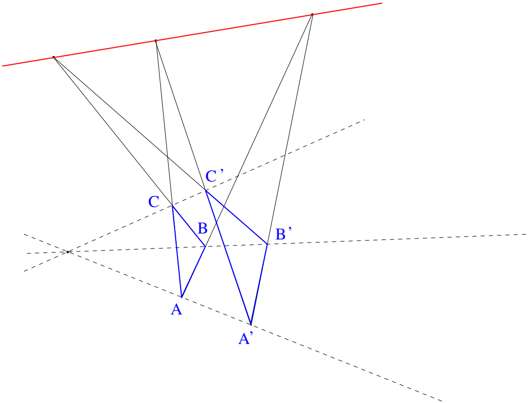

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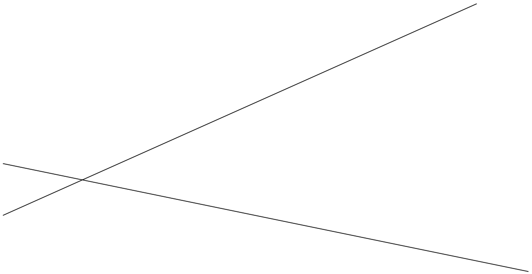

P1-P5' $\Rightarrow$ coordinatization in terms of a field (comutative division ring).

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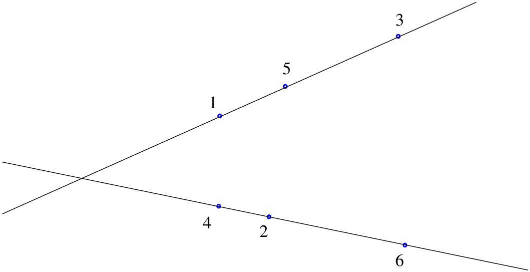

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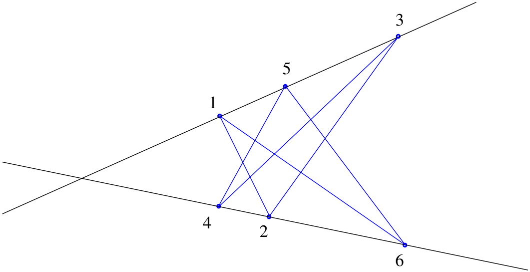

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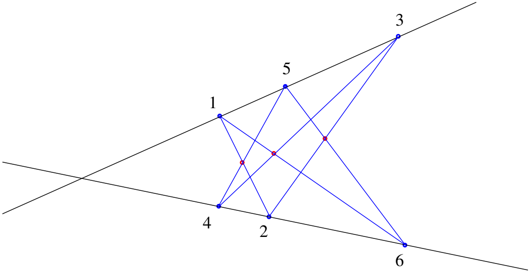

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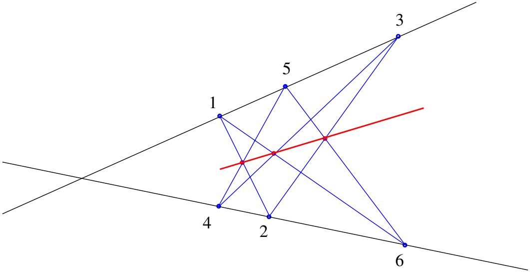

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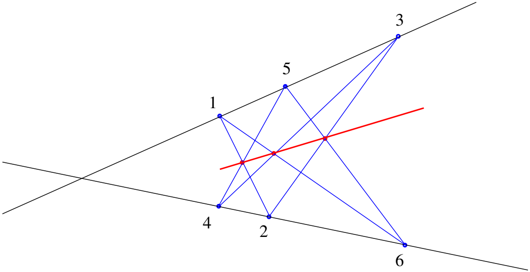

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The projective 3-space axioms

A projective 3-space is a set whose elements are called points, together with certain subsets called lines, and certain other subsets called planes, which satisfy the following axioms:

- **S1** Two distinct points lie on one and only line.
- **S2** Three noncollinear points lie on a unique plane.
- **S3** A line meets a plane in at least one point.
- **S4** Two planes have at least a line in common.
- **S5** There exist four noncoplanar points, no three of which are collinear.
- **S7** Every line has at least three points.

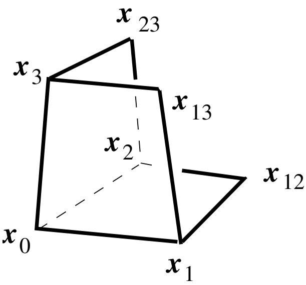
Theorem

Desargues' "axiom" holds in any projective 3-space, where we do not necessarily assume that all the points lie in a plane.

Geometric Integrability Scheme

Given generic points x_0, x_1, x_2 and x_3 in a projective 3-space, let x_{ij}, $1 \leq i < j \leq 3$, be generic points of the planes $\langle x_0, x_i, x_j \rangle$.

Then there exists exactly one point x_{123} which belongs simultaneously to the planes $\langle x_3, x_{13}, x_{23} \rangle$, $\langle x_2, x_{12}, x_{23} \rangle$ and $\langle x_1, x_{12}, x_{13} \rangle$.



Definition

A quadrilateral lattice is a map $x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D})$, $3 \leq N \leq M$, whose all elementary quadrilaterals are planar.

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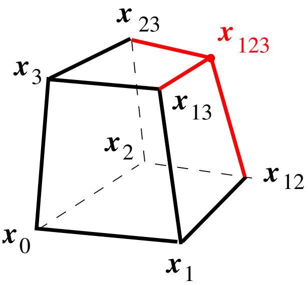

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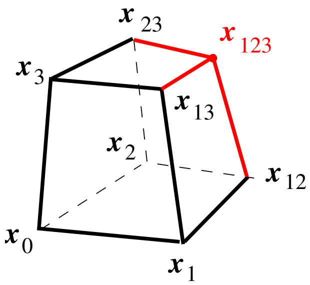

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The discrete Darboux equations (affine version)

In non-homogeneous coordinates $\boldsymbol{x} : \mathbb{Z}^N \to \mathbb{D}^M \sim \mathbb{P}(\mathbb{D}) \setminus H_\infty$,

$$\Delta_i \Delta_j \boldsymbol{x} = (\Delta_i \boldsymbol{x}) a^{ij} + (\Delta_j \boldsymbol{x}) a^{ji}, \quad 1 \le i < j \le N,$$

$$a^{ij} : \mathbb{Z}^N \to \mathbb{D}, \quad i \ne j.$$

Notation:

$\boldsymbol{x}_{(i)}(n_1, \ldots, n_i, \ldots, n_N) = \boldsymbol{x}(n_1, \ldots, n_i + 1, \ldots, n_N)$, $\Delta \boldsymbol{x} = \boldsymbol{x}_{(i)} - \boldsymbol{x}$.

The compatibility condition

$$\Delta_k a^{ij} + a^{ik} a^{ij}_{(k)} = a^{ij} a^{jk}_{(i)} + a^{ik} a^{kj}_{(i)}, \qquad i \ne j \ne k \ne i.$$

The $j \leftrightarrow k$ symmetry of the RHS implies the existence of functions $h^i : \mathbb{Z}^N \to \mathbb{D}$ such that $a^{ij} = (h^i)^{-1} \Delta_j h^i$, $i \ne j$.

In terms of

$$\boldsymbol{X}^i = (\Delta_i \boldsymbol{x})(h^i)^{-1}, \qquad \beta^{ij} = (\Delta_i h^j)(h^i_{(j)})^{-1}, \qquad i \ne j,$$

we have

$$\Delta_j \boldsymbol{X}^i = \boldsymbol{X}^j \beta^{ij}, \qquad \Delta_k \beta^{ij} = \beta^{kj} \beta^{ik}_{(j)}, \qquad i \ne j \ne k \ne i.$$

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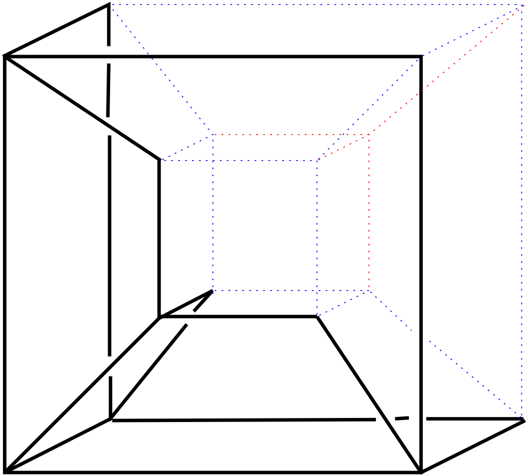
Multidimensional consistency of the quadrilateral lattice



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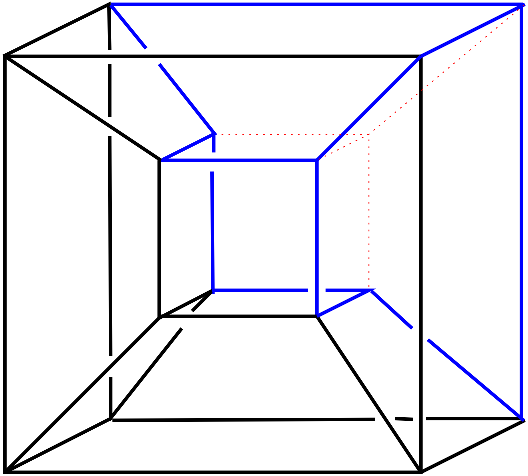

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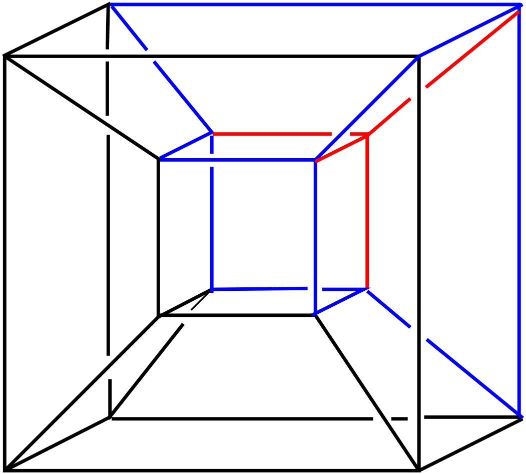


The vectorial fundamental transformation of Jonas

Given the column-vector solution $\boldsymbol{Y}^i : \mathbb{Z}^N \to \mathbb{D}^K$ of the linear problem

$$\Delta_j \boldsymbol{Y}^i = \boldsymbol{Y}^j \beta^{ij}, \qquad i \neq j,$$

and given the row-vector solution $\boldsymbol{Z}^i : \mathbb{Z}^N \to \mathbb{D}^K$ of its adjoint

$$\Delta_i \boldsymbol{Z}^j = \beta^{ij} \boldsymbol{Z}^i_{(j)}, \qquad i \neq j,$$

they allow to construct the $K \times K$ matrix-valued potential $\Omega[\boldsymbol{Y}, \boldsymbol{Z}]$ defined by

$$\Delta_i \Omega[\boldsymbol{Y}, \boldsymbol{Z}] = \boldsymbol{Y}^i \boldsymbol{Z}^i;$$

similarly one defines $\Omega[\boldsymbol{X}, \boldsymbol{Z}]$ and $\Omega[\boldsymbol{Y}, h]$. Then

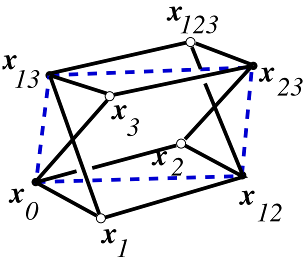
$$\tilde{\boldsymbol{x}} = \boldsymbol{x} - \Omega[\boldsymbol{X}, \boldsymbol{Z}] \Omega[\boldsymbol{Y}, \boldsymbol{Z}]^{-1} \Omega[\boldsymbol{Y}, h]$$

is a new quadrilateral lattice with the rotation coefficients

$$\tilde{\beta}^{ij} = \beta^{ij} - \boldsymbol{Z}^j \Omega[\boldsymbol{Y}, \boldsymbol{Z}]^{-1}_{(j)} \boldsymbol{Y}^i_{(j)}, \qquad i \neq j.$$

The B-quadrilateral lattice

Under hypotheses of the Geometric Integrability Scheme, assume that $\mathbb{D}$ is commutative and x_0, x_{12}, x_{13} and x_{23} are coplanar.



Then the points x_1, x_2, x_3 and x_{123} are coplanar as well.

Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{F})$, is called the B-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the points x, $x_{(ij)}$, $x_{(ik)}$ and $x_{(jk)}$ are coplanar.

A. D., 2007

The B-constraint implies existence of a function $\tau^B : \mathbb{Z}^N \to \mathbb{F}$ which satisfies Miwa's discrete BKP equation

$$\tau^B \tau^B_{(ijk)} = \tau^B_{(ij)}\tau^B_{(k)} - \tau^B_{(ik)}\tau^B_{(j)} + \tau^B_{(jk)}\tau^B_{(i)}, \quad 1 \leq i < j < k \leq N,$$

T. Miwa, 1982

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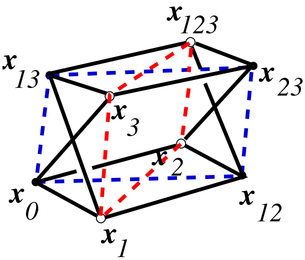

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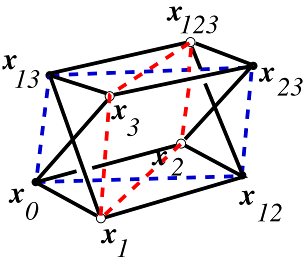

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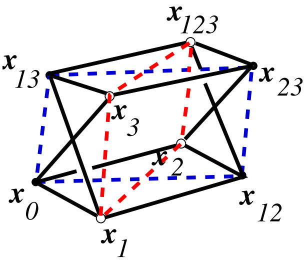

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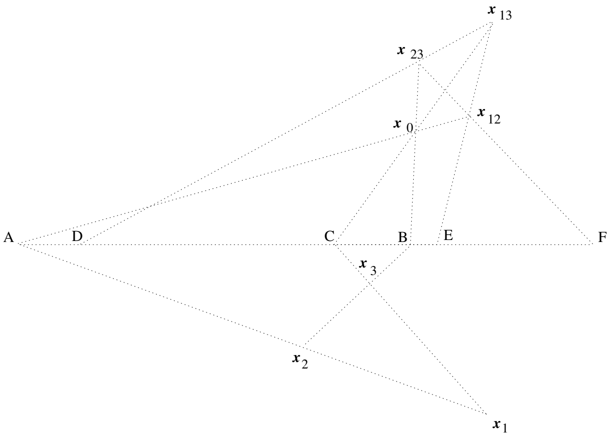

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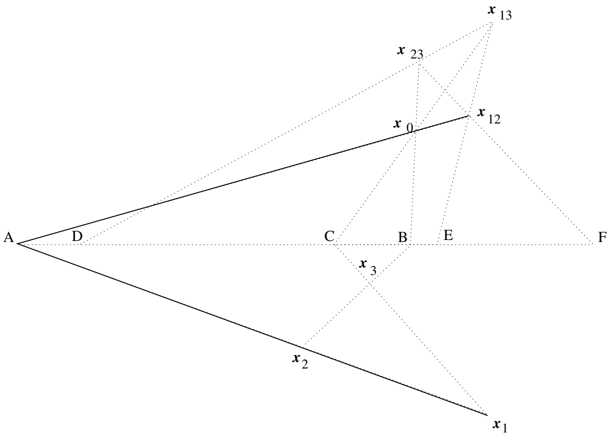

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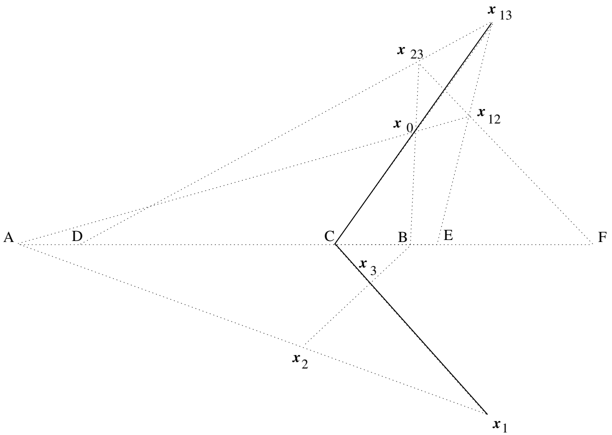

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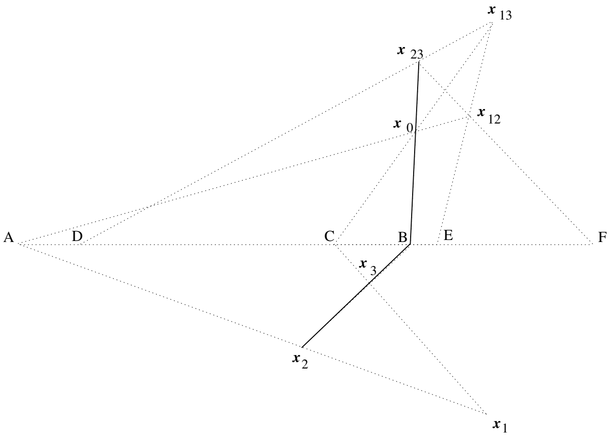

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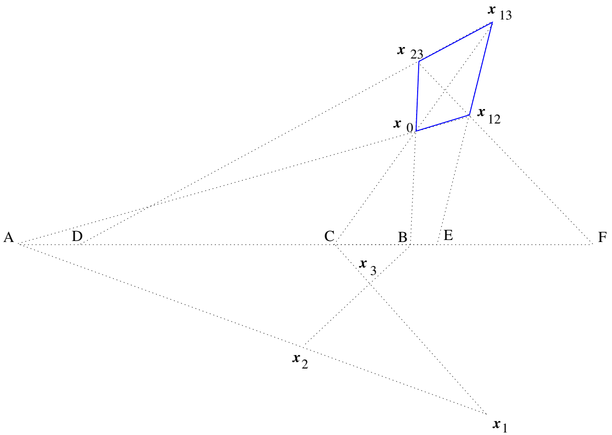

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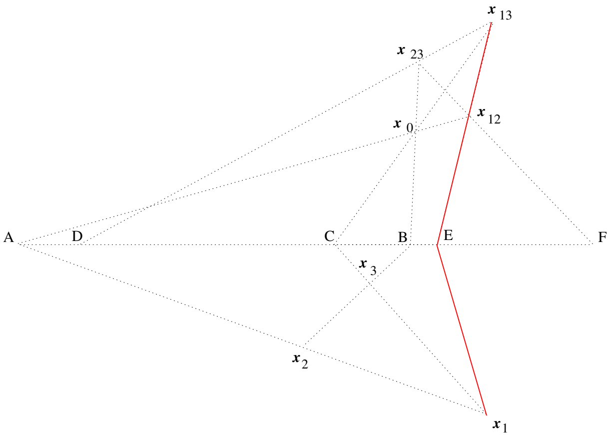

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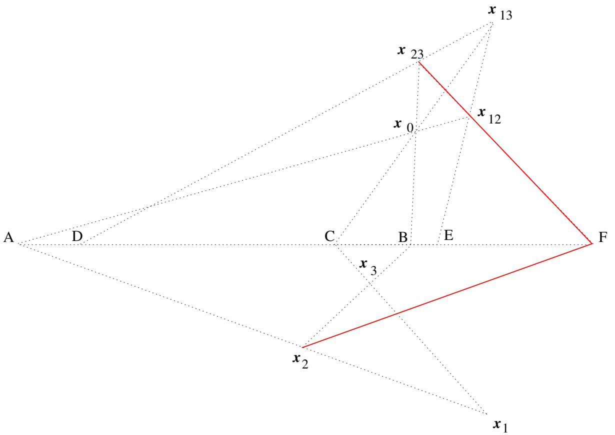
The Möbius theorem (1828)



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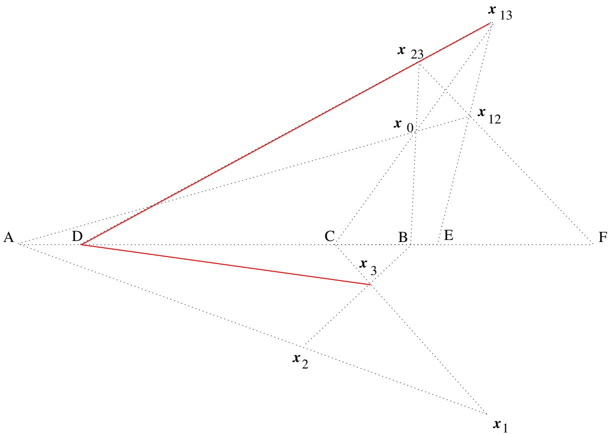

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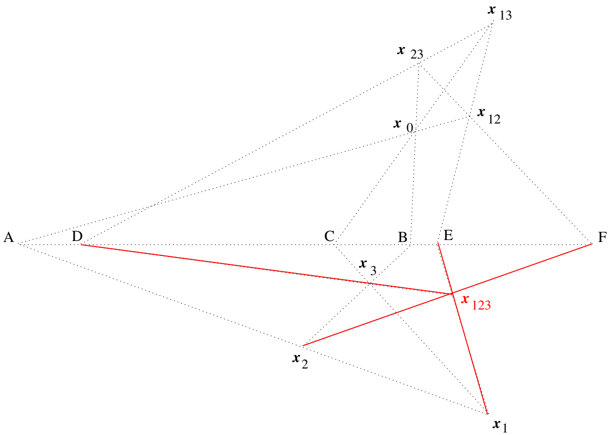

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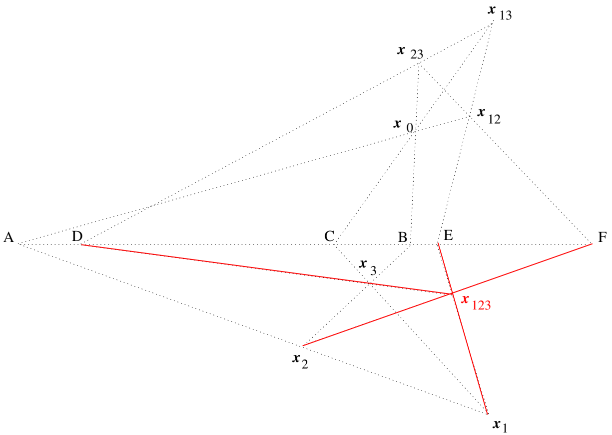

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The C-quadrilateral lattice

Definition

A quadrilateral lattice $x : \mathbb{Z}^N \to \mathbb{A}^M(\mathbb{F}) = \mathbb{P}^M(\mathbb{F}) \setminus H_\infty$, is called the C-quadrilateral lattice if for any triple of different indices $1 \leq i < j < k \leq N$ the three intersection points of the common lines of the opposite planes of the corresponding hexahedron with the hyperplane at infinity are collinear.

3D constraint needs checking its 4D consistency

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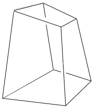
The CQL constraint



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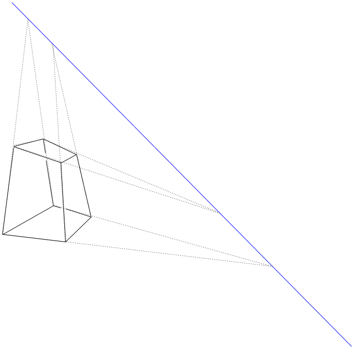

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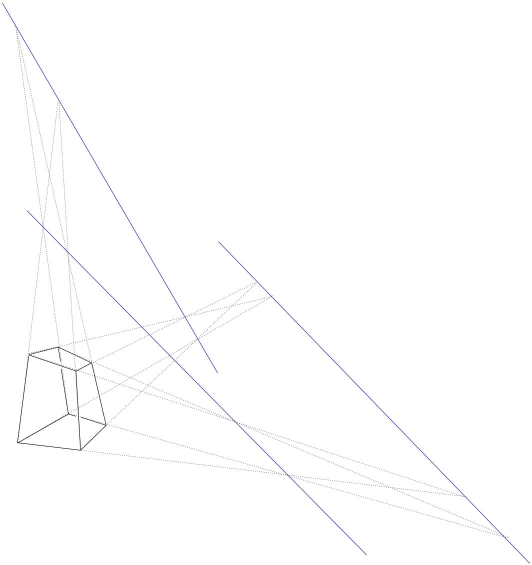

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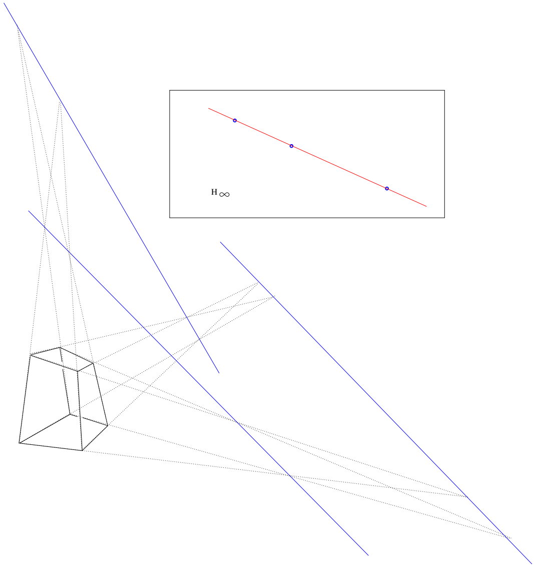

The discrete CKP equation

Algebraic characterization of the C-quadrilateral lattice

A quadrilateral lattice is subject to the C- reduction if and only if its rotation coefficients satisfy the constraint

$$\beta^{ij}\beta^{jk}\beta^{ki} = \beta^{kj}\beta^{ik}\beta^{ji}, \qquad i,j,k \quad \text{distinct.}$$

The symmetric lattice *W. K. Schief, A. D. & P. M. Santini, 2000*

The discrete CKP system *W. K. Schief, 2003*

$$\left(\tau\tau_{(ijk)} - \tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} - \tau_{(k)}\tau_{(ij)}\right)^2 = 4\left(\tau_{(i)}\tau_{(j)}\tau_{(ik)}\tau_{(jk)} + \tau_{(i)}\tau_{(k)}\tau_{(ij)}\tau_{(jk)} + \tau_{(j)}\tau_{(k)}\tau_{(ik)}\tau_{(ij)} - \tau_{(i)}\tau_{(j)}\tau_{(k)}\tau_{(ijk)} - \tau\tau_{(ij)}\tau_{(jk)}\tau_{(ik)}\right), \qquad i,j,k \quad \text{distinct.}$$

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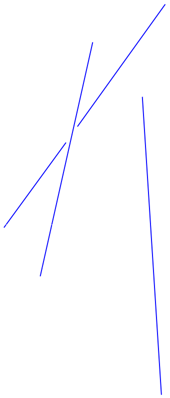
The discrete CKP system *W. K. Schief, 2003*

$$\begin{aligned}
&(\tau\,\tau_{(ijk)} - \tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} - \tau_{(k)}\tau_{(ij)})^2 = \\
&\quad 4(\tau_{(i)}\tau_{(j)}\tau_{(ik)}\tau_{(jk)} + \tau_{(i)}\tau_{(k)}\tau_{(ij)}\tau_{(jk)} + \tau_{(j)}\tau_{(k)}\tau_{(ik)}\tau_{(ij)} - \\
&\qquad \tau_{(i)}\tau_{(j)}\tau_{(k)}\tau_{(ijk)} - \tau\,\tau_{(ij)}\tau_{(jk)}\tau_{(ik)}), \qquad i,j,k \quad \text{distinct.}
\end{aligned}$$

The Gallucci Theorem

If three skew lines all meet three other skew lines, any transversal to the first set of three meets any transversal to the second set.



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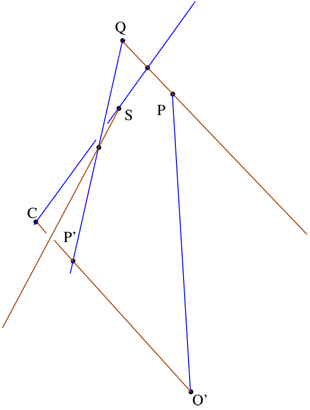

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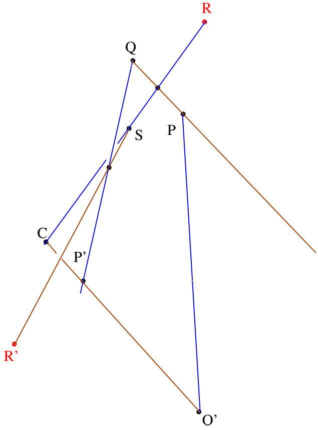

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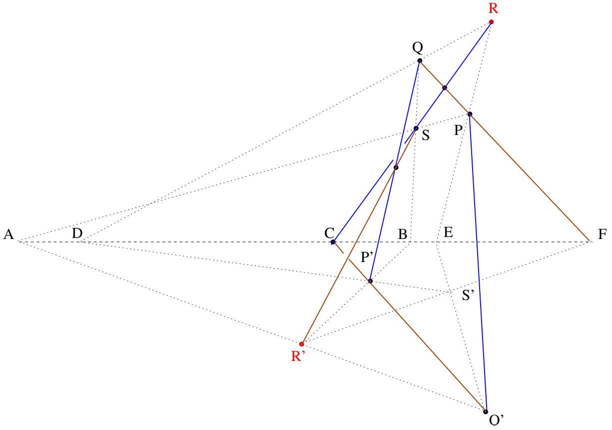

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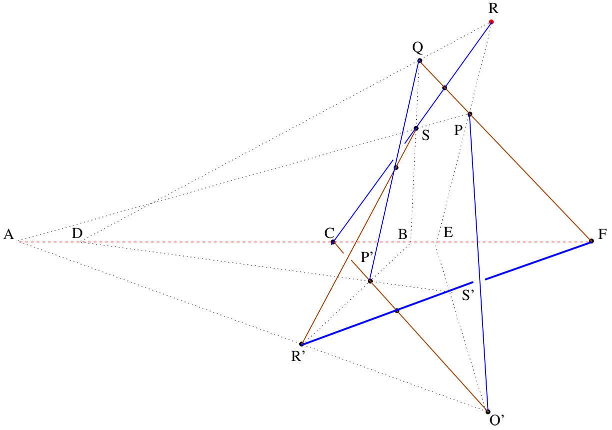

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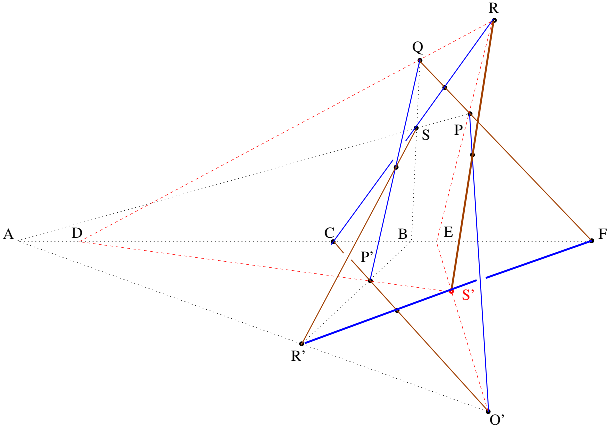

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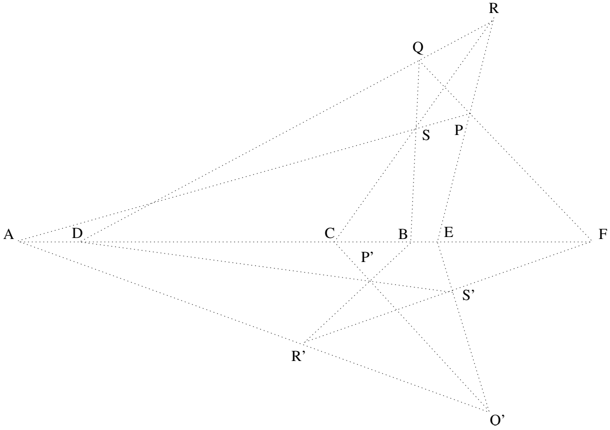

Theorem (16 point theorem)

Let $\mathbb{P}$ be a 3-dimensional projective space over the division ring $\mathbb{D}$. Let $\{g_1, g_2, g_3\}$ and $\{h_1, h_2, h_3\}$ be sets of skew lines with the property that each line g_i meets each line h_j. Then the following is true: $\mathbb{D}$ is commutative (hence a field) if and only if each transversal $g \notin \{g_1, g_2, g_3\}$ of $\{h_1, h_2, h_3\}$ intersects each transversal $h \notin \{h_1, h_2, h_3\}$ of $\{g_1, g_2, g_3\}$.

A