Island-3
Algebraic Aspects of Integrable Systems
Islay, July 5, 2007

On critical behaviour in Hamiltonian PDEs

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By-product of a perturbative approach to the classification of integrable Hamiltonian PDEs

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + B_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + B_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots$$

long-wave expansion (cf. small amplitude perturbative approach of A.Mikhailov, J.Sanders, Jing Ping Wang et al.)

$$\mathbf{u} = (u^1(x,t), \dots, u^n(x,t))$$

$$\deg B_k(\mathbf{u}; \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k$$

$$\deg \frac{\partial^m u^i}{\partial x^m} = m, \quad m > 0, \quad \deg u^i = 0$$

Question: properties of solutions

- how do they change when adding higher derivatives?

Especially at **critical regimes**:

points of gradient catastrophe of the leading order approximation

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

Problem of **universality** of critical behaviour:

to classify the types of critical behaviour of the quasilinear system

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x$$

and of the associated perturbed system

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + B_2(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}) + B_3(\mathbf{u}; \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots$$

# of dependent variables	Unperturbed	Perturbed
n=1	A_3 singularity $x = u\,t - \frac{u^3}{6}$	special solution to P_1^2 $x = u\,t - \frac{u^3}{6} - \left[\frac{1}{24}\left(u'^2 + 2u\,u''\right) + \frac{u^{IV}}{240}\right]$
n=2, hyperbolic case	Whitney singularity	special solution to P_1^2 + linear function
n=2, elliptic case	Elliptic umbilic	*tritronquée* solution to P_1

Remaining part of the talk: n=2, elliptic case

for the example of focusing nonlinear Schrödinger equation

(joint work with T.Grava & C.Klein)

Nonlinear Schrödinger equation

$$i\,\psi_t + \frac{1}{2}\,\Delta\,\psi + |\psi|^2\psi = 0$$

- Dispersive PDE

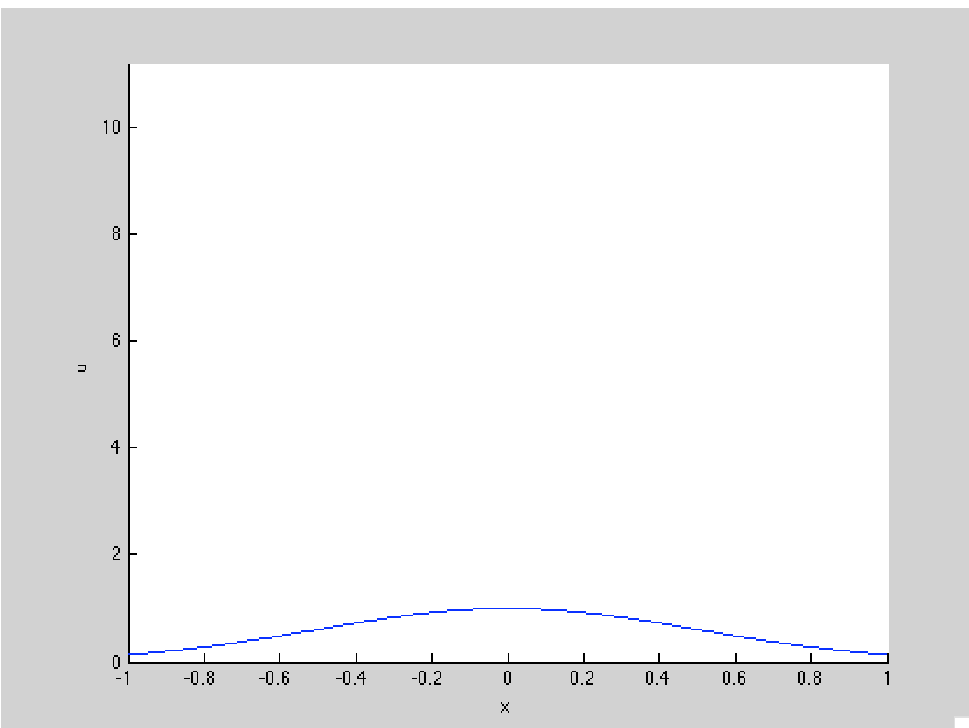
- Hamiltonian system

$$H = \frac{1}{2}\int \left(|\nabla\psi|^2 - |\psi|^4\right)\, d^n x$$

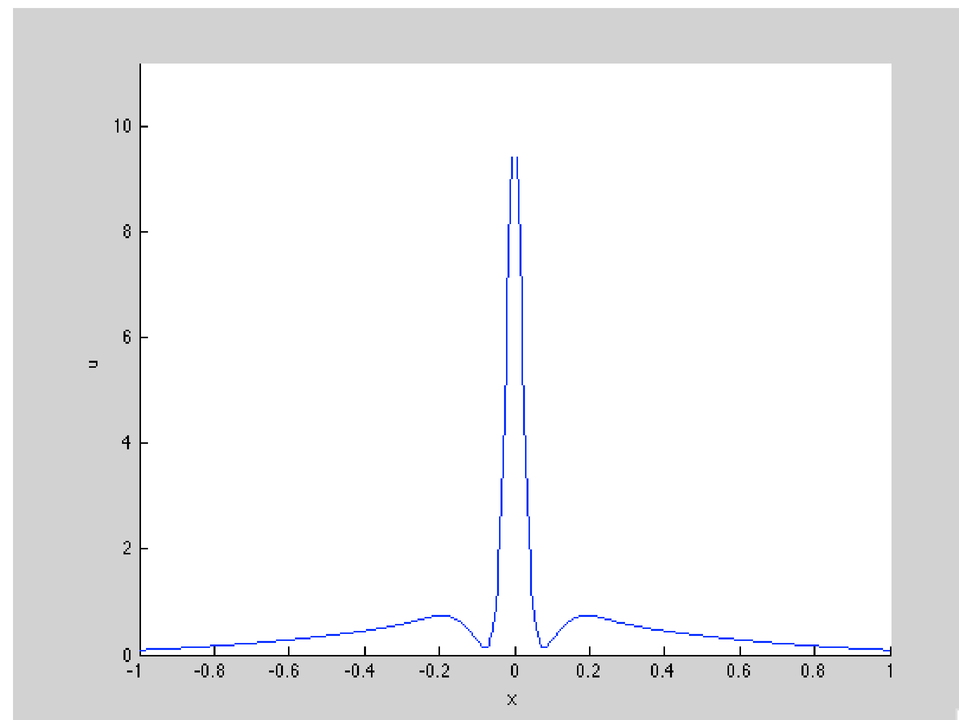
$$\left\{\psi(x), \bar{\psi}(y)\right\} = i\,\delta(x-y)$$

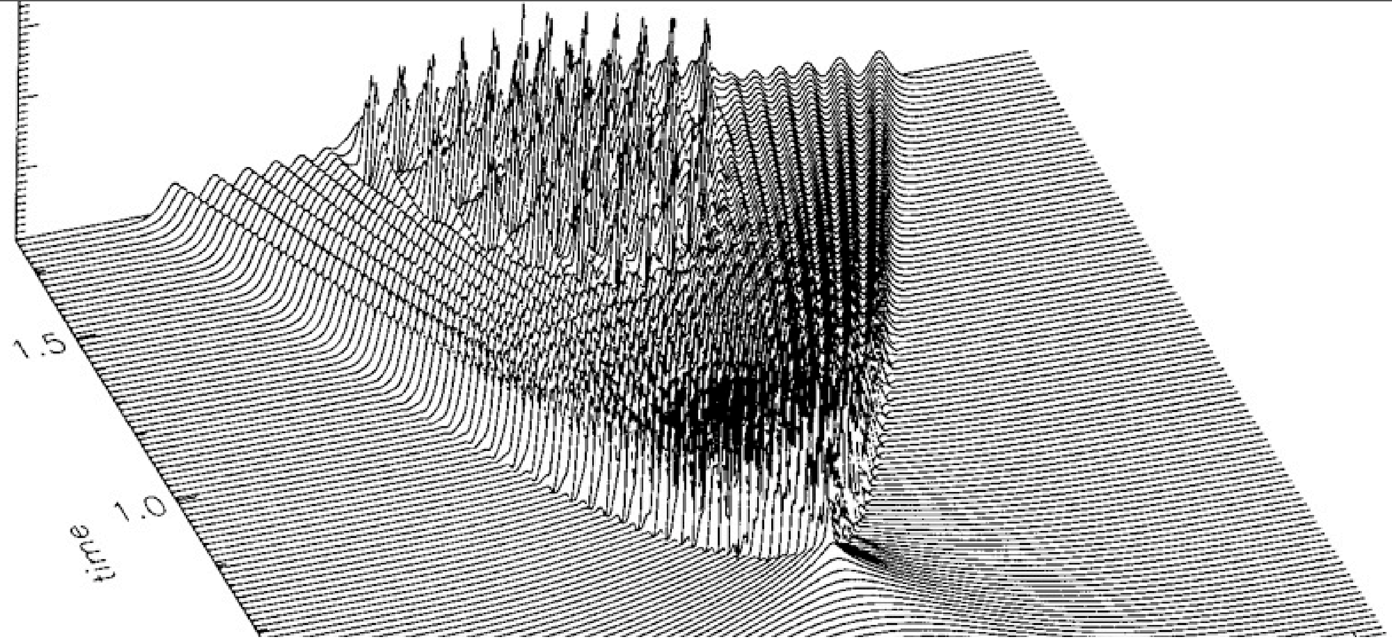
Self-focusing:

for a bump-like initial condition

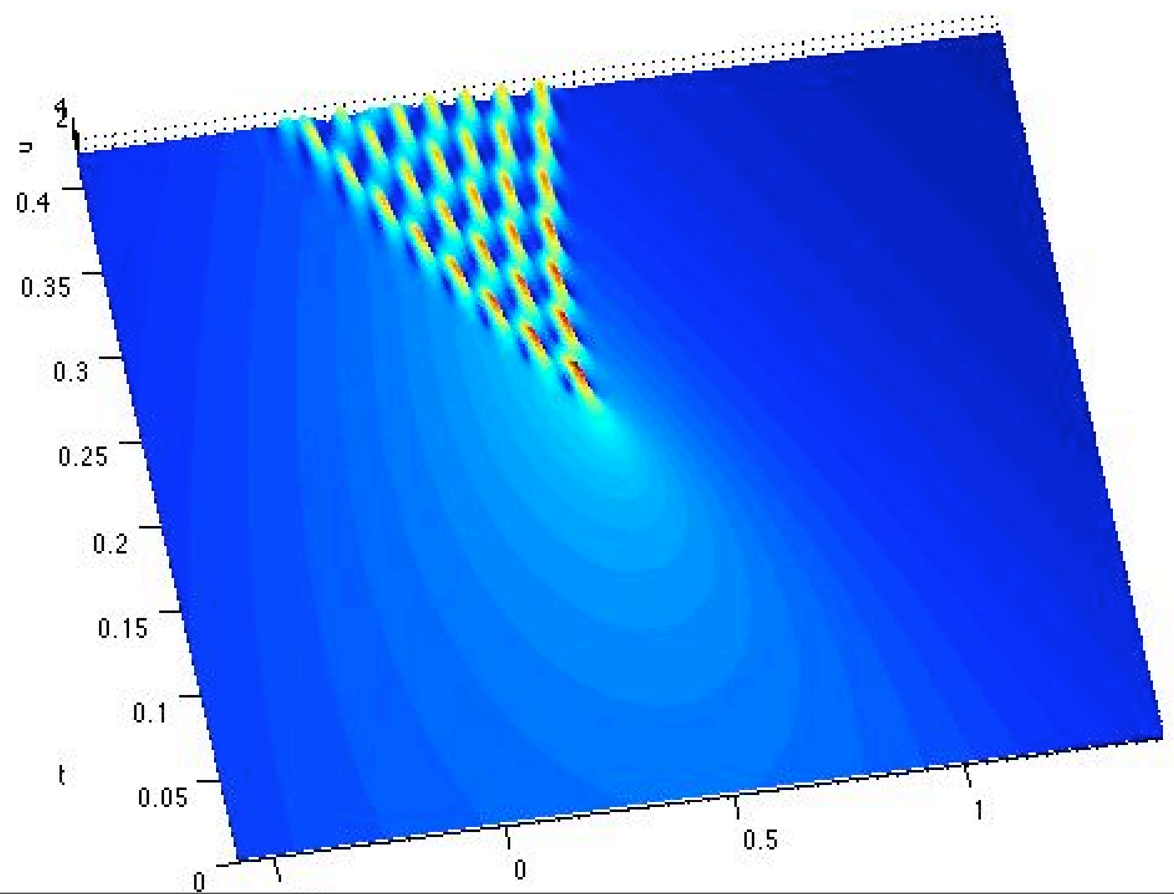


one arrives at





1.5
1.0
time



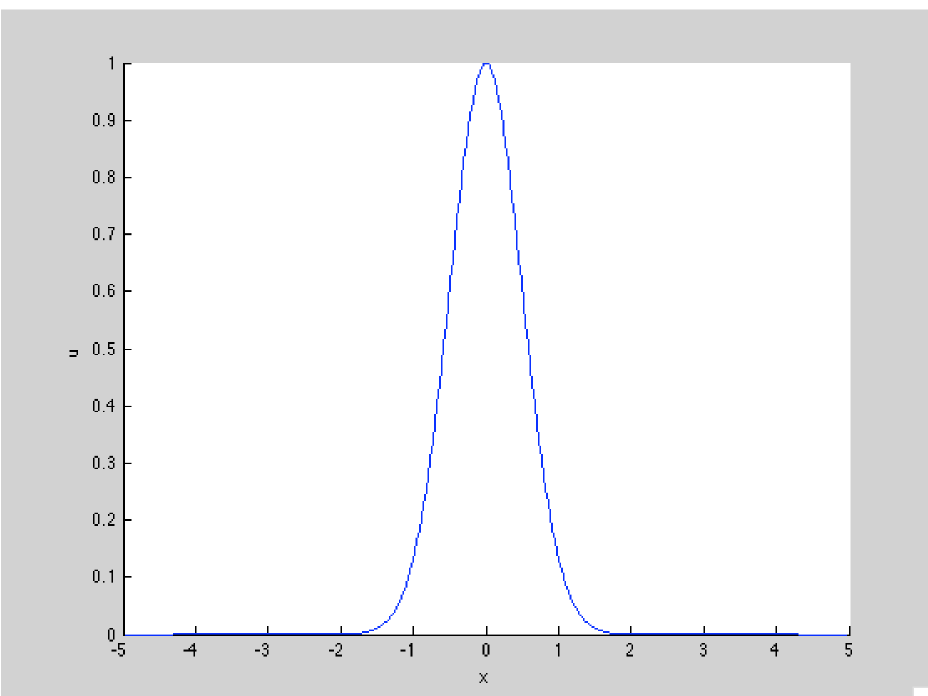
4
2
u
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
t
0
0.5
1

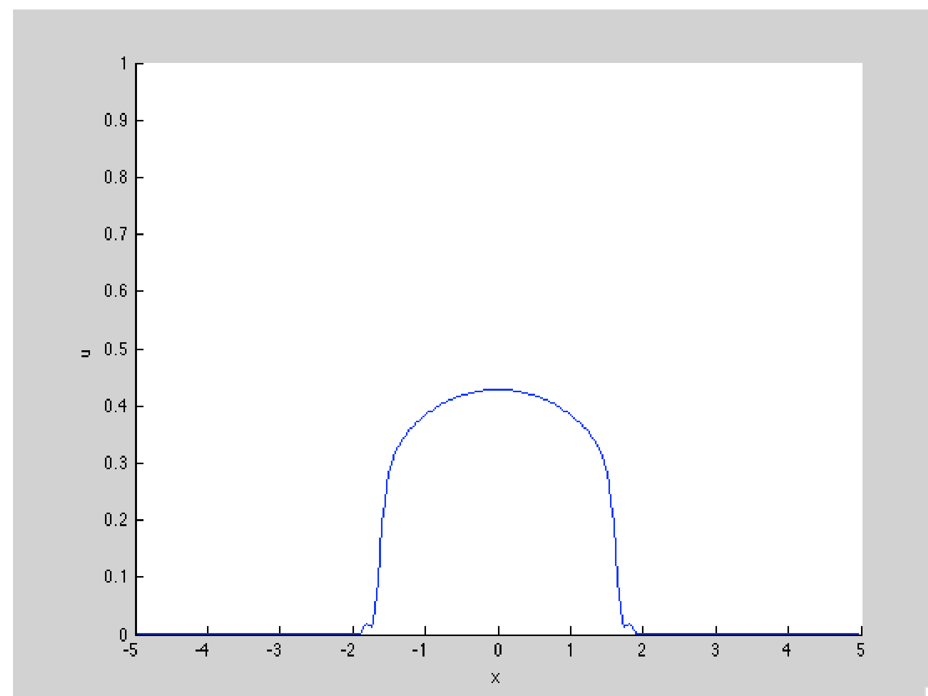
Compare with **defocusing** NLS:

$$i\,\psi_t + \frac{1}{2}\,\Delta\,\psi - |\psi|^2\psi = 0$$

initial condition

result of evolution





One-dimensional case

$$i\,\psi_t + \frac{1}{2}\,\psi_{xx} + |\psi|^2\psi = 0$$

Lax operator is not self-adjoint

$$L = \begin{pmatrix} i\,\partial_x & 0 \\ 0 & -i\,\partial_x \end{pmatrix} + \begin{pmatrix} 0 & -\psi \\ \bar{\psi} & 0 \end{pmatrix}$$

Small parameter ϵ

$$i\,\epsilon\,\psi_t + \frac{1}{2}\,\epsilon^2\psi_{xx} + |\psi|^2\psi = 0$$

Semiclassical ansatz $\psi = \sqrt{u}\,e^{\frac{i}{\epsilon}\int v\,dx}$ yields

$$\left.\begin{aligned} u_t + (u\,v)_x &= 0 \\ v_t + v\,v_x - u_x &= \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\,\frac{u_x^2}{u^2}\right) \end{aligned}\right\}$$

In the "dispersionless" limit $\epsilon \to 0$

one obtains a system

$$\left.\begin{aligned} u_t + (u\,v)_x &= 0 \\ v_t + v\,v_x - u_x &= 0 \end{aligned}\right\}$$

of **elliptic** type: the eigenvalues of the matrix

$$\begin{pmatrix} v & u \\ -1 & v \end{pmatrix}$$

are complex conjugate for $u > 0$

(modulation instability)

For small t the solutions to the Cauchy problem

$$u(x,0) = U(x), \quad v(x,0) = V(x)$$

for the systems

$$\left.\begin{array}{rcl} u_t + (u\,v)_x & = & 0 \\ \\ v_t + v\,v_x - u_x & = & \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\,\frac{u_x^2}{u^2}\right) \end{array}\right\}$$

and

$$\left.\begin{array}{rcl} u_t + (u\,v)_x & = & 0 \\ \\ v_t + v\,v_x - u_x & = & 0 \end{array}\right\}$$

are indistinguishable

The deviation begins near the **critical point** (x_0, t_0)

where derivatives of a solution to

$$\left.\begin{array}{rcl} u_t + (u\,v)_x & = & 0 \\ v_t + v\,v_x - u_x & = & 0 \end{array}\right\}$$

blow up

$$u \longrightarrow u_0, \quad v \longrightarrow v_0, \quad u_x,\ v_x \longrightarrow \infty$$

(similar to **gradient catastrophe**
in nonlinear hyperbolic systems)

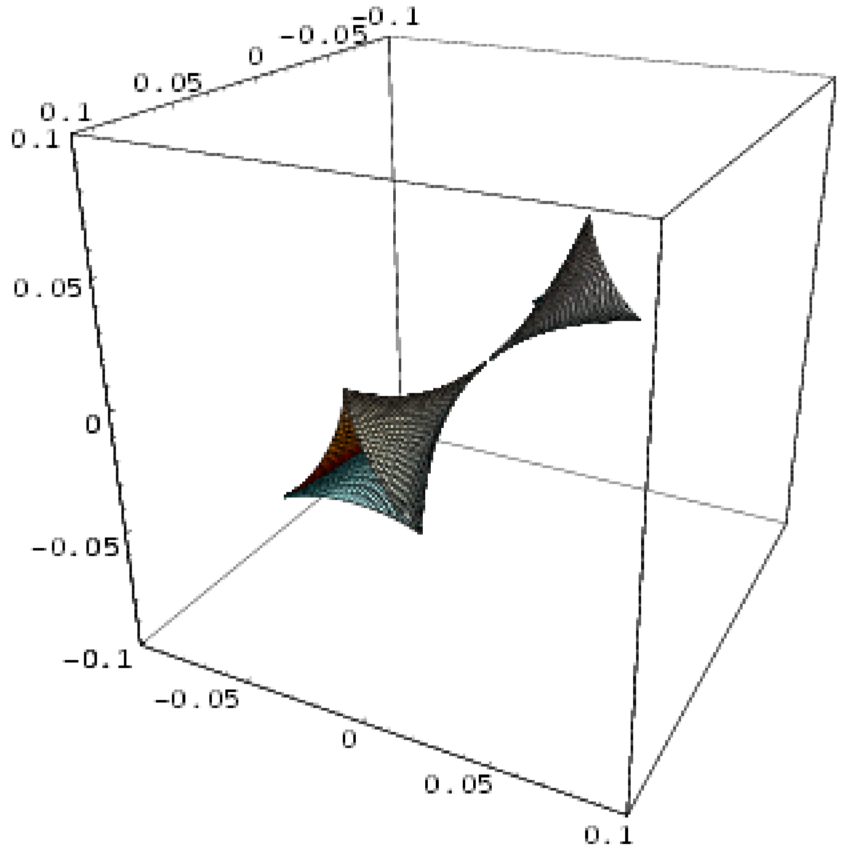
First problem: to study the local behaviour of a generic solution to the dispersionless system

$$\left.\begin{array}{rcl} u_t + (u\,v)_x & = & 0 \\ \\ v_t + v\,v_x - u_x & = & 0 \end{array}\right\}$$

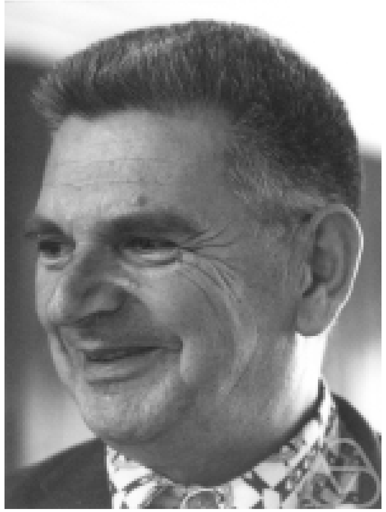
near the point of catastrophe $\quad x_0, \quad t_0$

$$u \longrightarrow u_0, \quad v \longrightarrow v_0, \quad u_x,\ v_x \longrightarrow \infty$$

Theorem 1. This local behaviour does not depend on the initial data. It is described by elliptic umbilic catastrophe.



Elliptic umbilic singularity (R.Thom)



(also $D_{4,\ -}$) function

$$F(X,Y) = \frac{1}{3}X^3 - XY^2$$

Level surface

$$F(X,Y) = 0$$

has singularity at $X = Y = 0$

Another description: singularity of the gradient map

$$(X, Y) \mapsto \operatorname{grad}\Phi(X, Y) = (U, V)$$

of a harmonic function $\quad \Delta\,\Phi = 0$

$$\det\begin{pmatrix} \partial U/\partial X & \partial U/\partial Y \\ \partial V/\partial X & \partial V/\partial Y \end{pmatrix} = \det\begin{pmatrix} \Phi_{XX} & \Phi_{XY} \\ \Phi_{XY} & \Phi_{YY} \end{pmatrix} = -\left[\Phi_{XX}^2 + \Phi_{XY}^2\right] = 0$$

$$\Leftrightarrow \quad d^2\Phi = 0$$

Miniversal unfolding

$$F(X,Y) = \frac{1}{3}\,X^3 - X\,Y^2 - a\,X - b\,Y + \frac{1}{2}\,c\,(X^2 + Y^2)$$

depending on the parameters $a,\ b,\ c$

Bifurcation diagram: look for the critical points (c=0)

$$\left.\begin{aligned} F_X &= X^2 - Y^2 - a + c\,X = 0 \\ F_Y &= -2\,X\,Y - b + c\,Y = 0 \end{aligned}\right\}$$

$\Rightarrow \quad X = X(a,b), \quad Y = Y(a,b)$ singularity at $a = b = 0$

Proof is based on

Lemma. Any analytic solution to

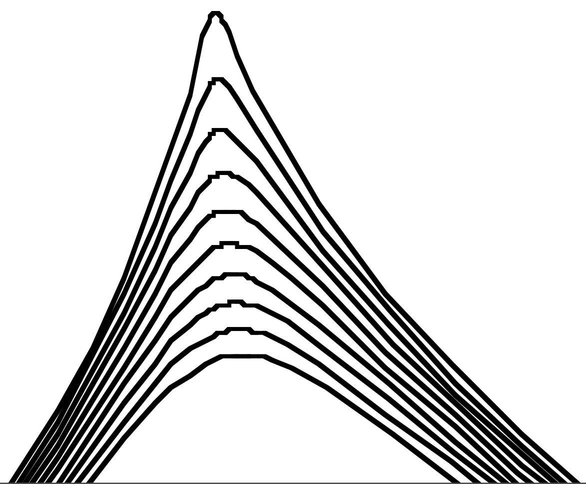
$$\left.\begin{aligned} u_t + (u\,v)_x &= 0 \\ v_t + v\,v_x - u_x &= 0 \end{aligned}\right\}$$

satisfying $(u_x)^2 + (v_x)^2 \neq 0$ can be obtained from the system

$$\left.\begin{aligned} x &= v\,t + f_u \\ 0 &= u\,t + f_v \end{aligned}\right\}$$

where $f = f(u, v)$ satisfies

$$f_{vv} + u\,f_{uu} = 0$$



Painlevé-I equation $\qquad w'' = 6\,w^2 - z$

Solutions $\quad w(z) \quad$ are

- meromorphic functions in $z \in \mathbf{C}$
- new transcendental functions

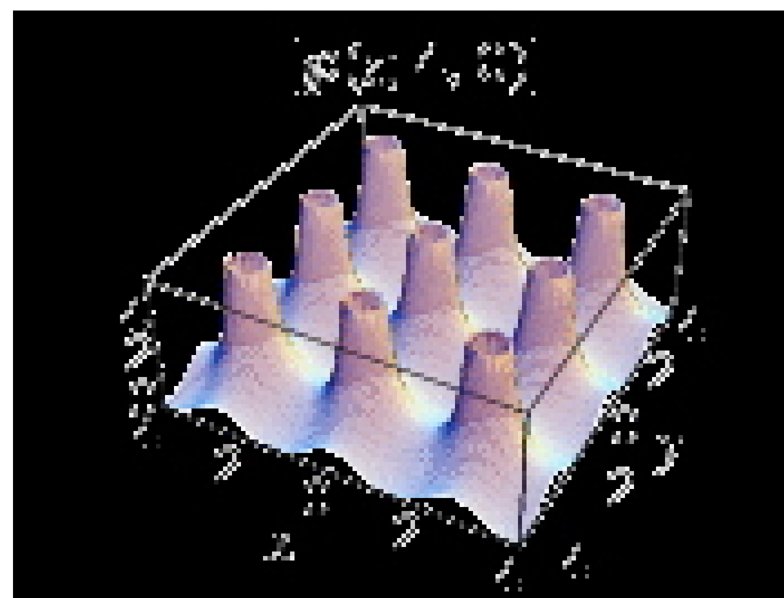
Cf. $$w'' = 6\,w^2 - z$$

and $$W'' = 6\,W^2 - \frac{g_2}{2}, \quad g_2 = \mathrm{const}$$

the Weierstrass elliptic function $W = \wp(Z; g_2, g_3)$

$$\wp(Z) = \frac{1}{Z^2} + \sum_{m^2+n^2\neq 0} \left[\frac{1}{(Z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \quad \Omega_{m,n} = 2\,m\,\omega_1 + 2\,n\,\omega_2$$

doubly periodic, lattice of poles



$$(\wp')^2 = 4\,\wp^3 - g_2\wp - g_3$$

Boutroux ansatz (1913): solutions to P-I

$$w(z) \simeq \sqrt{z}\,\wp\left(\frac{4}{5}\,z^{5/4}\right) \quad \text{for large } |z|$$

$$w = z^{1/2} W$$

Hint: do a substitution

$$Z = \tfrac{4}{5}\,z^{5/4}$$

to arrive at

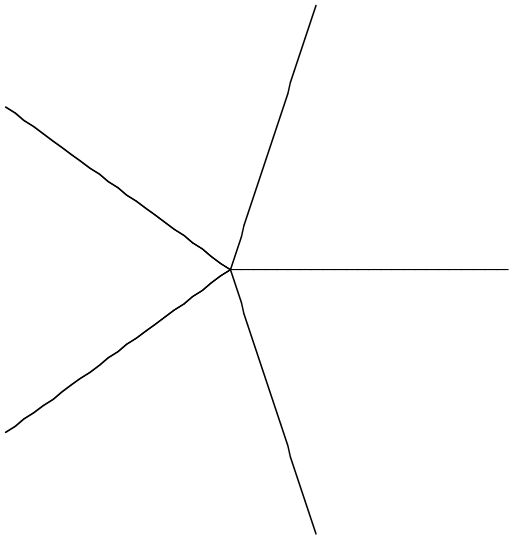
$$W'' = 6\,W^2 - 1 - \frac{1}{Z} W' + \frac{4}{25}\,\frac{W}{Z^2}$$

Theorem (Boutroux).

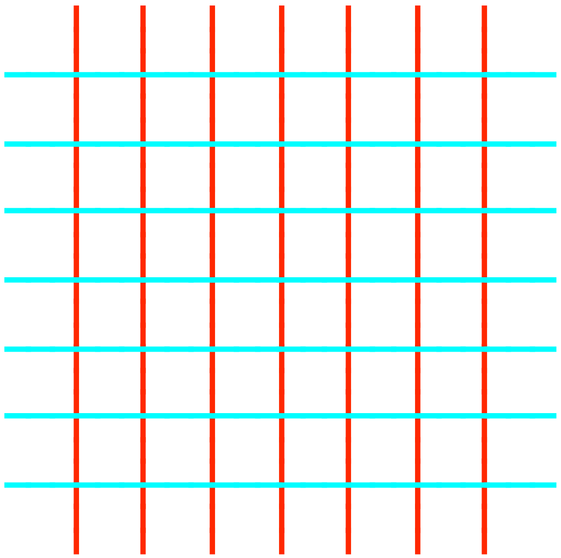
- Poles of a generic solution to P-I accumulate along the rays

$$\arg z = \frac{2\pi\, n}{5}, \quad n = 0,\, \pm 1,\, \pm 2$$

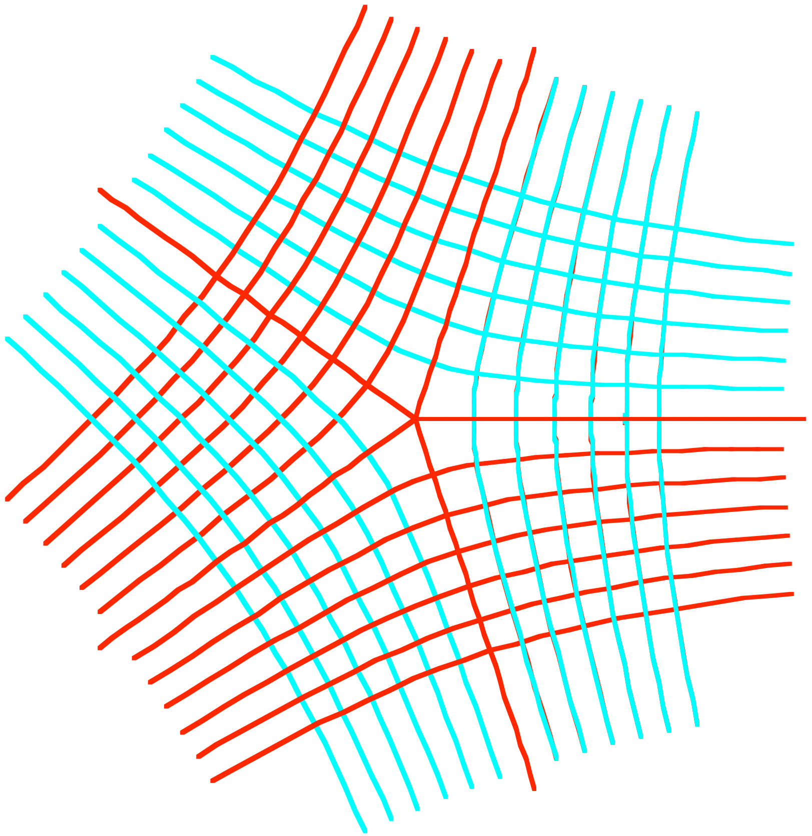
- For any three consecutive rays there exists a **unique *tritronquée*** solution $w_0(z)$ such that the lines of poles truncate along these three rays for large $|z|$



Z-plane

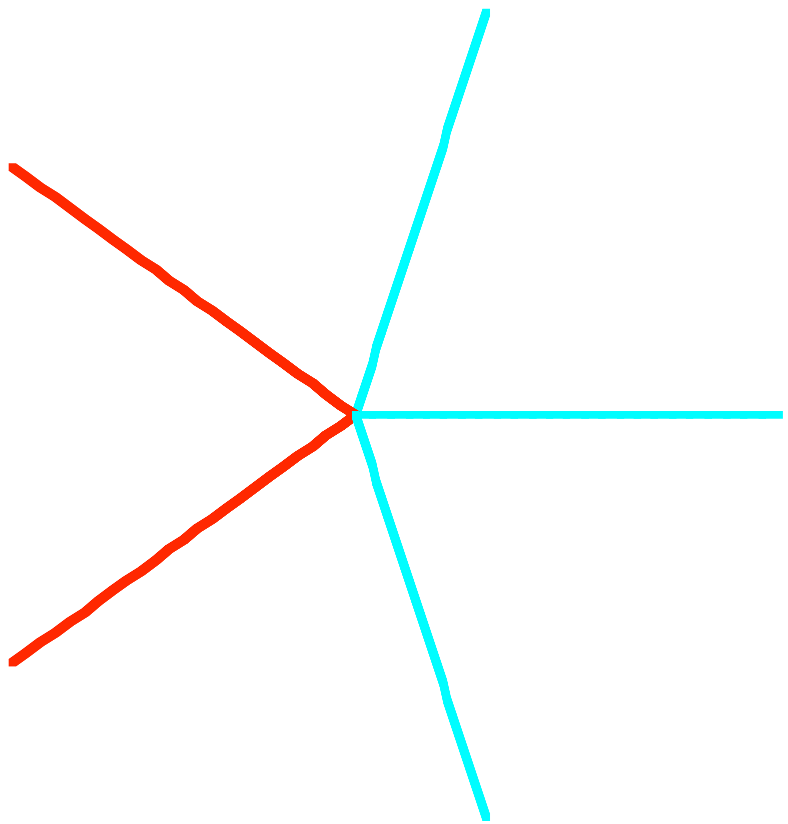


z-plane



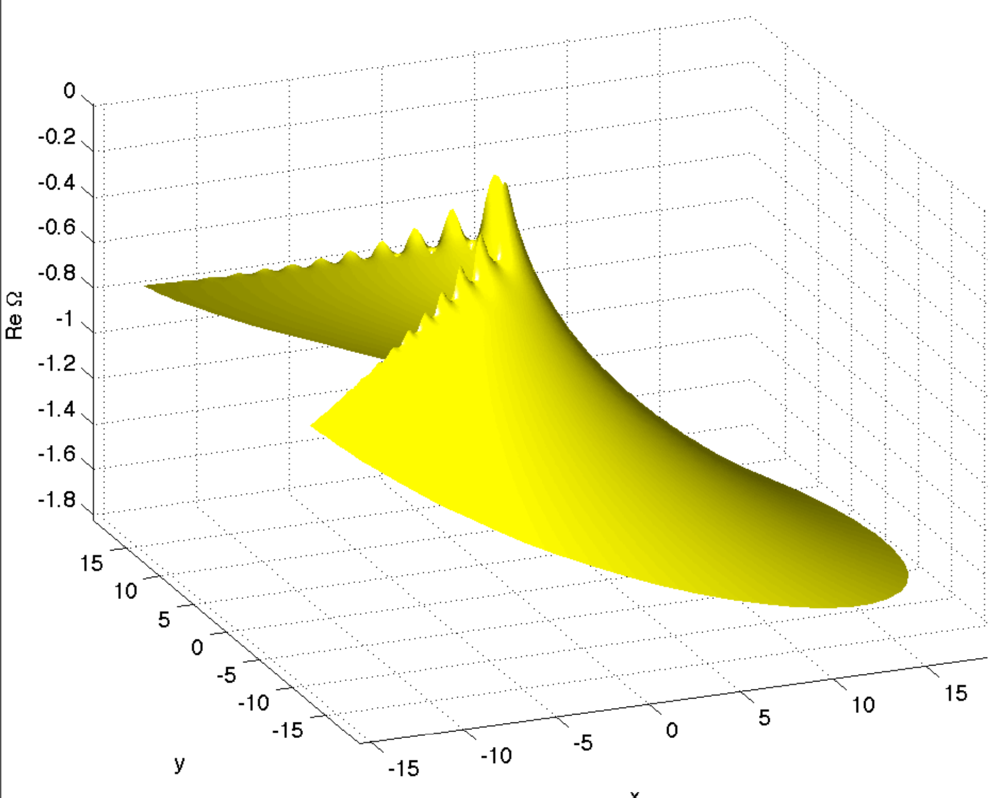
The tritronqué solution $\mathbf{w_0(z)}$ has no poles along green rays for large $|z|$, (cf. N.Joshi. A.Kitaev, 2001)

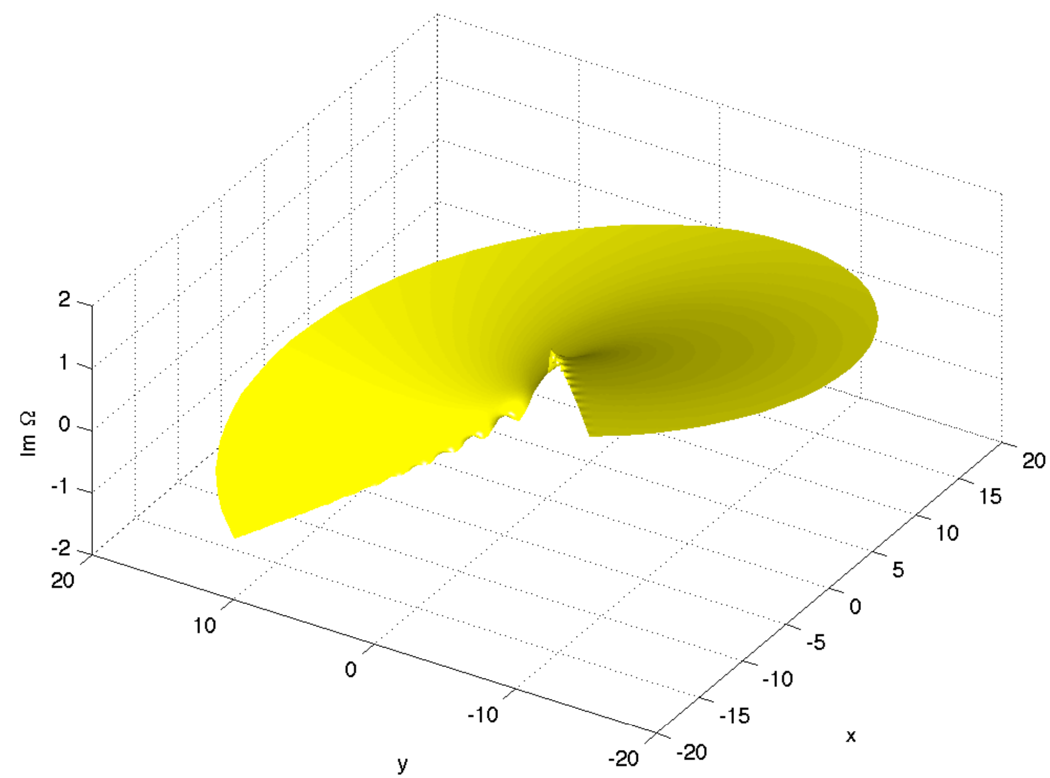
$$\mathbf{w_0(z)} \sim -\sqrt{\frac{z}{6}}, \quad |\arg z| < \frac{4\pi}{5}, \quad |z| >> 1$$



Conjecture 1: the tritronquée solution $\mathbf{w_0(z)}$

has no poles in the **entire sector** $|\arg z| < \frac{4\pi}{5}$





Main Conjecture:

local behaviour of a solution to the focusing NLS near the critical point (x_0, t_0, u_0, v_0): put

$$\bar{x} = x - x_0, \quad \bar{t} = t - t_0, \quad \bar{u} = u - u_0, \quad \bar{v} = v - v_0$$

$$\bar{u} + i\sqrt{u_0}\,\bar{v} \simeq -r\,e^{i\,\psi}\bar{t} + 2\epsilon^{2/5}\left(3r\sqrt{u_0}\right)^{2/5} e^{\frac{2i\psi}{5}}\mathbf{w_0}(\mathbf{z}) + O\left(\epsilon^{4/5}\right)$$

$$\mathbf{z} = \left(\frac{3r}{u_0^2}\right)^{1/5} e^{\frac{i\,\psi}{5}} \cdot \frac{i\sqrt{u_0}(\bar{x} - v_0\bar{t}) - u_0\bar{t} + \frac{1}{2}re^{i\psi}\bar{t}^2}{\epsilon^{2/5}}$$

where $\mathbf{w_0}(\mathbf{z})$ is the tritronquée solution to P-1,

$$f^0_{uuv} + i\sqrt{u_0}f^0_{uuu} =: r\,e^{-i\psi}$$

Main steps:

- any solution to $f_{vv} + u\, f_{uu} = 0$ gives a first integral

of
$$\left.\begin{array}{rcl} u_t + (u\, v)_x & = & 0 \\ v_t + v\, v_x - u_x & = & 0 \end{array}\right\}$$

$$H_f^0 = \int f(u,v)\, dx, \quad \partial_t H_f^0 = 0$$

• for any $f(u,v)$ there exists a unique extension to a first integral of NLS

$$\left.\begin{aligned} u_t + (u\,v)_x &= 0 \\ v_t + v\,v_x - u_x &= \frac{\epsilon^2}{4}\left(\frac{u_{xx}}{u} - \frac{1}{2}\,\frac{u_x^2}{u^2}\right) \end{aligned}\right\}$$

$$H_f^\epsilon = \int h_f\,dx, \quad \partial_t H_f^\epsilon = 0$$

$$\begin{aligned}
h_f = f - \frac{\epsilon^2}{12}\left[\left(f_{uuu} + \frac{3}{2u}f_{uu}\right)u_x^2 + 2f_{uuv}u_xv_x - uf_{uuu}v_x^2\right] \\
+\epsilon^4\left\{\frac{1}{120}\left[\left(f_{uuuu} + \frac{5}{2u}f_{uuu}\right)u_{xx}^2 + 2f_{uuuv}u_{xx}v_{xx} - uf_{uuuu}v_{xx}^2\right]\right. \\
-\frac{1}{80}f_{uuuu}u_{xx}v_x^2 - \frac{1}{48u}f_{uuuv}v_{xx}u_x^2 - \frac{1}{3456u^3}\left(30f_{uuu} - 9uf_{uuuu} + 12u^2f_{5u} + 4u^3f_{6u}\right)u_x^4 \\
-\frac{1}{432u^2}\left(-3f_{uuuv} + 6uf_{uuuuv} + 2u^2f_{5u\,v}\right)u_x^3v_x + \frac{1}{288u}\left(9f_{uuuu} + 9uf_{5u} + 2u^2f_{6u}\right)u_x^2v_x^2 \\
\left.+\frac{1}{2160}\left(9f_{uuuuv} + 10uf_{5u\,v}\right)u_xv_x^3 - \frac{u}{4320}\left(18f_{5u} + 5uf_{6u}\right)v_x^4\right\} + O(\epsilon^6)
\end{aligned}$$

- for any $f(u,v)$ satisfying $$f_{vv} + u\, f_{uu} = 0$$

the system

$$\left. \begin{aligned} x &= v\,t + \frac{\delta H_f^\epsilon}{\delta u(x)} \\ 0 &= u\,t + \frac{\delta H_f^\epsilon}{\delta v(x)} \end{aligned} \right\}$$

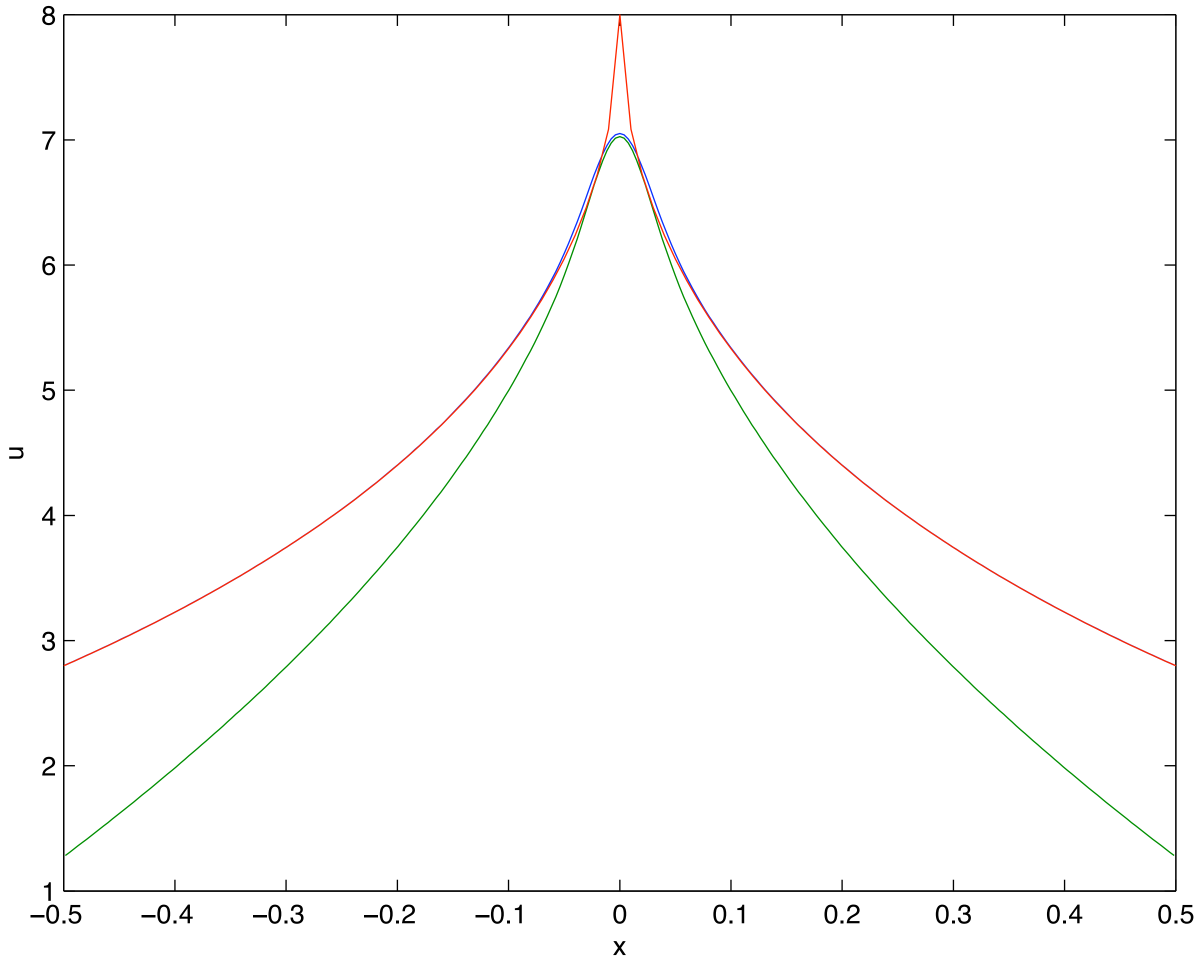
determines a solution to NLS. For small t

it is close to the solution

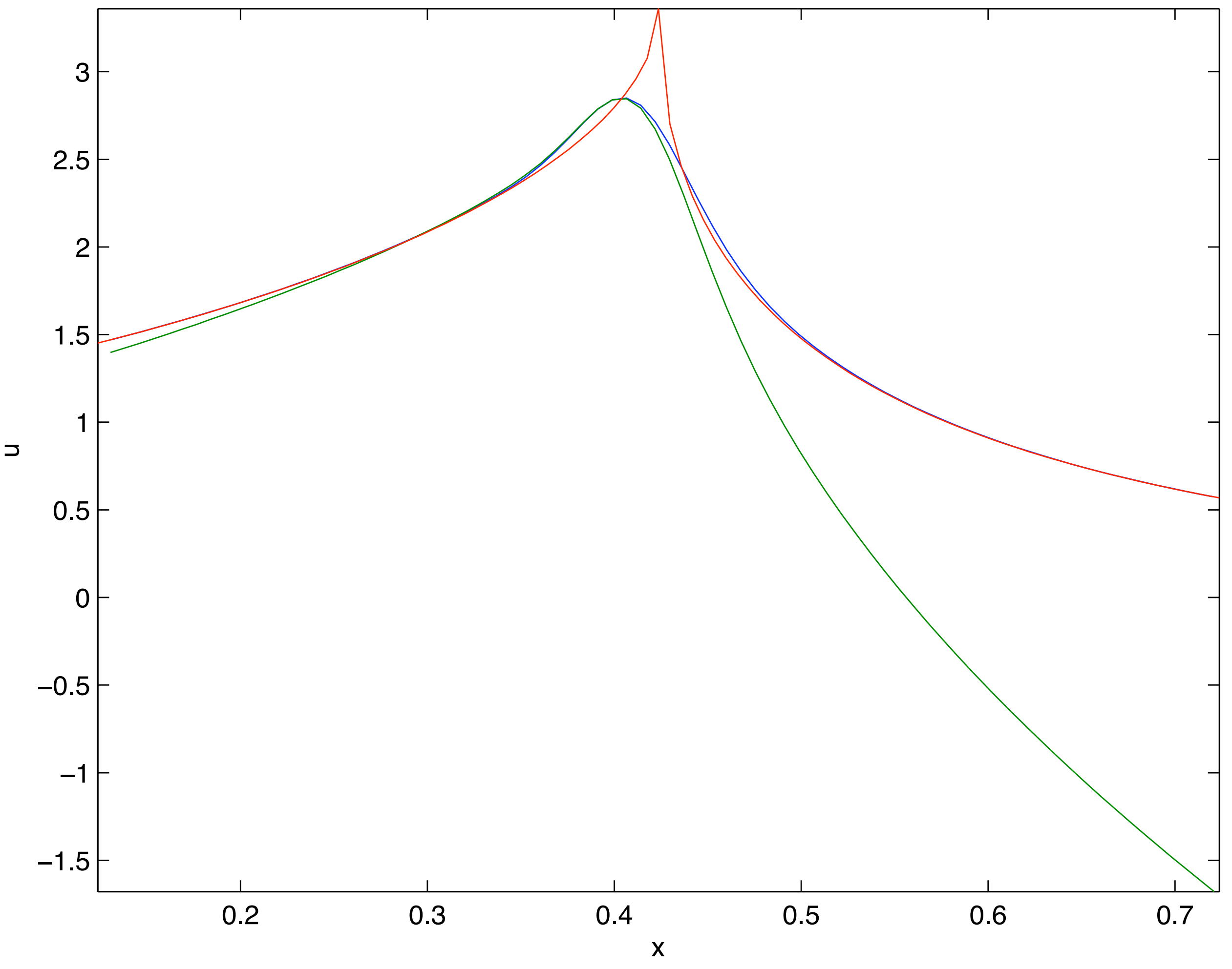
$$\left. \begin{aligned} x &= v\,t + f_u \\ 0 &= u\,t + f_v \end{aligned} \right\}$$

of the dispersionless NLS

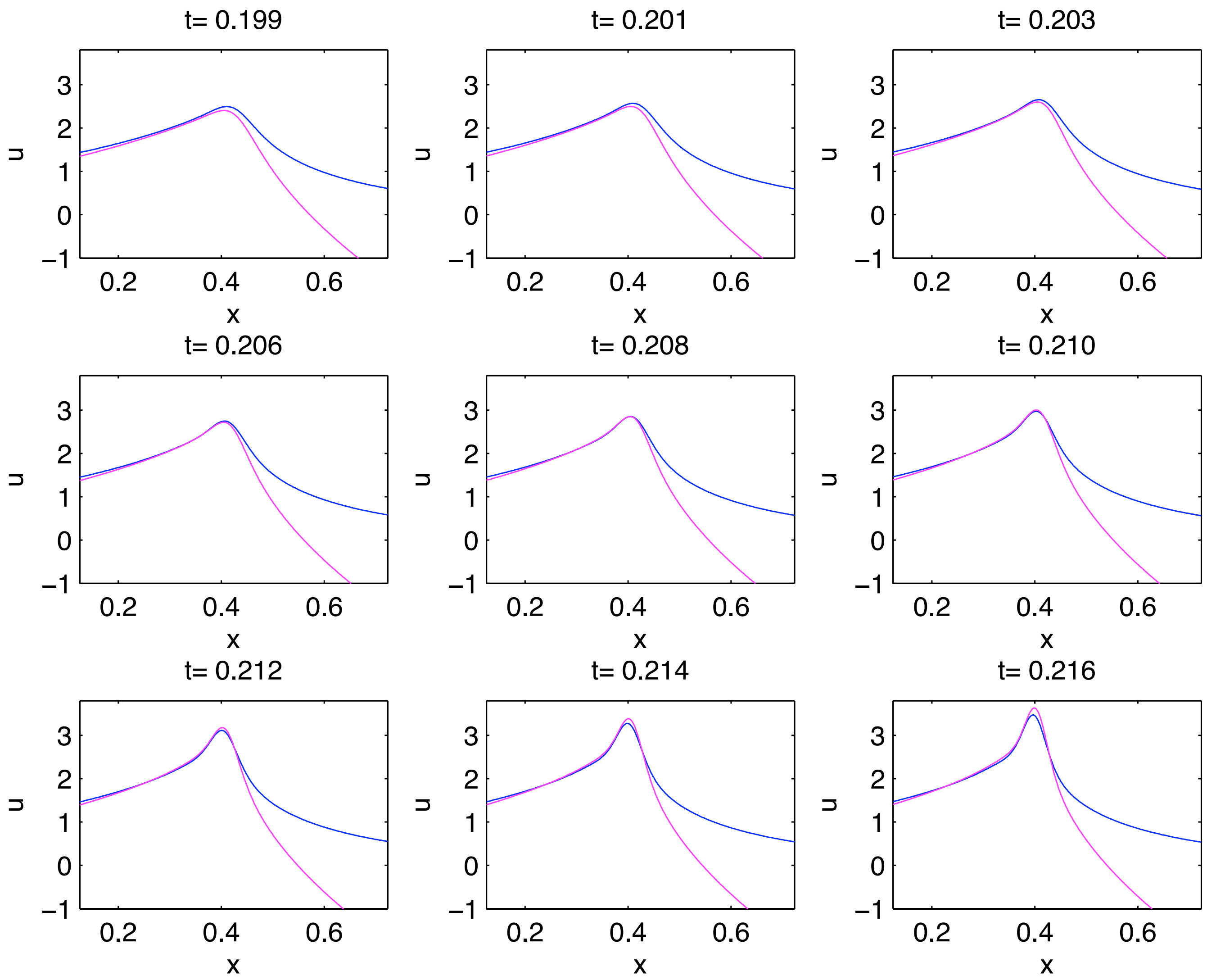
Comparison between NLS, dispersionless NLS, and the tritronquée solution to P-I



Same for nonsymmetric initial data



NLS versus tritronquée; the critical time is t=0.208 (in the center)



Details in

B.Dubrovin, S.-Q.Liu, Y.Zhang, Comm. Pure Appl. Math. **59** (2006) 559-615

B.Dubrovin, Comm. Math. Phys. **267** (2006) 117-139

T.Grava, C.Klein, arXiv:math-ph/0702038

B.Dubrovin, T.Grava, C.Klein, arXiv:0704.0501

Further programme:

- distribution of poles of the tritronquée solution inside the sector

$$|\arg z| > \frac{4\pi}{5}$$

- asymptotics of NLS inside the oscillatory zone

- matching of asymptotics

- generalization to Hamiltonian perturbations of general first order quasilinear systems of elliptic type

Thank you!