Trigonometric Solitons and an Adelic Flag Manifold

Luc Haine

University of Louvain

ISLAND 3 - Islay, July 6 2007

1. THE ADELIC GRASSMANNIAN

Classify all rank 1 commutative algebras $\mathcal{A}$ of differential operators, for which the joint eigenfunction $\psi(x,z)$ which satisfies

$$A(x,\partial/\partial x)\psi(x,z) = f_A(z)\psi(x,z),\ \forall A \in \mathcal{A},$$

also satisfies a differential equation in the spectral variable

$$B(z,\partial/\partial z)\psi(x,z) = g(x)\psi(x,z).$$

The solutions (**G. Wilson, 1993**) are parametrized by the so-called adelic Grassmannian

$$Gr^{ad} = \{W \in Gr^{rat} : \text{spec}(A_W) = \text{spec}(\mathcal{A}) \text{ is unicursal}\},$$
$$A_W = \{p(z) \in \mathbb{C}[z] :\ p(z).W \subset W\}.$$

There exists a bispectral involution $b : Gr^{ad} \to Gr^{ad}$:

$$\psi_{b(W)}(x,z) = \psi_W(z,x), \quad W \in Gr^{ad}.$$

In 1998, **G. Wilson** showed that Gr^{ad} can also be described as the union $\cup_{N\geq 0} C_N$ of the Calogero-Moser spaces

$$C_N = \Big\{(X,Z) \in gl(N,\mathbb{C}) \times gl(N,\mathbb{C}) : \\ \text{rank}([X,Z]+I) = 1\Big\}/GL(N,\mathbb{C}).$$

The correspondence can be seen as given by the map

$$\beta : (X,Z) \to \psi_W(x,z) = e^{xz} \det\{I - (xI - X)^{-1}(zI - Z)^{-1}\}.$$

The bispectral involution becomes transparent, as it is given by

$$b^C(X,Z) = (Z^t, X^t),$$

i.e. we have a commutative diagram

$$\begin{array}{ccc} \cup_{N\geq 0} C_N & \xrightarrow{\beta} & Gr^{ad} \\ \downarrow{\scriptstyle b^C} & & \downarrow{\scriptstyle b} \\ \cup_{N\geq 0} C_N & \xrightarrow{\beta} & Gr^{ad} \end{array}$$

2. THE ADELIC FLAG MANIFOLD (H-Iliev IMRN 2000, No. 6)

Theorem 1 *If* $\tau(t_1, t_2, t_3, \ldots)$ *is a tau function of the KP-hierarchy, then*

$$\tau(n, t_1, t_2, t_3, \ldots) = \tau\Big(t_1 + n, t_2 - \frac{n}{2}, t_3 + \frac{n}{3}, \ldots\Big),$$

is a tau function of the discrete KP-hierarchy

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L],$$

$$L = \Delta + \sum_{i=0}^{\infty} a_i(n)\Delta^{-j}, \quad \Delta f(n) = f(n+1) - f(n).$$

$$\Big(\Rightarrow \frac{\partial}{\partial t_2}\Delta a_0 = \frac{\partial}{\partial t_1}(\Delta a_0^2 - 2\Delta a_0) + \frac{\partial^2}{\partial t_1^2}(\Delta a_0 + 2a_0)\Big)$$

Starting from any $W \in Gr^{ad}$ and its tau function $\tau_W(t_1, t_2, \ldots)$, the corresponding wave function of the discrete KP hierarchy is

$$\psi(n,t,z) = (1+z)^n \exp\Big(\sum_{k=1}^{\infty} t_k z^k\Big) \times \frac{\tau_W\Big(t_1 + n - \frac{1}{z}, t_2 - \frac{n}{2} - \frac{1}{2z^2}, t_3 + \frac{n}{3} - \frac{1}{3z^3}, \ldots\Big)}{\tau_W\Big(t_1 + n, t_2 - \frac{n}{2}, t_3 + \frac{n}{3}, \ldots\Big)}.$$

Define a flag of subspaces of $L^2(S^1, \mathbb{C})$

$$\mathcal{V} : \ldots \subset V_{n+1} \subset V_n \subset V_{n-1} \subset \ldots,$$

with V_n the closure in $L^2(S^1, \mathbb{C})$ of the space

$$V_n^{alg} = \text{span of } \{\psi(n,0,z), \psi(n+1,0,z), \psi(n+2,0,z), \ldots\}.$$

We call the set of these flags the adelic flag manifold Fl^{ad}.

Let $\psi_{\mathcal{V}}(n,z) \equiv \psi(n,0,z)$ and define

$$A_{\mathcal{V}} = \Big\{ \text{rational functions } f(z) \text{ with poles only at } z=-1 \text{ and } z=\infty, \text{ such that } \exists\, k \in \mathbb{Z} \text{ for which } f(z).V_n \subset V_{n+k},\ \forall\, n \Big\}.$$

Theorem 2 *Any $\mathcal{V} \in Fl^{ad}$ gives rise to a rank one commutative algebra $\mathcal{A}$ of difference operators $A \in \mathcal{A}$, isomorphic to $A_{\mathcal{V}}$,*

$$A\psi_{\mathcal{V}}(n,z) \equiv \sum_{\text{finitely many } j \in \mathbb{Z}} a_j(n)\psi_{\mathcal{V}}(n+j,z) = f_A(z)\psi_{\mathcal{V}}(n,z),$$

for which there is a differential operator $B(z,\partial/\partial z)$ such that

$$B(z,\partial/\partial z)\psi_{\mathcal{V}}(n,z) = g(n)\psi_{\mathcal{V}}(n,z).$$

Proof: Define $\psi_{b(\mathcal{V})}(x,z) = \psi_{\mathcal{V}}(z, e^x - 1)$. Then, $\psi_{b(\mathcal{V})}(x,z)$ is the wave function of a susbspace $b(\mathcal{V}) \in Gr^{rat}$. ■

Example: $\lambda, \alpha \in \mathbb{C}, |\lambda| < 1$.

$$W = \frac{1}{z-\lambda}\Big\{g \in \mathbb{C}[z] : \ g'(\lambda) = \alpha g(\lambda)\Big\} \in Gr^{ad}.$$

$$u = (z-\lambda)^2, \ v = (z-\lambda)^3 \in A_W \Rightarrow \mathrm{spec}(A_W) : v^2 = u^3.$$

$$\tau_W(t_1, t_2, \ldots) = \sum_{i=1}^{\infty} i t_i \lambda^{i-1} - \alpha$$

$$\Downarrow$$

$$\tau(n; t_1, t_2, \ldots) =$$

$$\sum_{i=1}^{\infty} i\left(t_i + \frac{(-1)^{i-1} n}{i}\right)\lambda^{i-1} - \alpha = \sum_{i=1}^{\infty} i t_i \lambda^{i-1} - \alpha + n(1+\lambda)^{-1}.$$

$$\psi_{\mathcal{V}}(n,z) = \left(1 - \frac{1+\lambda}{(z-\lambda)(n-\beta(1+\lambda))}\right)(1+z)^n, \quad \mathcal{V} \in Fl^{ad},$$

$$\beta = \alpha - \sum_{i=1}^{\infty} i t_i \lambda^{i-1}.$$

$$u = (z-\lambda)^2, \; r = \frac{(z-\lambda)^2}{z+1} \in A_{\mathcal{V}} \Rightarrow$$

$$\mathrm{spec}(A_{\mathcal{V}}) : y^2 = r^3\Big(r + 4(1+\lambda)\Big), \; y = u - r\Big(r + 2(1+\lambda)\Big).$$

$$\psi_{b(\mathcal{V})}(x,z) \equiv \psi_{\mathcal{V}}(z, e^x - 1) = \left(1 - \frac{\delta}{(e^x - \delta)(z - \tilde{\lambda})}\right)e^{xz},$$

$$\delta = 1 + \lambda, \; \tilde{\lambda} = \beta(1+\lambda),$$

$$b(\mathcal{V}) = \frac{1}{z - \tilde{\lambda}}\Big\{g \in \mathbb{C}[z] : g(\tilde{\lambda}) = \delta g(\tilde{\lambda} - 1) = 0\Big\} \in Gr^{trig} \subset Gr^{rat}.$$

$$\mathrm{spec}\Big(A_{b(\mathcal{V})}\Big) : g = \pm f\sqrt{4f+1}.$$

3. TRIGONOMETRIC GRASSMANNIAN
(Sigma 3, 2007, 015)

The generic space in Gr^{trig} is defined to be

$$W = \frac{1}{\prod_{i=1}^{N}(z-\lambda_i)}\Big\{g \in \mathbb{C}[z] : g(\lambda_i) = \delta_i g(\lambda_i - 1)\Big\},$$
$$\lambda_i \neq \lambda_j, i \neq j, \quad \lambda_i - \lambda_j \neq 1, \quad \delta_i \neq 0.$$

$$Gr^{trig} = \Big\{W \in Gr^{rat} : A_W \text{ is isomorphic to a commutative ring of differential operators with coefficients in } \mathbb{C}(e^x)\Big\}.$$

These rings can be described as specific Darboux transformations of constant coefficients differential operators. In the generic case

$$L_0 = \prod_{i=1}^{N}(\partial_x - \lambda_i)(\partial_x - \lambda_i + 1) = Q_0 P_0 \to L = P_0 Q_0.$$

Theorem 3 *There is a commutative diagram*

$$\begin{array}{ccc} \cup_{N\geq 0} C_N & \stackrel{\beta}{\longrightarrow} & Gr^{ad} \equiv Fl^{ad} \\ \Big\downarrow b^C & & \Big\downarrow b \\ \cup_{N\geq 0} C_N^{trig} & \xrightarrow{\beta^{trig}} & Gr^{trig} \end{array}$$

with

$$C_N^{trig} = \Big\{(X,Z) \in GL(N,\mathbb{C}) \times gl(N,\mathbb{C}) : \\ \textit{rank}\,(XZX^{-1} - Z + I) = 1\Big\}/GL(N,\mathbb{C}),$$

and

$$b^C(X,Z) = \Big(I + Z^t, X^t(I + Z^t)\Big).$$

Lemma 1 *The bijection* $\beta : \cup_{N\geq 0} C_N \to Fl^{ad}$ *is given by the map*

$$(X,Z) \to \psi_{\mathcal{V}}(n,z) = (1+z)^n \det\Big\{I + \Big(X - n(I+Z)^{-1}\Big)^{-1}\Big(zI - Z\Big)^{-1}\Big\}.$$

Proof: The tau function of a space $W = \beta(X,Z) \in Gr^{ad}$ is

$$\tau_W(t_1,t_2,t_3,\ldots) = \det\Big\{X - \sum_{k=1}^{\infty} k t_k Z^{k-1}\Big\}, \quad (X,Z) \in C_N.$$

Assuming the spectrum of Z is inside the unit circle, one finds

$$\tau_{\mathcal{V}}(n,t_1,t_2,\ldots) \equiv \tau_W\Big(t_1+n, t_2-\frac{n}{2}, t_3+\frac{n}{3},\ldots\Big),$$
$$= \det\Big\{X - \sum_{k=1}^{\infty} k t_k Z^{k-1} - n(I+Z)^{-1}\Big\}.$$

■

Proof: The (stationary) Baker-Akhiezer function $\psi_W(x,z)$ of $W \in Gr^{trig}$, in terms of $(X,Z) \in C_N^{trig}$ is

$$\psi_W(x,z) = e^{xz}\,\mathrm{det}\Big\{I - X(e^x I - X)^{-1}(zI - Z)^{-1}\Big\},$$

which defines $\beta^{trig} : \cup_{N\geq 0} C_N^{trig} \to Gr^{trig}$. Then,

$$\begin{aligned}
\psi_{b^{-1}(W)}(n,z) &= \psi_W\Big(\log(1+z), n\Big)\\
&= (1+z)^n \mathrm{det}\Big\{I - X\Big((1+z)I - X\Big)^{-1}\Big(nI - Z\Big)^{-1}\Big\}\\
&= (1+z)^n \mathrm{det}\Big\{I + \Big(zI - \tilde{Z}\Big)^{-1}\Big(\tilde{X} - n(I+\tilde{Z})^{-1}\Big)^{-1}\Big\},
\end{aligned}$$

with $\tilde{X} = ZX^{-1}$ and $\tilde{Z} = X - I$.

$\mathrm{rank}([\tilde{Z},\tilde{X}] + I) = \mathrm{rank}([X, ZX^{-1}] + I) = \mathrm{rank}(XZX^{-1} - Z + I) = 1$,

$$\text{i.e.}\quad (b^C)^{-1}(X,Z) = \Big((X^{-1})^t Z^t, X^t - I\Big).$$

$\blacksquare$

4. RUIJSENAARS' DUALITY

Theorem 4 *Let* $(X,Z) \in C_N^{trig}$ *be such that both* X *and* Z *are diagonalizable, and let* $W = \beta^{trig}(X,Z) \in Gr^{trig}$. *Then*

$$\tau_W(x + t_1, t_2, t_3, \dots) = \prod_{i=1}^{N} 2\sinh \frac{\big(x - x_i(t_1, t_2, t_3, \dots)\big)}{2},$$

with $x_i(t_1, t_2, t_3, \dots)$ *a solution of the trigonometric (hyperbolic) Calogero-Moser hierarchy, and*

$$\tau_{b^{-1}(W)}\big(n, t_1, t_2, \dots\big) = \prod_{i=1}^{N} \big(n - \lambda_i(t_1, t_2, \dots)\big),$$

with $\lambda_i(t_1, t_2, \dots)$ *a solution of the rational Ruijsenaars-Schneider hierarchy.*

Proof:

- $(X, Z) \equiv \Big(K = diag(e^{x_1}, \ldots, e^{x_N}),$

$$L_{ij}(x,y) = \delta_{ij} y_i + (1-\delta_{ij}) \frac{1}{2\sinh\left(\frac{1}{2}(x_i - x_j)\right)} \Big).$$

$$H_k(x,y) = \varphi_k\Big(L(x,y)\Big) =$$

$$\frac{1}{k+1} \operatorname{tr}\Big\{ \Big(L(x,y) - I\Big)^{k+1} - L^{k+1}(x,y) \Big\}, \quad k = 1, 2, \ldots$$

$$\prod_{i=1}^{N} \left(e^x - e^{x_i(t)}\right) = \det\left\{ e^x I - K \exp\left\{ \sum_{k=1}^{\infty} t_k \nabla\varphi_k(L) \right\} \right\}$$

$$= e^{Nx} \det\left\{ I - K \exp\left\{ -(t_1 + x) I + \sum_{k=2}^{\infty} t_k \Big((L-I)^k - L^k \Big) \right\} \right\}.$$

$$\bullet\ (X,Z) \equiv \Big(L^{RS}_{ij} = \frac{\mu_i\mu_j}{1+\lambda_j-\lambda_i},\ K^{RS} = diag(\lambda_1,\dots,\lambda_N)\Big),$$

$$\lambda_i \neq \lambda_j,\ \forall i \neq j, \qquad \mu_i = e^{\frac{\theta_i}{2}} \prod_{k\neq i}\left[1-\frac{1}{(\lambda_i-\lambda_k)^2}\right]^{1/4} \neq 0.$$

$$H_k(\lambda,\theta) = \varphi_k\Big(L^{RS}(\lambda,\theta)\Big) = -\mathrm{tr}\Big\{L^{RS}(\lambda,\theta)-I\Big\}^k, \qquad k=1,2,\dots$$

$$\tau_{b^{-1}(W)}(n,t_1,t_2,\dots) =$$

$$(-1)^N(\det X)^{-1}\det\Big\{nI - Z + \sum_{k=1}^{\infty} kt_k(X-I)^{k-1}X\Big\} =$$

$$(-1)^N(\det L^{RS})^{-1}\det\Big\{nI - K^{RS} - \sum_{k=1}^{\infty} t_k\, L^{RS}\nabla\varphi_k(L^{RS})\Big\} =$$

$$(-1)^N e^{-\sum_{i=1}^N\theta_i}\prod_{i=1}^N\Big(n-\lambda_i(t_1,t_2,\dots)\Big).$$

$\blacksquare$

Thank you!