Backlund Transformations for Noncommutative Anti-Self-Dual Yang-Mills (ASDYM) Equations

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NOTE

- **In my poster, the word ``noncommutative (=NC)'' means noncommutative spaces.**
  (motivation of me: a physisist)

$$[x^\mu, x^\nu] = \sqrt{-1}\theta^{\mu\nu}$$

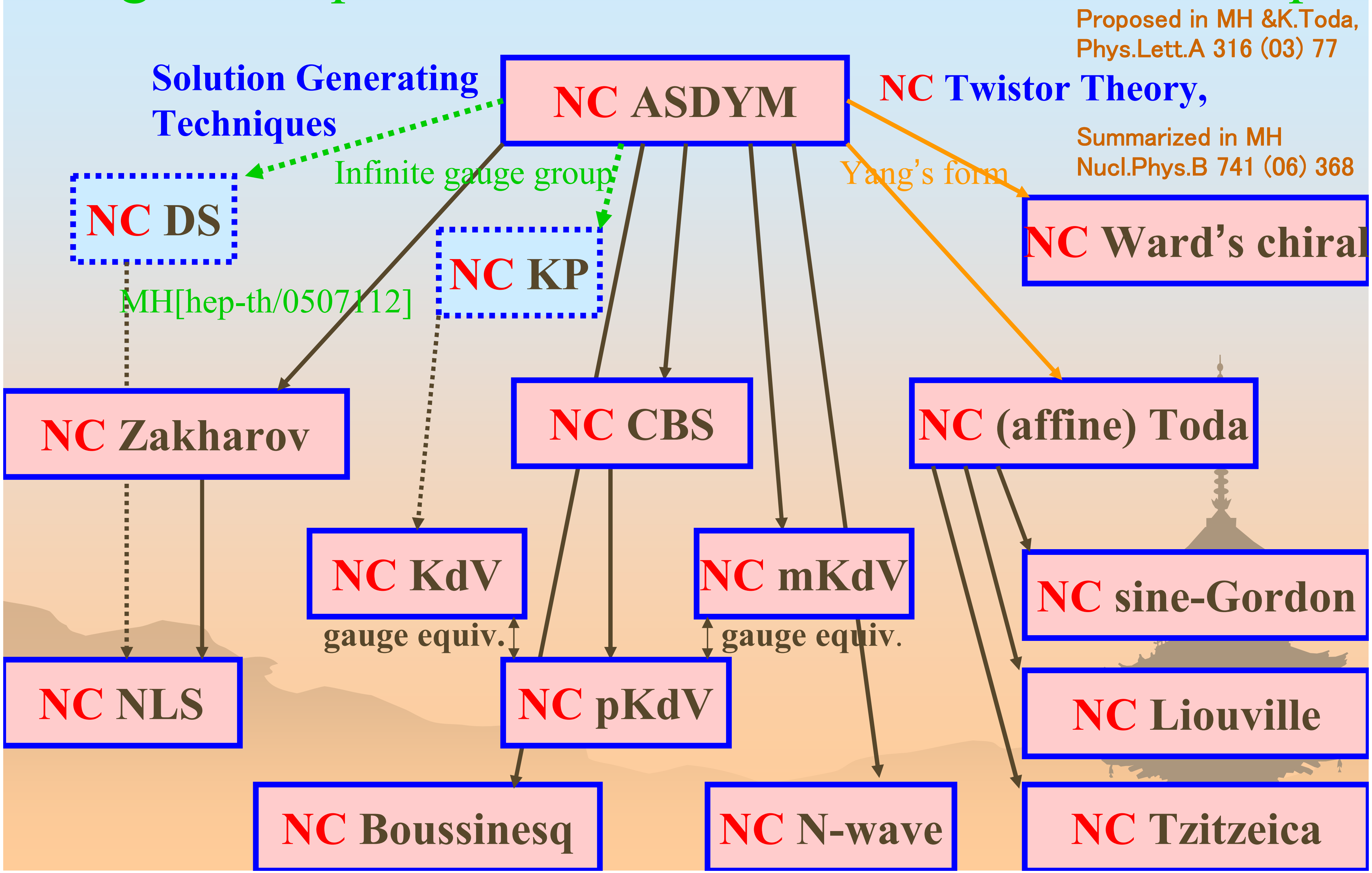
- **But most of the results can be extended to more general noncommutative situation.**

(motivation of Claire and Jon: mathematicians)

- **Here all products of variables becomes NC**

$$e.g. \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a*e+b*g & a*f+b*h \\ c*e+d*g & c*f+d*h \end{pmatrix}$$

NC Ward's conjecture: Many (perhaps all?) NC integrable eqs are reductions of the NC ASDYM eqs.



- NC ASDYM eq. for GL(2) (Yang's form):

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0 \qquad J : 2 \times 2$$

- Yang's J matrix can be decomposed as follows

$$J = \begin{pmatrix} A^{-1} - \tilde{B} * A * B & -\tilde{B} * \tilde{A} \\ \tilde{A} * B & \tilde{A} \end{pmatrix}$$

- The following transformations leave NC ASDYM eq. as it is (Backlund trfs.):

$$\beta : \begin{cases} \partial_z B^{new} = A * \tilde{B}_{\tilde{w}} * \tilde{A}, \ \partial_w B^{new} = A * \tilde{B}_{\tilde{z}} * \tilde{A}, \\ \partial_{\tilde{z}} \tilde{B}^{new} = \tilde{A} * B_w * A, \ \partial_{\tilde{w}} \tilde{B}^{new} = \tilde{A} * B_z * A, \\ A^{new} = \tilde{A}^{-1}, \ \tilde{A}^{new} = A^{-1} \end{cases}$$

(NC Corrigan-Fairlie-Yates-Goddard (CFYG) trf.)

$$\gamma_0 : \begin{pmatrix} A^{-1\,new} & \tilde{B}^{new} \\ B^{new} & \tilde{A}^{-1\,new} \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1} & B \\ \tilde{B} & A^{-1} \end{pmatrix}^{-1}$$

Backlund trf. for NC ASDYM eq.

- **Let's consider the combined Backlund trf.**

$$J_{[1]} \xrightarrow{\alpha=\gamma_0\circ\beta} J_{[2]} \xrightarrow{\alpha} J_{[3]} \xrightarrow{\alpha} \cdots$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

- **If we take a seed sol.,** $A_{[1]} = \tilde{A}_{[1]} = B_{[1]}^{-1} = \tilde{B}_{[1]}^{-1} = \Delta_0, \quad \partial^2 \Delta_0 = 0$

the generated solutions are :

$$A_{[n]} = \left|D_{[n]}\right|_{11}, \tilde{A}_{[n]} = \left|D_{[n]}\right|_{nn}, B_{[n]} = \left|D_{[n]}\right|_{1n}^{-1}, \tilde{B}_{[n]} = \left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

NC Atiyah-Ward ansatz sols.

Quasideterminants !
(a kind of NC determinants)

[Gelfand-Retakh]

We could generate various (complicated) solutions of NC ASDYM eq. from a (simple) seed solution Δ_0 by using the previous Backlund trf. $\alpha = \gamma_0 \circ \beta$

A seed solution:

$\Delta_0 = 1 + ``\frac{1}{z\tilde{z} - w\tilde{w}}"$ → **NC instantons**

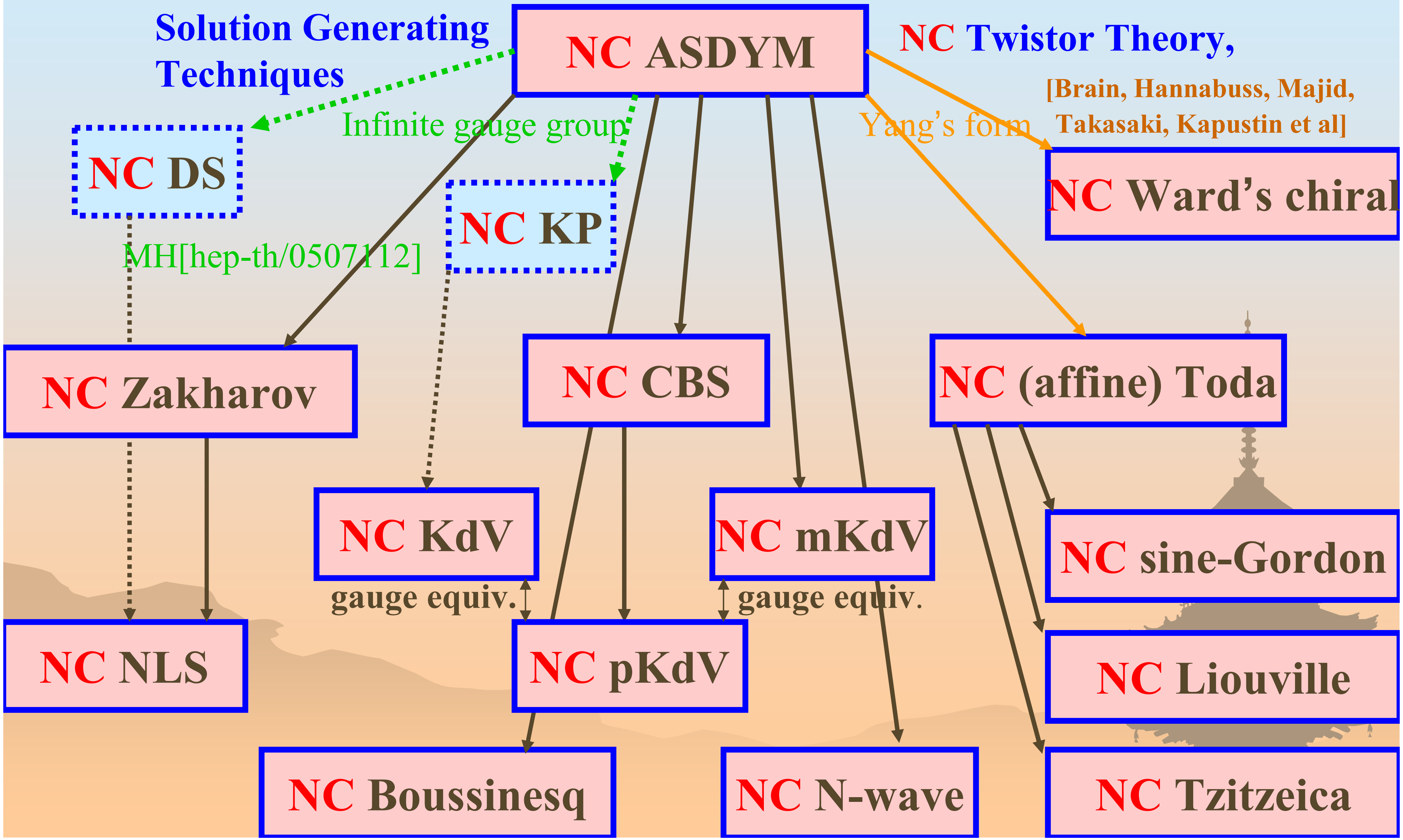
$\Delta_0 = \exp(\textit{linear of } z, \tilde{z}, w, \tilde{w})$ → **NC Non-Linear plane-waves**

NC CFYG trf. would relate to a Darboux transform for NC ASDYM [Gilson&Nimmo&Ohta et. al] and `weakly non-associative' algebras, (cf. Quasideterminants sols. for NC (m)KP are naturally derived from a Darboux trf. [Gilson-Nimmo] [Craig's poster] and the `weakly non-associative' algebras. [Dimakis&Mueller-Hoissen])

NC twistor can give an origin of NC CFYG transform.

$$\beta : F^{new} = \Xi^{-1} * F * \Xi, \quad \gamma_0 : F^{new} = C^{-1} F C,$$

We could reduce the present results into lower-dim. integrable eqs. such as KdV via NC Ward's conjecture.



Reduction to NC KdV eq.

MH, PLB625, 324 [hep-th/0507112]

- (1) Take a dimensional reduction and gauge fixing:

$$(z,\tilde{z},w,\tilde{w}) \to (t,x) = (z, w+\tilde{w}), \qquad A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The reduced NC ASDYM is:

$$(i) \quad [A_w, A_{\tilde{z}}]_* = 0$$

$$(ii) \quad A'_w - A'_{\tilde{w}} + [A_z, A_{\tilde{z}}]_* - [A_w, A_{\tilde{w}}]_* = 0$$

$$(iii) \quad A'_z - \dot{A}_w + [A_w, A_z]_* = 0$$

- (2) Take a further reduction condition:

NOT traceless !

$$A_w = \begin{pmatrix} q & -1 \\ q' + q*q & -q \end{pmatrix}, A_{\tilde{w}} = O, A_z = \begin{pmatrix} \frac{1}{2}q'' + \underline{q'*q} & -q' \\ f(q,q',q'',q''') & -\frac{1}{2}q'' \underline{- q*q'} \end{pmatrix}$$

We can get NC KdV eq. in such a miracle way !

$$(iii) \quad \Rightarrow \quad \dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u'*u + u*u') \qquad u = 2q', \qquad [t,x] = i\theta$$

Note: $A, B, C \in gl(2) \xrightarrow{\theta \to 0} sl(2)$ U(1) part is !

Explicit Atiyah-Ward ansatz solutions of NC Yang's eq. G=GL(2)

$$A_{[1]} = \tilde{A}_{[1]} = B_{[1]}^{-1} = \tilde{B}_{[1]}^{-1} = \Delta_0, \quad \partial^2 \Delta_0 = 0$$

$$A_{[2]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}, \tilde{A}_{[2]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}, B_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1}, \tilde{B}_{[1]} = \begin{vmatrix} \Delta_0 & \Delta_{-1} \\ \Delta_1 & \Delta_0 \end{vmatrix}^{-1},$$

$$\partial_z \Delta_0 = -\partial_{\tilde{w}} \Delta_1, \partial_z \Delta_{-1} = -\partial_{\tilde{w}} \Delta_0, \partial_w \Delta_0 = -\partial_{\tilde{z}} \Delta_1, \partial_w \Delta_{-1} = -\partial_{\tilde{z}} \Delta_0$$

$$A_{[n]} = \left|D_{[n]}\right|_{11}, \tilde{A}_{[n]} = \left|D_{[n]}\right|_{nn}, B_{[n]} = \left|D_{[n]}\right|_{1n}^{-1}, \tilde{B}_{[n]} = \left|D_{[n]}\right|_{n1}^{-1}$$

$$\frac{\partial \Delta_r}{\partial z} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_r}{\partial w} = -\frac{\partial \Delta_{r+1}}{\partial \tilde{z}}$$

$$D_{[n]} = \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-(n-1)} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n-1} & \Delta_{n-2} & \cdots & \Delta_0 \end{pmatrix}$$

$$J_{[n]} = \begin{pmatrix} A_{[n]}^{-1} - \tilde{B}_{[n]} * A_{[n]} * B_{[n]} & -\tilde{B}_{[n]} * \tilde{A}_{[n]} \\ \tilde{A}_{[n]} * B_{[n]} & \tilde{A}_{[n]} \end{pmatrix}$$

Quasi-determinants

- **Quasi-determinants are not just a NC generalization of commutative determinants, but rather related to inverse matrices.**
- **For an n by n matrix** $X=(x_{ij})$ **and the inverse** $Y=(y_{ij})$ **of X, quasi-determinant of X is directly defined by**

$$|X|_{ij} = y_{ji}^{-1} \qquad \left(\xrightarrow{\theta\to 0} \frac{(-1)^{i+j}}{\det X^{ij}} \det X \right)$$

some factor

X^{ij}: **the matrix obtained from X deleting i-th row and j-th column**

- **Recall that**

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Rightarrow$$

$$Y = X^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}$$

→ **We can also define quasi-determinants recursively**

Quasi-determinants

- **Defined inductively as follows**

[For a review, see Gelfand et al., math.QA/0208146]

$$|X|_{ij} = x_{ij} - \sum_{i',j'} x_{ii'}((X^{ij})^{-1})_{i'j'} x_{j'j}$$

$$= x_{ij} - \sum_{i',j'} x_{ii'}(|X^{ij}|_{j'i'})^{-1} x_{j'j}$$

$n=1: \quad |X|_{ij} = x_{ij}$

$n=2: \quad |X|_{11} = x_{11} - x_{12} \cdot x_{22}^{-1} \cdot x_{21}, \; |X|_{12} = x_{12} - x_{11} \cdot x_{21}^{-1} \cdot x_{22},$

$\quad |X|_{21} = x_{21} - x_{22} \cdot x_{12}^{-1} \cdot x_{11}, \; |X|_{22} = x_{22} - x_{21} \cdot x_{11}^{-1} \cdot x_{12},$

$n=3: \quad |X|_{11} = x_{11} - x_{12} \cdot (x_{22} - x_{23} \cdot x_{33}^{-1} \cdot x_{32})^{-1} \cdot x_{21} - x_{13} \cdot (x_{32} - x_{33} \cdot x_{23}^{-1} \cdot x_{22})^{-1} \cdot x_{21}$

$\quad - x_{12} \cdot (x_{23} - x_{22} \cdot x_{32}^{-1} \cdot x_{33})^{-1} \cdot x_{31} - x_{13} \cdot (x_{33} - x_{32} \cdot x_{22}^{-1} \cdot x_{23})^{-1} \cdot x_{31}$

$\quad \ldots$

Here we discuss G=GL(N) NC ASDYM eq. from the viewpoint of linear systems with a spectral parameter ζ .

(All products are star-products.)

- **Linear systems (NC case):**

$$L*\psi=(D_w-\zeta D_{\tilde{z}})*\psi=0,$$
$$M*\psi=(D_z-\zeta D_{\tilde{w}})*\psi=0.$$

e.g. $\begin{pmatrix}\tilde{z} & w\\ \tilde{w} & z\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}x^0+ix^1 & x^2-ix^3\\ x^2+ix^3 & x^0-ix^1\end{pmatrix}$

- **Compatibility condition of the linear system:**

$$[L,M]_*=[D_w,D_z]_*+\zeta([D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*)+\zeta^2[D_{\tilde{z}},D_{\tilde{w}}]_*=0$$

$$\Leftrightarrow\begin{cases}F_{zw}=[D_z,D_w]_*=0,\\ F_{\tilde{z}\tilde{w}}=[D_{\tilde{z}},D_{\tilde{w}}]_*=0,\\ F_{z\tilde{z}}-F_{w\tilde{w}}=[D_z,D_{\tilde{z}}]_*-[D_w,D_{\tilde{w}}]_*=0\end{cases}$$

:NC ASDYM equation

$$(F_{\mu\nu}:=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]_*)$$

$$\theta^{\mu\nu}=\begin{bmatrix}0 & \theta^1 & & \\ -\theta^1 & 0 & & O\\ & & 0 & \theta^2\\ O & & -\theta^2 & 0\end{bmatrix}$$

Yang's form and NC Yang's equation

- **NC ASDYM eq. can be rewritten as follows**

$$\begin{cases} F_{zw} = [D_z, D_w]_* = 0, & \Rightarrow \quad \exists h,\ D_z * h = 0,\ D_w * h = 0 \\ F_{\tilde{z}\tilde{w}} = [D_{\tilde{z}}, D_{\tilde{w}}]_* = 0, & \Rightarrow \quad \exists \tilde{h},\ D_{\tilde{z}} * \tilde{h} = 0,\ D_{\tilde{w}} * \tilde{h} = 0 \\ F_{z\tilde{z}} - F_{w\tilde{w}} = [D_z, D_{\tilde{z}}]_* - [D_w, D_{\tilde{w}}]_* = 0 & \end{cases}$$

If we define Yang's matrix: $J := \tilde{h}^{-1} * h$

then we obtain from the third eq.:

$$\partial_z (J^{-1} * \partial_{\tilde{z}} J) - \partial_w (J^{-1} * \partial_{\tilde{w}} J) = 0$$ **:NC Yang's eq.**

The solution J reproduces the gauge fields as

$$A_z = -h_z * h^{-1},\quad A_w = h_w * h^{-1},\quad A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} * \tilde{h}^{-1},\quad A_{\tilde{w}} = \tilde{h}_{\tilde{w}} * \tilde{h}^{-1}$$

Note: In the present formalism, star products can be replaced with general NC associative products.

We could generate various (non-trivial) solutions of NC Yang's eq. from a (trivial) seed solution by using the previous Backlund trf. together with a simple trf. $\gamma_0 : J^{new} = C^{-1} J C, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\Leftrightarrow \gamma_0 : \begin{pmatrix} A^{-1^{new}} & \widetilde{B}^{new} \\ B^{new} & \widetilde{A}^{-1^{new}} \end{pmatrix} = \begin{pmatrix} \widetilde{A}^{-1} & B \\ \widetilde{B} & A^{-1} \end{pmatrix}^{-1}$$

This combined trf. would generate a group of hidden symmetry of NC Yang's eq., which would be also applied to lower-dimension.

For G=GL(2), we can present the transforms more explicitly and give an explicit form of a class of solutions (Atiyah-Ward ansatz).

1. Introduction (my motivation)

Successful points in NC theories

- Appearance of new physical objects
- Description of real physics (in gauge theory)
- Various successful applications
  to D-brane dynamics etc.

Construction of exact solitons are important.
(partially due to their integrablity)

Final goal: NC extension of all soliton theories
(Soliton eqs. can be embedded in gauge theories
via Ward’s conjecture !)

(Q) How we get NC version of the theories?

(A) We have only to replace all products of fields in ordinary commutative gauge theories with star-products: $f(x)g(x) \rightarrow f(x) * g(x)$

- **The star product: (NC and associative)**

$$f(x) * g(x) := f(x) \exp\left(\frac{i}{2} \theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu \right) g(x) = f(x)g(x) + i \frac{\theta^{\mu\nu}}{2} \partial_\mu f(x) \partial_\nu g(x) + O(\theta^2)$$

A deformed product

Presence of background magnetic fields

Note: coordinates and fields themselves are usual c-number functions. But commutator of coordinates becomes…

$$[x^\mu, x^\nu]_* := x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}$$

NC !

3. Backlund transforms for NC KdV eq.

- In this section, we give an exact soliton solutions of NC KdV eq. by a Darboux transformation.
  [Gilson-Nimmo, JPA40(07)3839, nlin.si/0701027]
- We see that ingredients of quasi-determinants are naturally generated by the Darboux transformation. (an origin of quasi-determinants)
- We also make a comment on asymptotic behavior of soliton scattering process [MH, JHEP 02 (2007) 094 [hep-th/0610006].

Lax pair of NC KdV eq.

- **Linear systems:**

$$L*\psi=(\partial_x^2+u-\lambda^2)*\psi=0,$$

$$M*\psi=(\partial_t-\partial_x^3-(3/2)u\partial_x-(3/4)u_x)*\psi=0.$$

- **Compatibility condition of the linear system:**

$$[L,M]_*=0 \quad \Leftrightarrow \quad \dot{u}=\frac{1}{4}u_{xxx}+\frac{3}{4}(u*u_x+u_x*u)$$

:NC KdV equation

- **Darboux transform for NC KdV**

Let us take an eigen function W of L and define $\Phi=W*\partial_x W^{-1}$
Then the following trf. leaves the linear systems as it is:

$$\widetilde{L}=\Phi*L*\Phi^{-1},\quad \widetilde{M}=\Phi*M*\Phi^{-1},\quad \widetilde{\psi}=\Phi*\psi$$

and $\widetilde{u}=u+2(W_x*W^{-1})_x \quad (\xrightarrow{\theta\to 0} u+2\partial_x^2\log W)$

The Darboux transformation can be iterated

- **Let us take eigen fcns.** $(f_1,\cdots,f_N)$ **of** L **and define**

$$\Phi_i = W_i * \partial_x W_i^{-1} = \partial_x - W_{i,x} * W_i^{-1} \qquad (W_1 \equiv f_1, \Phi_1 = f_1 * \partial_x f_1)$$

$$W_{i+1} = \Phi_i * f_{i+1} = f_{i+1,x} - W_{i,x} * W_i^{-1} * f_{i+1} \qquad (i = 1,2,3,\cdots)$$

$$\psi_{i+1} = \Phi_i * \psi_i = \psi_{i,x} - W_{i,x} * W_i^{-1} * \psi_i$$

- **Iterated Darboux transform for NC KdV**

The following trf. leaves the linear systems as it is

$$L_{[i+1]} = \Phi_i * L_{[i]} * \Phi_i^{-1}, \quad M_{[i+1]} = \Phi_i * M_{[i]} * \Phi_i^{-1}, \quad \psi_{[i+1]} = \Phi_i * \psi_{[i]}$$

$$(L_{[1]}, M_{[1]}, \psi_{[1]}) \xrightarrow{\Phi_1} (L_{[2]}, M_{[2]}, \psi_{[2]}) \xrightarrow{\Phi_2} \cdots$$

$$|||$$

$$(L, M, \psi)$$

In fact, (W_i, ψ_i) **are quasi-determinants of Wronski matrices !**

and

$$u_{[N+1]} = u + 2\sum_{i=1}^{N} (W_{i,x} * W_i^{-1})_x \quad \left(\xrightarrow{\theta\to 0} u + 2\partial_x^2 \log W(f_1,\cdots,f_N)\right)$$

Exact N-soliton solutions of the NC KdV eq.

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} \partial_x^2 \log\det W(f_1, \cdots, f_N)$$

Etingof–Gelfand–Retakh,
[q-alg/9701008]

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

$$f_i = \exp\left(\xi(x, \lambda_i)\right) + a_i \exp\left(-\xi(x, \lambda_i)\right)$$

$$\xi(x, t, \lambda) = x_1 \lambda + t \lambda_i^3 \qquad (M * f_i = (\partial_t - \partial_x^3) f_i = 0)$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

Quasi-det solutions can be extended to NC integrable hierarchy

Exact N-soliton solutions of the NC KP hierarchy

$$L = \Phi * \partial_x \Phi^{-1}$$ solves the NC KP hierarchy !

$$= \partial_x + \frac{u}{2}\partial_x^{-1} + \cdots$$

$$\Phi f := \left| W(f_1, \ldots, f_N, f) \right|_{N+1, N+1}$$

quasi-determinant of Wronski matrix

Etingof-Gelfand-Retakh, [q-alg/9701008]

$$f_i = \exp \xi(x, \alpha_i) + a_i \exp \xi(x, \beta_i)$$

$$\xi(x, \alpha) = x_1 \alpha + x_2 \alpha^2 + x_3 \alpha^3 + \cdots$$

$$u = 2\partial_x \sum_{i=1}^{N} (\partial_x W_i) * W_i^{-1} \xrightarrow{\theta \to 0} 2\partial_x^2 \log\det W(f_1, \cdots, f_N)$$

$$W_i := \left| W(f_1, \ldots, f_i) \right|_{i,i}$$

Wronski matrix:

$$W(f_1, f_2, \cdots, f_m) = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \\ \partial_x f_1 & \partial_x f_2 & \cdots & \partial_x f_m \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{m-1} f_1 & \partial_x^{m-1} f_2 & \cdots & \partial_x^{m-1} f_m \end{bmatrix}$$

More generalization is possible.

[MH, hep-th/0610006]

Interpretation of the exact N-soliton solutions

- We have found exact N-soliton solutions for the wide class of NC hierarchies.
- Physical interpretations are non-trivial because when $f(x), g(x)$ are real, $f(x)*g(x)$ is not in general.
- However, the solutions could be **real** in some cases.
  - (i) **1-soliton solutions are all the same as commutative ones because of** **Dimakis-Muller-Hoissen, [hep-th/0007015]**

$$f(x-vt)*g(x-vt)=f(x-vt)g(x-vt)$$

  - (ii) **In asymptotic region, configurations of multi-soliton solutions could be real in soliton scatterings and the same as commutative ones.**

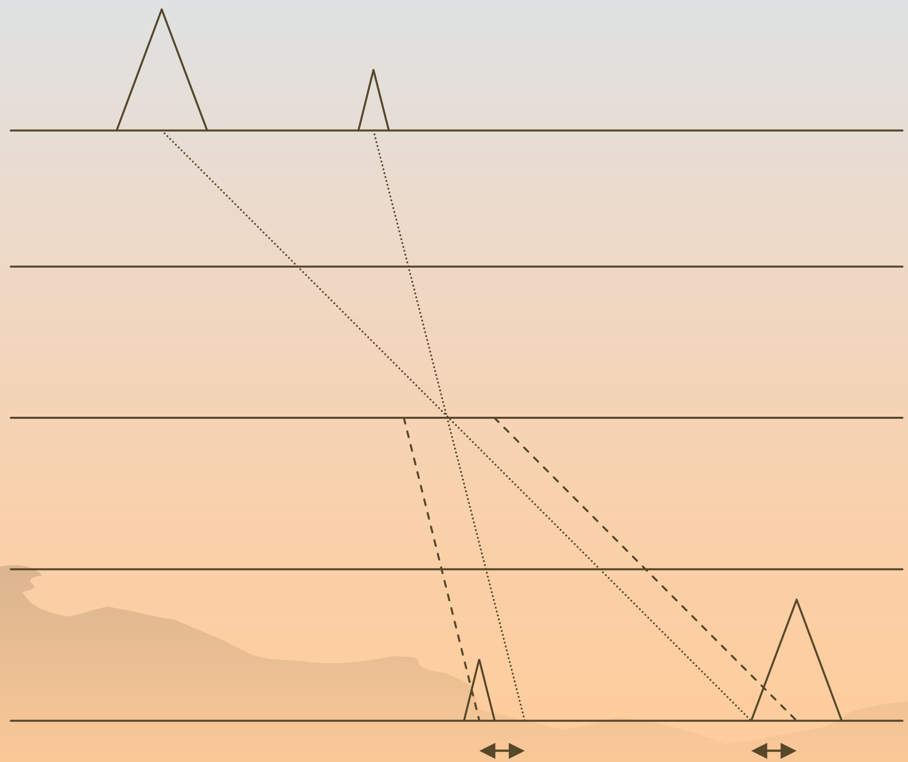
MH, JHEP[hep-th/0610006]

❁ 2-soliton solution of KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\,2}\cosh^{-2}(k_i x - 4k_i^{\,3} t), \quad \underset{\text{velocity}}{v_i} = 4k_i^{\,2}, \quad \underset{\text{height}}{h_i} = 2k_i^{\,2}$$

Scattering process (commutative case)



The shape
and velocity
is preserved ! (stable)

The positions are shifted ! (Phase shift)

MH, JHEP02 (2007) 094
[hep-th/0610006]
cf Paniak, hep-th/0105185

❁ 2-soliton solution of NC KdV

each packet has the configuration:

$$u^{(i)} = 2k_i^{\ 2}\cosh^{-2}(k_i x - 4k_i^{\ 3} t), \quad \underset{\text{velocity}}{v_i} = 4k_i^{\ 2}, \quad \underset{\text{height}}{h_i} = 2k_i^{\ 2}$$

Scattering process (NC case)