Aspects of finite-gap integration of the $SU(2)$ Bogomolny equations

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Yang-Mills-Higgs Lagrangian in Minkoswki space

$$-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}D_\mu\Phi D_\mu\Phi$$

$$F_{ij}=\partial_i a_j-\partial_j a_i+[a_i,a_j],\qquad D_i\Phi=\partial_i\Phi+[a_i,\Phi].$$

F, curvature of a connections $a_i(t,\boldsymbol{x})$ and Higgs field $\Phi(t,\boldsymbol{x})$ are from $SU(2)$, D_i -covariant derivative, $\boldsymbol{x}\in\mathbb{R}^3$

Static configuration, $\partial_t a_i(t,\boldsymbol{x})=\partial_t\Phi(t,\boldsymbol{x})=0$, minimizing the energy of the system,

$$D_i\Phi=\pm\sum_{j,k=1}^{3}\epsilon_{ijk}F_{jk}$$

Bogomolny equation together with boundary conditions

$$|\Phi(r)|_{r\to\infty}\sim 1-\frac{n}{2r}+O(r^{-2}),\ r=\sqrt{x_1^2+x_2^2+x_3^2}$$

That is **monopole of the charge** n

We develop **Atiyah-Drinfeld-Manin-Hitchin-Nahm** construction

Step 1. Solve **Nahm equations**

$$\frac{\mathrm{d}T_i(s)}{\mathrm{d}s} = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j(s), T_k(s)], \quad s \in [0,2]$$

$T_i(s)$: regular $s \in (0,2)$, simple pole $s = 0, 2$, $\mathrm{Res}_{s=0}T_i(s)$ irreducible n-dim. representation of $su(2)$ also $T_i(s) = -T_i^\dagger(s)$, $T_i(s) = T_i^\dagger(2-s)$.

Step2. Solve **Weyl equation**, $s \in (0,2)$

$$\left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}s} + \sum_{j=1}^{3}(T_j(s) + \imath x_j)\otimes\sigma_j\right)\boldsymbol{\Upsilon}(\boldsymbol{x}, s) = 0.$$

Here σ_j - Pauli matrices

Step 3. Choose orthonormal basis

$$\int_0^2 \boldsymbol{\Upsilon}_\mu^\dagger(\boldsymbol{x}, s)\boldsymbol{\Upsilon}_\nu(\boldsymbol{x}, s)\mathrm{d}s = \delta_{\mu\nu}$$

$$\Phi(\boldsymbol{x})_{\mu\nu} = \imath\int_0^2 s\boldsymbol{\Upsilon}_\mu^\dagger(\boldsymbol{x}, s)\boldsymbol{\Upsilon}_\nu(\boldsymbol{x}, s)\mathrm{d}s,$$

$$a_i(\boldsymbol{x})_{\mu\nu} = \imath\int_0^2 \boldsymbol{\Upsilon}_\mu^\dagger(\boldsymbol{x}, s)\frac{\partial}{\partial x_i}\boldsymbol{\Upsilon}_\nu(\boldsymbol{x}, s)\mathrm{d}s, \quad i = 1,2,3$$

[Panagopoulos,1983] computed antiderivatives

Hitchin construction (1982,1983) Lax representation

$$\frac{\mathrm{d}A}{\mathrm{d}s} = [A, M]$$

$$A = A_{-1}\zeta^{-1} + A_0 + A_1\zeta, \quad M = \frac{1}{2}A_0 + \zeta A_1$$

$$A_{\pm 1} = T_1 \pm \imath T_2, \quad A_0 = 2\imath T_3$$

The curve $\mathcal{C}$ of genus $g_{\mathcal{C}} = (n-1)^2$

$$\mathcal{C} = (\eta, \zeta) \quad : \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta) = 0$$

is subjected to **Hitchin constraints**

Let L^λ – holomorphic line bundle on $T\mathbb{P}^1$ defined by transition function $g_{0,\infty} = \exp(-\lambda\eta/\zeta)$ on $U_0 \cap U_\infty$ and similarly defined $L^\lambda(m) = L^\lambda \otimes \mathcal{O}(m)$,$g_{0,\infty} = \zeta^m \exp(-\lambda\eta/\zeta)$

H1. $\mathcal{C}$ admits the involution $(\zeta, \eta) \to \left(-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2\right)$

H2. L^2 is trivial on $\mathcal{C}$ and $L(n-1)$ is real

H3. $H^0(\mathcal{C}, L^\lambda(n-2)) = 0$ for $\lambda \in (0, 2)$

H2.[Ercolani & Sinha, 1989, Braden & E, 2006]

$$\gamma_\infty(P)_{P\to\infty_i} = \left(\frac{\rho_i}{\xi^2} + O(1)\right)\mathrm{d}\xi, \quad \oint_{\mathfrak{a}_k} \gamma_\infty = 0$$

$$\boldsymbol{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_n}\gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau\boldsymbol{m},$$

$\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$- **Ercolani-Sinha vectors**

[Braden & E,2006]: For the genus 4 curve

$$\mathcal{C} = (\zeta, \eta): \quad \eta^3 = \chi \prod_{k=1}^{6} (\zeta - \lambda_k)$$

H2 means that the $\mathcal{C}$ covers N-sheetedly elliptic curve $\mathcal{E}$, with modulus τ_1, $\pi : \mathcal{C} \to \mathcal{E}$, and such homology basis exists that

$$\tau \sim \begin{pmatrix} \tau_1 & 1/N & 0 & 0 \\ 1/N & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}, \quad \boldsymbol{U} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and N (**Hopf number**) is given in terms of Ercolani-Sinha vectors $\boldsymbol{n}, \boldsymbol{m}$.

Resemblance to the condition for KdV-solution to be an elliptic soliton [Belokolos & E, 1994].

Equivalently **H2** means that the cover π is **tangential** in the sense of [Treibich & Verdier, 1990]

Example: Tetrahedral monopole

$$\eta^3 + \chi(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0$$

[Hitchin & Manton & Murray, 1995] This curve admits tetrahedral symmetry. τ-matrix is reduced to the form [Braden & E,2006]

$$\tau \sim \begin{pmatrix} \frac{1}{4}\rho & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{5\rho}{4} & \rho & 0 \\ 0 & \rho & 2\rho & \rho \\ 0 & 0 & \rho & \frac{2}{7} + \frac{6}{7}\rho \end{pmatrix}, \; \rho^3 = 1, \quad \boldsymbol{U} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Conjecture: H2 $\equiv \pi : \mathcal{C} \to \mathcal{E}$ **is tangential**

Conjecture: H2$\equiv$ τ-matrix is expressible in terms of $\boldsymbol{n}, \boldsymbol{m}$ and $\mathrm{Res}_{P\to\infty_i}\eta/\zeta$ only

To explain **H3** we shall first develop integration of the Nahm equations

Hitchin Lax representation leads to the non-standard spectral problem

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{2}A_0(z)\right)\boldsymbol{\psi} = -\zeta A_1(z)\boldsymbol{\psi}$$

$$A_1(z) = C(z)A_1(0)C^{-1}(z) \quad \text{iff} \quad A_1' = [\underbrace{C'C^{-1}}_{A_0/2}, A_1]$$

[Ercolani & Sinha, 1989] $\boldsymbol{\psi} = C(z)\boldsymbol{\phi}$ leads to the **standard matrix spectral problem**

$$\boldsymbol{\phi}' + Q_0\boldsymbol{\phi} = -\zeta A_1(0)\boldsymbol{\phi}$$

with

$$Q_0 = C^{-1}A_0C$$
$$A_1(0) = \mathrm{Diag}(\rho_1, \ldots, \rho_n),$$
$$\rho_i = \mathrm{Res}_{P=\infty_i}\frac{\eta}{\zeta}$$

If Q_0 is known then C should be found from

$$C' = \frac{1}{2}CQ_0$$

[Braden & E, 2006] Components of the eigenfunction $\boldsymbol{\phi}$, i.e. sections of $L^{z+1}(n-1)$ and the matrix $Q_0(z)$ are

$$\phi_j(z,P) = g_j(P)\exp\left\{\int_{P_0}^{P}\gamma_\infty - z\nu_j\right\} \times \frac{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\int_{\infty_j}^{P}\boldsymbol{\omega} + z\boldsymbol{U} - \boldsymbol{K}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}(-\boldsymbol{K})}{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\int_{\infty_j}^{P}\boldsymbol{\omega} - \boldsymbol{K}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}(z\boldsymbol{U} - \boldsymbol{K})}$$

Here $g_j(P)$ form a basis of the holomorphic sections $L(n-1)$

$$Q_0(z)_{j,l} = \pm\frac{\rho_j - \rho_l}{E(\infty_j,\infty_l)}\exp\left\{\imath\pi\boldsymbol{q}\cdot\int_{\infty_j}^{\infty_l}\boldsymbol{\omega}\right\} \times \frac{\theta\left(\int_{\infty_j}^{\infty_l}\boldsymbol{\omega} + (z+1)\boldsymbol{U} - \boldsymbol{K}\right)}{\theta((z+1)\boldsymbol{U} - \boldsymbol{K})}\mathrm{e}^{z(\nu_l - \nu_j)}$$

Here $E(P,Q)$-prime-form.

Resemblance to Euler top equation

$$\dot{M} = [\Omega, M]$$
$$M = I\Omega + \Omega I, \quad I = \mathrm{Diag}(I_1, \ldots, I_N)$$

$$n = 2, \quad \text{Nahm eqns.}: \quad \frac{\mathrm{d}f_1}{\mathrm{d}s} = f_2 f_3, \text{ etc.}$$
$$N = 3, \quad \text{Euler eqns.}: \quad \frac{\mathrm{d}\Omega_1}{\mathrm{d}t} = \frac{I_2 - I_3}{I_1}\Omega_2\Omega_3, \text{ etc.}$$

In [Dubrovin, 1977] Manakov's Lax representation was used to integrate the problem

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - [I, V] + \zeta I, \ \zeta I^2 - [I^2, V]\right] = 0, \quad [I, V] = \Omega$$

Standard matrix spectral problem

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} - \Omega\right\}\psi = -\zeta I \psi$$

Associated algebraic curve,

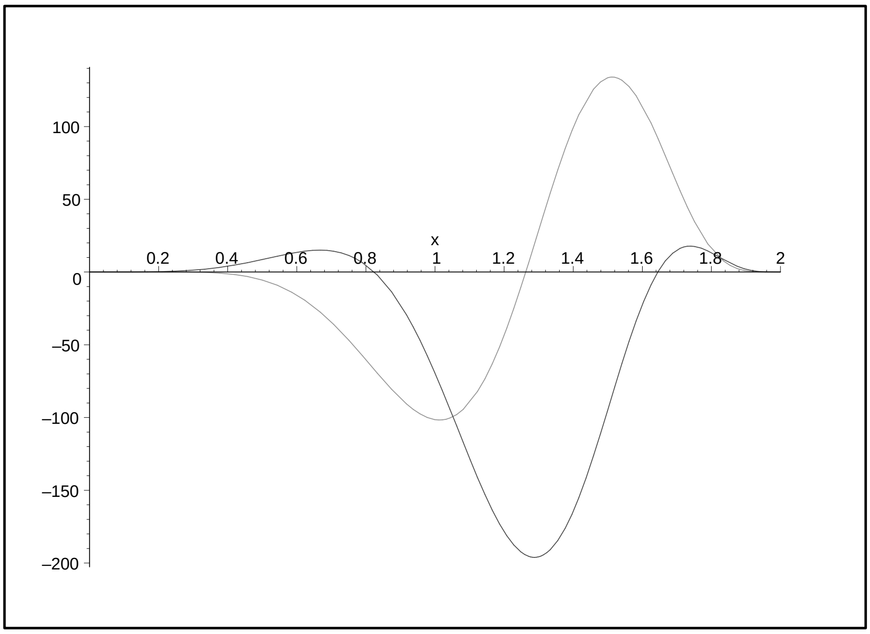
$$\det(\zeta I^2 - [I^2, V] + \eta 1_N) = 0$$

is of genus $(N-1)(N-2)/2$

H3. There is no sections of $L^{z+1}(n-2)$ at $-1 < z < 1$, i.e. there is no poles of $\phi_j(P,z)$ at $-1 < z < 1$.

$\boldsymbol{U}s - \boldsymbol{K}$ does not intersect θ-divisor

$$\theta(\boldsymbol{U}s - \boldsymbol{K};\tau) \neq 0 \qquad \text{at} \quad s \in (0,2)$$



Plot of the real and imaginary parts of the function $\theta(\boldsymbol{U}x - \boldsymbol{K};\tau)$ in the case of tetrahedral monopole curve

Nahm ansatz [Nahm, 1982] Adjoint Weyl eq.:

$$\left(-\imath 1_{2n}\frac{\mathrm{d}}{\mathrm{d}s} - \sum_{j=1}^{3}(T_j(s)+\imath x_j)\otimes\sigma_j\right)\Psi(\boldsymbol{x},s)=0.$$

$\Psi = (\boldsymbol{\Psi}_1,\ldots,\boldsymbol{\Psi}_n)$ and $\Upsilon = (\boldsymbol{\Upsilon}_1,\ldots,\boldsymbol{\Upsilon}_n)$

$$\Upsilon = (\Psi^\dagger)^{-1}$$

Introduce two vectors, $\boldsymbol{u}\in\mathbb{R}^3$, $|\boldsymbol{u}|=1$ and $\boldsymbol{v}\in\mathbb{C}^3$

$$\boldsymbol{u}=\frac{1}{|\zeta|^2+1}(\imath(\zeta-\overline{\zeta}),\ -(\zeta+\overline{\zeta}),\ |\zeta|^2-1),$$

$$\boldsymbol{u}\times\boldsymbol{v}=-\imath\boldsymbol{v}$$

Nahm's Ansatz

$$\boldsymbol{\Psi}=\left(1_2+\sum_{m=1}^{3}\sigma_m u_m\right)\otimes\boldsymbol{\psi},\qquad |\boldsymbol{u}|=1$$

leads to the non-standard spectral problem

$$\left[\imath 1_n\frac{\mathrm{d}}{\mathrm{d}s}+(\boldsymbol{T}(s)+\imath\boldsymbol{x}1_n)\cdot\boldsymbol{u}\right]\boldsymbol{\psi}=0,\qquad s\in(0,2)$$

subjected to the **Atiyah-Ward constraint**

$$\det(\boldsymbol{v}\cdot(\boldsymbol{T}(s)+\imath\boldsymbol{x}1_n))=0$$

Set $\eta = \imath \boldsymbol{v} \cdot \boldsymbol{x}$, then obtain Hitchin curve $\mathcal{C} = (\eta, \zeta)$, i.e. Atiyah-Ward constraint means that values of ζ are roots $\zeta_k, k = 1, \ldots, n$ of $2n$-polynomial

$$(\imath \boldsymbol{v} \cdot \boldsymbol{x})^n + a_1(\zeta)(\imath \boldsymbol{v} \cdot \boldsymbol{x})^{n-1} + \ldots + a_n(\zeta) = 0$$

Non-standard spectral problem by Nahm can be reduced to the standard spectral problem by the same trick as above,

$$\left(\frac{\mathrm{d}}{\mathrm{d}s} + A_{-1}(s)\right)\boldsymbol{\psi}(s,\zeta) = \zeta^2 A_1(s)\boldsymbol{\psi}(s,\zeta)$$

Conclusion: To calculate gauges $a_i(\boldsymbol{x})$ and Higgs field $\Phi(\boldsymbol{x})$ we need to know only values of the Baker-Akhiezer function

$$\boldsymbol{\psi}(s,\zeta)|_{s=0,2,\zeta=\zeta_k}$$

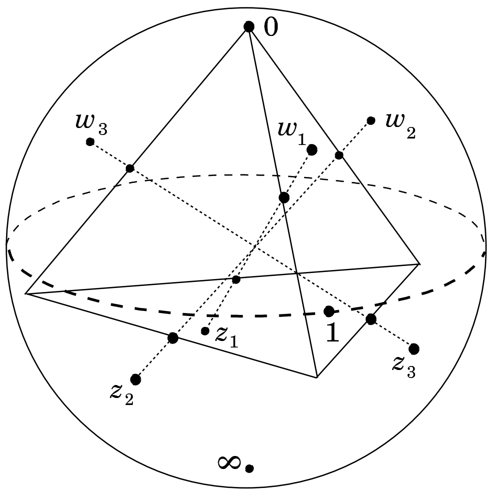
at boundaries of the segment [0,2] and spectral parameter fixed at the appropriate solutions of the Atiayh-Ward constrain.

Charge three monopole General 3-monopole curve be

$$\eta^3 + \eta a_2(\zeta) + a_6(\zeta) = 0$$

Consider particular case

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b \in \mathbb{R}$$



At $b = 5\sqrt{2} \Rightarrow$ Platonic solid: tetrahedron

[Braden & E]: **H1** & **H2** are satisfied iff

$$\frac{F\left(\frac{1}{3},\frac{2}{3};1;t\right)}{F\left(\frac{1}{3},\frac{2}{3};1;1-t\right)} = f(t) \in \mathbb{Q}, \quad t = \frac{-b+\sqrt{b^2+4}}{2\sqrt{b^2+4}}$$

In particular,

$$f(t_0) = 2 \Longrightarrow t_0 = \frac{1}{2} - \frac{5\sqrt{3}}{18},$$
$$b = 5\sqrt{2} \Longrightarrow \text{Tetrahedron}$$

[Ramanujan, 1914] Second Notebook: Let r (signature) and $n \in \mathbb{N}$

$$\frac{F\left(\frac{1}{r},\frac{r-1}{r};1;1-x\right)}{F\left(\frac{1}{r},\frac{r-1}{r};1;x\right)} = n\frac{F\left(\frac{1}{r},\frac{r-1}{r};1;1-y\right)}{F\left(\frac{1}{r},\frac{r-1}{r};1;y\right)}.$$

Then $\mathcal{P}(x,y)=0$ is algebraic equation, find it!

Ramanujan theory for signature 3, $r=3$, $n=2$

$$(xy)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}}(1-y)^{\frac{1}{3}} = 1$$

Set $y=\frac{1}{2}$, $n=2$ to obtain $b=5\sqrt{2}$.

Other signatures: [Berndt & Bhargava & Garvan, 1995]

Conjecture: Tetrahedral curve is the only monopole curve in the given class of curves

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b \in \mathbb{R}$$

Details: Braden and Enolski, arXiv: math-ph/060104, math-ph/0704.3939